Nonextensive statistical effects on the relativistic nuclear equation of state

A. Drago\textsuperscript{a}, A. Lavagno\textsuperscript{b}, P. Quarati\textsuperscript{b}

\textsuperscript{a}Dipartimento di Fisica, Università di Ferrara and INFN, 44100 Ferrara, Italy
\textsuperscript{b}Dipartimento di Fisica, Politecnico di Torino and INFN, 10129 Torino, Italy

Following the basic prescriptions of the Tsallis’ nonextensive thermodynamics, we study the relativistic nonextensive thermodynamics and the equation of state for a perfect gas at the equilibrium. The obtained results are used to study the relativistic nuclear equation of state in the hadronic and in the quark-gluon plasma phase. We show that small deviations from the standard extensive statistics imply remarkable effects into the shape of the equation of state.

Key words: Relativistic thermodynamics, Equation of state, Quark-gluon plasma
PACS: 05.20.Dd, 05.90.+m, 25.75.Nq

1 Introduction

It has been shown that the nonextensive generalization of the Boltzmann-Gibbs thermostatistics, proposed by Tsallis, can be very relevant in many physical applications where long-range interactions, long-range microscopic memories and/or fractal space-time constraints are present [1]. In particular, recently, many authors outline the possible connection to the nonextensive statistical framework with nuclear and high energy physical applications [2–7]. The aim of this work is to generalize the basic concepts of the nonextensive statistical mechanics to the relativistic regime and to investigate, through the obtained relativistic thermodynamic relations, the relevance of nonextensive statistical effects on the hadronic and on the quark-gluon plasma (QGP) equation of state (EOS). As we will see, small deviations from the extensive thermostatistics produce a significant modification into the shape of the hadronic and QGP equation of state with important consequence on the deconfined phase transition and on several nuclear properties.
In this section we present the basic macroscopic thermodynamics variables in the language of the nonextensive relativistic kinetic theory. Let us start by introducing the particle four-flow in the phase space as [8]

\[ N^\mu(x) = \frac{1}{Z_q} \int \frac{d^3p}{p^0} p^\mu f(x, p), \] (1)

and the energy-momentum flow as

\[ T^{\mu\nu}(x) = \frac{1}{Z_q} \int \frac{d^3p}{p^0} p^\mu p^\nu f^q(x, p), \] (2)

where we have set \( \hbar = c = 1 \), \( x \equiv x^\mu = (t, \mathbf{x}) \), \( p \equiv p^\mu = (p^0, \mathbf{p}) \), \( p^0 = \sqrt{\mathbf{p}^2 + m^2} \) is the relativistic energy and \( f(x, p) \) is the particle distribution function. The four-vector \( N^\mu = (n, \mathbf{j}) \) contains the probability density \( n = n(x) \) (which is normalized to unity) and the probability flow \( j = j(x) \). The energy-momentum tensor contains the normalized \( q \)-mean expectation value of the energy density, as well as the energy flow, the momentum and the momentum flow per particle. Its expression follows directly from the definition of the mean \( q \)-expectation value in nonextensive statistics [1]; for this reason it is given in terms of \( f^q(x, p) \).

Furthermore, in the framework of the nonextensive thermostatistics, it appears natural to generalize the nonextensive four-flow entropy \( S^\mu_q(x) \) as follows

\[ S^\mu_q(x) = -k_B \int \frac{d^3p}{p^0} p^\mu f^q(x, p) [\ln_q f(x, p) - 1]. \] (3)

