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GROUP THEORETICAL CALCULATION OF TRACES AND
DETERMINANTS OCCURRING IN DUAL THEORIES

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A B S T R A C T

A group theoretical calculation is given of all the functions, defined in terms of infinite matrices, which occur in the dual theory of fermions and off-mass shell states when treated in the operator formalism.

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We wish to report a new way of calculating certain quantities arising in recent work in dual theories ¹⁾. The method is simpler and more natural, in being group theoretical, though it relies upon a mysterious relationship between certain inequivalent representations of the Möbius group.

Consider the infinite matrices,

$$\begin{aligned} M_{rs} &= -(-\lambda)^{r+s} \frac{r}{r+s} \begin{pmatrix} -\frac{1}{2} \\ r-\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ s-\frac{1}{2} \end{pmatrix} \quad r, s = \frac{1}{2}, \frac{3}{2}, \dots \\ S_{MN} &= (-\lambda)^{M+N} \frac{M}{M+N} \begin{pmatrix} -\frac{1}{2} \\ M \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ N \end{pmatrix} \quad M, N = 1, 2, \dots \end{aligned} \quad (1)$$

and the vectors,

$$\begin{aligned} v_r &= \frac{1}{\sqrt{2}} (-\lambda)^{r-\frac{1}{2}} \begin{pmatrix} -\frac{1}{2} \\ r-\frac{1}{2} \end{pmatrix} \lambda^r \quad r = \frac{1}{2}, \dots \\ w_M &= \frac{1}{\sqrt{2}} (-\lambda)^M \begin{pmatrix} -\frac{1}{2} \\ M \end{pmatrix} \quad M = 1, 2, \dots \end{aligned} \quad (2)$$

The quantities of interest are $\det(1-M^2)$, $\det(1-S^2)$, $v^T(1-M^2)^{-1}v$ and $w^T(1-S^2)^{-1}w$, together with the similar quantities in which the matrices M and S are replaced by their antisymmetric parts, A and T .

The fundamental algebra of dual theories is the Virasoro algebra,

$$\begin{aligned} [L_N, L_M]_- &= (N-M) L_{N+M} + \delta_{N+M,0} N(N^2-1) C \\ L_N &= L_{-N}^+ \quad N, M = 0, \pm 1, \pm 2, \dots \end{aligned} \quad (3)$$

where C is a representation dependent c number which does not affect the Möbius subalgebra, $N, M = 0, \pm 1$.

Representations of the latter, bilinear in annihilation and creation operators, are easily obtained from the known irreducible representations of the Möbius group, conventionally labelled (J, k) , where $J(J+1)$ is the quadratic Casimir operator and k is the non-integral part of L_0 . The extension to the whole Virasoro algebra fixes $J=0$ or $-\frac{1}{2}$ (with the operators being Bose or Fermi, respectively) ²⁾. Asking for a singly infinite set of oscillators fixes $k=0$ or $\frac{1}{2}$ ³⁾, leaving the four possibilities tabulated below :

J	k	
	0	$\frac{1}{2}$
0	Veneziano, $Q^\mu(z)$	Off-shell states, $R^\mu(z)$
$-\frac{1}{2}$	Ramond, $\Gamma^\mu(z)$	Neveu-Schwarz, $H^\mu(z)$

The theory with fermions involves Q^μ together with either Γ^μ or H^μ depending on the sector; fermionic or mesonic ⁴⁾. The fermion emission vertex, $W(z)$, maps between the two $J = -\frac{1}{2}$ fields Γ^μ and H^μ and involves both the matrices M and S ^{5),6)}. In fact these are the reflection matrices for G and F gauges, respectively ⁷⁾.

It has recently been found ⁸⁾ that the corresponding quantity mapping between the two $J=0$ fields Q^μ and R^μ plays an important role in the construction of vertices for the emission of off-mass shell states. It also involves the matrices M and S ⁹⁾. Inside the off-shell vertex R^μ is used in conjunction with H or Γ , instead of Q^μ , and it is again found that the matrices M and S arise in the reflection of gauges ⁸⁾. Whereas the representations $(0,0)$, $(-\frac{1}{2},0)$, $(-\frac{1}{2},\frac{1}{2})$ can be interpreted in terms of a massless string ¹⁰⁾, this is not yet the case when the representation $(0,\frac{1}{2})$ is involved.