It is possible to show [8] that such an entropy, together a generalized relativistic Boltzmann equation, satisfies the relativistic local \( H \)-theorem and implies the following Tsallis-like equilibrium probability distribution

\[ f_{eq}(p) = \frac{1}{Z_q} \left[ 1 - (1 - q) \frac{p^\mu U_\mu}{k_B T} \right]^{1/(1-q)}, \] (4)

where \( U_\mu \) is the hydrodynamic four-velocity [9] and \( f_{eq} \) depends only on the momentum in absence of an external field. At this stage, \( k_B T \) is a free parameter and only in the derivation of the equation of state it will be identified with the physical temperature.
We are able now to evaluate explicitly all other thermodynamic variables and provide a complete macroscopic description of a relativistic system at the equilibrium. Let us first calculate the probability density defined as

\[ n = N^\mu U_\mu = \frac{1}{Z_q} \int \frac{d^3p}{p^0} p^\mu U_\mu f_{eq}(p) . \]  

(5)

Since \( n \) is a scalar, it can be evaluated in the rest frame where \( U^\mu = (1, 0, 0, 0) \). Setting \( \tau = p^0/k_B T \) and \( z = m/k_B T \), the above integral can be written as

\[ n = \frac{4\pi}{Z_q} (k_B T)^3 \int_0^\infty d\tau (\tau^2 - z^2)^{1/2} \tau e^{-\tau} . \]  

(6)

Let us introduce the \( q \)-modified Bessel function of the second kind as follows

\[ K_n(q, z) = \frac{2^n n!}{(2n)!} z^n \int_0^\infty d\tau (\tau^2 - z^2)^{n-1/2} \left( e^{-\tau} \right)^q , \]  

(7)

then, by means of a partial integration of Eq.(6), the particle density can be cast into the compact form

\[ n = \frac{4\pi}{Z_q} m^2 k_B T K_2(q, z) . \]  

(8)

Considering the decomposition of the energy-momentum tensor \([9]\): \( T^{\mu\nu} = \epsilon U^\mu U^\nu - p \Delta^{\mu\nu} \), where \( \epsilon \) is the energy density, \( p \) the pressure and \( \Delta^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu \), the equilibrium pressure can be calculated as

\[ p = -\frac{1}{3} T^{\mu\nu} \Delta_{\mu\nu} = -\frac{1}{3} \frac{1}{Z_q} \int \frac{d^3p}{p^0} p^\mu p^\nu \Delta_{\mu\nu} f_{eq}(p) , \]  

(9)

and can be expressed as

\[ p = \frac{4\pi}{Z_q} m^2 (k_B T)^2 K_2(q, z) . \]  

(10)

Comparing Eq.(8) with Eq.(10), we obtain

\[ p = n k_B T , \]  

(11)

which is the equation of state of a perfect gas if we identify \( T \) as the physical temperature of the system.
We proceed now to calculate the energy density $\epsilon$ as

$$\epsilon = T^{\mu\nu}U_\mu U_\nu = \frac{1}{Z_q} \int \frac{d^3 p}{p^0} (p^\mu U_\mu)^2 f_q(p). \quad (12)$$

Inserting the previously defined variables $\tau$ and $z$ and using the definition in Eq.(7), we obtain

$$\epsilon = \frac{4\pi}{Z_q} m^4 \left[ 3 \frac{K_2(q, z)}{z^2} + \frac{K_1(q, z)}{z} \right]. \quad (13)$$

Thus the energy per particle $e = \epsilon/n$ is

$$e = 3 k_B T + m \frac{K_1(q, z)}{K_2(q, z)}, \quad (14)$$

which has the same structure of the relativistic expression obtained in the framework of the equilibrium Boltzmann-Gibbs statistics [9].

For a system of particles in degenerate regime the above classical distribution function (4) has to be modified by including the fermion and boson quantum statistical prescriptions. For a dilute gas of particles and/or for small deviations from the standard extensive statistics ($q \approx 1$) the equilibrium distribution function, in the grand canonical ensemble, can be written as [10]

$$n(k, \mu) = \frac{1}{[1 + (q - 1)(E(k) - \mu)/T]^{1/(q-1)} \pm 1}, \quad (15)$$

where the sign + stands for fermion and − for boson particle. All the previous results can be easily extended to the case of quantum statistical mechanics.

3 Nonextensive statistics in hadronic matter and QGP

The motivation of the importance of non-standard statistical effects in nuclear and high energy physics lies in the fact the extreme conditions of density and temperature in high energy nuclear collisions give rise to memory effects and long–range color interactions. These conditions imply the presence of non–Markovian processes in the kinetic equation affecting the thermalization process toward equilibrium as well as the standard equilibrium distribution [11,12]. A rigorous determination of the conditions that produce a nonextensive behavior, due to memory effects and/or long–range interactions,
should be based on microscopic calculations relative to the parton plasma originated during the high energy collisions. At this stage we limit ourselves to consider the problem from a qualitative point of view. However, it is noteworthy to notice that in proximity of the hadronic-QGP phase transition, non-perturbative QCD calculations become important. Only a small number of partons is present in the Debye sphere: the ordinary mean field approximation of the plasma is no longer correct and memory effects are not negligible. In addition, we observe that at high density the color magnetic field remains unscreened (in leading order) and long-range color magnetic interaction should be present at all temperatures.

From the above considerations it appears reasonable that in regime of high density and temperature both hadronic and quark-gluon EOSs must be affected by nonextensive statistical effects. In the next two subsections, we will study the two EOSs separately on the basis on the previously obtained relativistic thermodynamic relations.

3.1 Nonextensive hadronic equation of state

Concerning the hadronic phase we use a relativistic non-linear model based on the interacting many-particle system consisting of nucleons, isoscalar and isovector mesons ($\sigma$, $\omega$ and $\rho$)[13]. On the basis of the Eqs.(2), (9) and (12), the pressure and the energy density can be written as

\[ P = \sum_{i=n,p} \frac{2}{3} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{E_i^*(k)} [n^q(k, \mu_i^*) + n^q(k, -\mu_i^*)] - \frac{1}{2} m_{\sigma, \omega, \rho}^2 \phi^2 - \frac{1}{3} a_\sigma \sigma^3 - \frac{1}{4} b_\rho \rho^4 + \frac{1}{2} m_\omega^2 \omega^2 + \frac{1}{2} m_\rho^2 \rho^2, \] (16)

\[ \epsilon = \sum_{i=n,p} 2 \int \frac{d^3k}{(2\pi)^3} E_i^*(k) [n^q(k, \mu_i^*) + n^q(k, -\mu_i^*)] + \frac{1}{2} m_{\sigma, \omega, \rho}^2 \] (17)

where $n(k, \mu_i)$ and $n(k, -\mu_i)$ are the fermion particle and antiparticle distribution (15). The nucleon effective energy is defined as $E_i^* = \sqrt{k^2 + M_i^* \rho^2}$, where $M_i^* = M_N - g_\sigma \sigma$. The effective chemical potentials $\mu_i^*$ are given in terms of the vector meson mean fields $\mu_i = \mu_i^* - g_\omega \omega \mp g_\rho \rho$ (– proton, + neutron), where $\mu_i$ are the thermodynamical chemical potentials $\mu_i = \partial \epsilon / \partial \rho_i$. At zero temperature they reduce to the Fermi energies $E_{F_i} \equiv \sqrt{k_{F_i}^2 + M_i^2}$ and the nonextensive statistical effects disappear. The isoscalar and isovector meson fields ($\sigma$, $\omega$ and $\rho$) are obtained as a solution of the field equations in mean
field approximation and the related couplings \(g_\sigma\), \(g_\omega\) and \(g_\rho\) are the free parameter of the model [13]. Finally, The baryon densities \(\rho_B\) are given by

\[
\rho_B = \gamma \int \frac{d^3k}{(2\pi)^3} \left[ n(k, \mu_i^*) - n(k, -\mu_i^*) \right],
\]

(18)

where \(\gamma\) is the spin/isospin multiplicity.