The $(-\frac{1}{2},0)$ and $(0,\frac{1}{2})$ fields $R^\mu(z)$ and $\Gamma^\mu(z)$ are, explicitly

$$R^\mu(z) = -i \sum_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{r}} (z^r \hat{c}_r^\mu - z^{-r} \hat{c}_r^\mu)$$

$$\Gamma^\mu(z) = \gamma^\mu + i\sqrt{2} \gamma^\mu \sum_1^{\infty} (z^n d_n^{\mu+} + z^{-n} d_n^\mu)$$

where

$$[c_r^\mu, \hat{c}_s^\mu]_- = -\delta_{rs} g^{\mu\nu}$$

$$[d_n^\mu, d_m^{\mu+}]_+ = -\delta_{nm} g^{\mu\nu}$$

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}, \quad (\gamma^\mu)^2 = 1$$

The corresponding Virasoro generators

$$L_N^c = \frac{1}{4\pi i} \oint dz z^{N+1} : \left(\frac{dR}{dz} \right)^2 : + \frac{D}{16} \delta_{N0}$$

$$L_N^d = \frac{1}{8\pi} \oint dz z^N : \Gamma(z) \cdot \frac{d\Gamma(z)}{dz} : + \frac{D}{16} \delta_{N0}$$

share the properties

$$L_1 |0\rangle = 0, \quad L_{-1} |0\rangle \neq 0, \quad L_0 |0\rangle = \frac{D}{16} |0\rangle \quad (4)$$

with D the dimension of space-time. For the other representations the state $|0\rangle$ is invariant under the group.

It is fundamental that ⁶⁾

$$W(z) |0\rangle_{NS} = e^{z L_{-1}^d} |0\rangle_R$$

and it is this which suggests our basic ploy of considering carefully chosen expectation values with respect to the state

$$e^{z L_{-1}^d} |0\rangle$$

In fact, as already noted long ago ⁶⁾,

$$e^{z L_{-1}^d} |0\rangle = \exp \left\{ \gamma^\mu \gamma_\mu \cdot d^+ - \frac{1}{2} d^+ T d^+ \right\} |0\rangle \quad (5)$$

while it can be proved similarly, (by first normal ordering), that

$$e^{z L_{-1}^c} |0\rangle = \exp \left\{ \frac{1}{2} c^+ P c^+ \right\} |0\rangle \quad (6)$$

where $T = \frac{1}{2}(S - S^T)$, $P = -N^{-1}MN$, $N_{rs} = \delta_{rs} \sqrt{r}$.

The expectation values are evaluated by means of the group relation (simply checked for the two-dimensional representation)

$$e^{\lambda L_{+1}} e^{\mu L_{-1}} = e^{\frac{\mu}{1-\lambda\mu} L_{-1}} (1-\lambda\mu)^{-2L_0} e^{\frac{\lambda}{1-\lambda\mu} L_{+1}} \quad (7)$$

For example, using Eqs. (4) and (7)

$$\langle 0 | e^{\lambda L_{+1}} e^{\lambda L_{-1}} | 0 \rangle = (1 - \lambda^2)^{-D/8}$$

which, by Eq. (6), is also

$$\langle 0 | e^{\frac{1}{2} c P c} e^{\frac{1}{2} c^{\dagger} P c^{\dagger}} | 0 \rangle = [\det(1 - P^2)]^{-D/2}$$

using another identity ^{11),12)}

$$\langle 0 | e^{\frac{1}{2} \gamma A \gamma} e^{\frac{1}{2} \gamma^{\dagger} B \gamma^{\dagger}} | 0 \rangle = [\det(1 \mp AB)]^{\mp D/2} \quad (8)$$

with the - or + signs, according to whether the γ operators are Bose or Fermi, respectively.

Hence our first result,

$$\det(1 - M^2) = (1 - \lambda^2)^{1/4} \quad (9)$$

This is the correction factor for the projected propagator in the four fermion amplitude ⁷⁾.

Since A , the antisymmetric part of M , equals $M - vv^T$, one can prove ¹²⁾

$$\det(1 - A^2) = \det(1 - M^2) \frac{1}{1 - (v^T(1 - A^2)^{-1}v)^2}$$

and

$$v^T(1 - M^2)^{-1}v = \frac{v^T(1 - A^2)^{-1}v}{1 - (v^T(1 - A^2)^{-1}v)^2}$$

Thus it remains to calculate $v^T(1 - M^2)^{-1}v$ which we do now by evaluating the quantity

$$F = \langle 0 | e^{\lambda L_{+1}} \text{z.c. y.c.} e^{\lambda L_{-1}} | 0 \rangle$$

in two ways, as before. Using Eqs. (6) and (8), we find

$$F = - [\det(1 - P^2)]^{-D/2} x^T (1 - P^2)^{-1} y$$

whilst repeated use of Eqs. (4) and (7) yields

$$F = (1 - \lambda^2)^{-D/8} \langle 0 | R(\lambda) x \cdot c \ y \cdot c^+ R^{-1}(\lambda) | 0 \rangle$$

where R is the representation in the c Fock space of the projective transformation

$$z \rightarrow \frac{1}{1 - \lambda^2} \frac{z - \lambda}{1 - \lambda z} \quad (10)$$

Thus, choosing

$$x = N v, \quad y = N^{-1} v$$

and using Eq. (9) and the fact that the R^u field transforms as a $(0, \frac{1}{2})$ representation of the projective group, we can deduce :

$$v^T (1 - M^2)^{-1} v = \sum_{k, l, r = \frac{1}{2}}^{\infty} \frac{v_k}{\sqrt{k}} D_{-l, -k}^{(0, \frac{1}{2})}(R) D_{l, r}^{(0, \frac{1}{2})}(R) \sqrt{r} v_r \quad (11)$$

where $D_{rs}^{(0, \frac{1}{2})}(R)$ is the (infinite) representation matrix of the transformation R in the representation $(0, \frac{1}{2})$. For an arbitrary transformation, α , we would have ¹³⁾

$$D_{mn}^{(0, \frac{1}{2})}(\alpha) = \sqrt{\frac{n}{m}} \oint_C \frac{dy}{2\pi i y} y^{m-n} \left(a + \frac{b}{y}\right)^m (cy + d)^{-m} \quad (12)$$

$m, n = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots$

if

$$\alpha(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1$$

The contour C must be as illustrated in the Figure :

When we use Eq. (12), the sums are easily performed in expression (11) to give :

$$v^T (1-M^2)^{-1} v = \frac{\lambda}{2 \sqrt{1-\lambda^2}}$$

From this we can deduce all the results calculated before ¹⁾.

The corresponding quantities involving S, T and w can be evaluated similarly. Thus, denoting

$$r = w^T (1-T^2)^{-1} w$$

we find

$$(1-r)^D [\det(1-T^2)]^{D/2} = (1-\lambda^2)^{-D/8} \quad (13)$$

$$(1+r)(1-r)^{D-1} [\det(1-T^2)]^{D/2} = K(\lambda) (1-\lambda^2)^{1/2 - D/8}$$

$$\det(1-S^2) = (1-r^2) \det(1-T^2) \quad (14)$$

by calculating matrix elements of either 1 or γ^μ with respect to the state

$$e^{L_{-1}} |0\rangle$$

The left-hand sides of Eq. (13) follow from direct calculations using Eqs. (5) and (8), [a useful isomorphism to note in this context is $L_1 \simeq \sqrt{5} \gamma \cdot b$, $L_{-1} \simeq \sqrt{5} \gamma \cdot b^+$, $L_0 \simeq -(b^+ b + D/2)$ if $[b^\mu, b^{+\nu}]_+ = -g^{\mu\nu}$].

the right-hand sides follow from the group theory, for example

$$\begin{aligned} \langle 0 | e^{\lambda L_1} \gamma^m e^{\lambda L_{-1}} | 0 \rangle &= \langle 0 | R(\lambda) \gamma^m R^{-1}(\lambda) | 0 \rangle (1-\lambda^2)^{-D/8} \\ &= D_{00}^{(-\frac{1}{2}, 0)}(R) \gamma^m (1-\lambda^2)^{-D/8} \end{aligned}$$

taking advantage of the fact that γ^μ belongs to a field transforming as a $(-\frac{1}{2}, 0)$ representation of the projective group. R is the transformation of Eq. (10) and the matrix $D^{(-\frac{1}{2}, 0)}(\alpha)$ is defined by ¹³⁾

$$D_{mn}^{(-\frac{1}{2}, 0)}(\alpha) = \oint_C \frac{dz}{2\pi i z} z^{m-n} \left(a + \frac{b}{z}\right)^{m-\frac{1}{2}} (cz+d)^{-m-\frac{1}{2}}, \quad m, n = 0, \pm 1, \dots$$

instead of Eq. (12). The contour C is that of the Figure again.

Thus

$$D_{00}^{(-\frac{1}{2}, 0)}(R) = K(\lambda) \sqrt{1-\lambda^2}$$

where $K(\lambda)$ is the complete elliptic integral of the first kind ¹⁴⁾

$$K(\lambda) = \frac{1}{2\pi i} \oint_C dz \frac{1}{\sqrt{z} \sqrt{z-1} \sqrt{1-\lambda^2 z}}$$

All the desired quantities follow from Eqs. (13) and (14), for example,

$$\det(1-S^2) = K(\lambda) (1-\lambda^2)^{1/4}$$

as previously calculated ¹⁾.

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