In Fig. 1, we report the obtained hadronic EOS: pressure as a function of the baryon number density (in units of the nuclear saturation density \(\rho_0 = 0.16\) fm\(^{-3}\)) for different values of \(q\). Because of previous phenomenological studies in heavy-ion collisions have brought to values of \(q\) greater than unity [2], we concentrate our analysis to \(q > 1\). The results are plotted at the temperature \(T = 100\) MeV and at fixed value of \(Z/A = 0.4\). The range of the considered baryon density and the chosen values of the parameters correspond to a physical situation that can realized in the recently proposed high energy heavy-ion collisions experiment (like in \(Au - Au\) collisions) (see, for example, Ref.[15]).

![Fig. 1. Hadronic equation of state: pressure versus baryon number density (in units of the nuclear saturation density \(\rho_0\)) for different values of \(q\).](image)

3.2 *Nonextensive QGP equation of state*

In the simple model of free quarks in a bag, the pressure, energy density and baryon number density for a Fermi gas of quarks in the framework of nonextensive statistics (see Eqs.(1), (2), (9) and (12)) can be written, respectively, as

\[
P = \sum_{f=u,d} \frac{1}{3} \frac{\gamma_f}{2\pi^2} \int_0^\infty k \frac{\partial \epsilon_f}{\partial k} \left[ n^q(k, \mu_f) + n^q(k, -\mu_f) \right] k^2 dk - B;
\]

(19)
\[ 
\epsilon = \sum_{f=u,d} \frac{\gamma_f}{2\pi^2} \int_0^\infty \epsilon_f [n^q(k,\mu_f) + n^q(k,-\mu_f)] k^2 dk + B, \tag{20} 
\]

\[ 
\rho = \sum_{f=u,d} \frac{1}{3} \frac{\gamma_f}{2\pi^2} \int_0^\infty [n(k,\mu_f) - n(k,-\mu_f)] k^2 dk, \tag{21} 
\]

where \( \epsilon_f = (k^2 + m_f^2)^{1/2} \), \( n(k,\mu_f) \) and \( n(k,\mu_f) \) are the particle and antiparticle quark distributions. The quark degeneracy for each flavor is \( \gamma_f = 6 \). Similar expressions for the pressure and the energy density can be written for the gluons as massless Bose gas with zero chemical potential and degeneracy factor \( \gamma_g = 16 \). Because of the fermion-boson nonextensive distribution (15), the results are not analytical, also in the massless quark approximation. Therefore, a numerical evaluations of the integral must be performed. Note that a similar calculation, only for the quark-gluon phase, was also performed in Ref.[14] by studying the phase transition diagram.

In Fig. 2, we report the EOS for massless quark \( u, d \) and gluons, for different values of \( q \). As in Fig. 1, the results are plotted at the temperature \( T = 100 \) MeV and at fixed value of \( Z/A = 0.4 \); the bag parameter is \( B^{1/4} = 170 \) MeV. In both the figures we can observe remarkable effects in the shape of the EOS for small deviations from the standard statistics.

![Fig. 2. Same of Fig. 1 for the case of the quark-gluon equation of state with \( B^{1/4} = 170 \) MeV.](image2.png)

4 Conclusion

In this work we have studied the relativistic thermodynamic relations and derived the EOS of a gas of free particle at the equilibrium in the framework of the nonextensive Tsallis thermostatistics. The results are applied to obtain a consistent generalization of the EOS of strongly interacting hadronic matter.
and of deconfined QGP. The range of density and temperature chosen are physical values estimated in the recently proposed high energy heavy-ion collisions experiments at finite baryon chemical potential [15]. We find that small deviations from the Boltzmann-Gibbs statistics implies a sensible modification of the two considered EOSs. A complete discussion on the nuclear physics consequence lies out the scope of this paper, however we want to outline that such a modification of the EOSs can strongly affect different physical properties, like, for example, the critical phase transition density, the symmetry energy, the nuclear compressibility, connected to experimental observables.

References


