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REGGEON FIELD THEORY ON A LATTICE : A FORMULATION

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A B S T R A C T

The principle of universality is taken from statistical mechanics and applied to the Reggeon field theory. Following a pedagogical review of the implications of universality for the Reggeon calculus, a lattice formulation of the Reggeon field theory is presented and discussed. Scaling laws for the thermodynamic quantities of such systems are described and relations among critical indices are derived. These relations enable high energy scattering behaviour to be calculated using lattice methods.

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1. INTRODUCTION

In recent years one of the descriptions most frequently used in the theory and phenomenology of high-energy scattering has been Regge theory. It has long been known that the analysis of diffractive processes at high energies is very complicated: theoretically a Regge pole with intercept $\alpha_p(0) \approx 1$ should be accompanied by an infinite series of coincident Regge cuts¹⁾, and indeed, experimentally a simple pole description with $\alpha_p(0) \approx 1$ fails to fit the data²⁾. An early and thorough analysis of the relations and interactions between Regge poles and cuts was given by Gribov and co-authors³⁾. They set up the Reggeon calculus and showed how it could be formulated as an analogue field theory in which near forward high-energy scattering could be translated into an infra-red problem. The full arsenal of field theory techniques could then be used to analyze high-energy diffractive scattering.

In recent months^{4, 5)} much progress has been made in this direction using the renormalization group and ϵ -expansion techniques to study the infra-red behavior of the analogue Reggeon field theory⁶⁾. In these analyses, the elastic scattering amplitude has the asymptotic form

$$A(s, t) \sim i s (\ln s)^\eta f(t (\ln s)^\nu)$$

and the average multiplicity behaves like

$$\langle n \rangle \sim (\ln s)^{1+\theta}$$

with η , ν and θ critical indices to be calculated. In this case $\epsilon = 4 - D$, where D is the number of transverse space dimensions, equal to 2 physically. The ϵ -expansions of the indices yield^{4, 5)}, to second order,

$$\begin{aligned}\eta &\approx \frac{1}{12} \epsilon + \frac{0.64}{12} \epsilon^2 \\ \nu &\approx 1 + \frac{1}{24} \epsilon + \frac{0.58}{24} \epsilon^2 \\ \gamma &\approx \frac{1}{3} \epsilon + \frac{0.2}{3} \epsilon^2\end{aligned}$$

where γ is an index governing the behavior of the triple Pomeron vertex^{3, 4)}. For physical $\epsilon = 2$ the $O(\epsilon^2)$ terms are in general larger than the $O(\epsilon)$ terms. It was known that the ϵ -expansion was probably only asymptotic, but experience with solid-state physics and statistical mechanics calculations⁷⁾ had been that

the divergence started at $O(\epsilon^3)$ for $\epsilon = 2$. Should we now believe the calculations to $O(\epsilon^2)$ which work very well in solid-state physics, or rather stop at $O(\epsilon)$, or mistrust the ϵ -expansion completely?

The answer may lie in pursuing further the resemblance between Reggeon field theory and solid-state physics. Many two- and three-dimensional statistical mechanics systems (e.g. Ising spin systems, liquid-gas mixtures) have a critical temperature at which they exhibit second-order phase transitions. It has long been known⁸⁾ that the critical indices describing the divergences of thermodynamic quantities near the critical temperature are apparently identical for large classes of different physical systems. This "universal"⁹⁾ behavior seems to depend solely on the dimensionality and general symmetry properties of the system, and on the fact that there are no explicit long range forces. It is independent of the specific nature of the short range forces, e.g. whether or not the interactions are of nearest neighbor type, whether the field or spin variables have discrete or continuous values, even of whether the system is discrete (e.g. an Ising model on a crystalline lattice) or continuous (e.g. a ϕ^4 field theory). Indeed, this "universality" underpins Wilson's⁷⁾ applications of the ϵ -expansion to calculate critical indices in the Ising model. In his approach, he replaces the Ising model by a ϕ^4 field theory which belongs to the same "universality class", and then uses the renormalization group and ϵ -expansion to calculate the critical indices of the field theory.

Here we will do the reverse: we start from the Reggeon field theory and set up an analogue theory on a lattice. If universality really is applicable to the Reggeon calculus, then this analogue shall be meaningful. There are two reasons for studying the lattice theory:

i) By studying the general properties of a lattice theory, and the nature of any phase transitions it may exhibit, one may hope to learn something about the applicability of universality ideas to the continuum limit. In general one expects that if a lattice theory exhibits a second, or higher order phase transition, then universality is applicable to the lattice theory and its continuum limit (if it exists).

ii) Certain techniques which are applicable only to lattice theories, notably, the high temperature expansion, have been used with great success in statistical mechanics for calculating critical indices⁸⁾. In view of the questionable accuracy of the ϵ -expansion for the determination of the indices η , ν and γ , an alternate calculational scheme seems desirable. Indeed, although the renormalization group and the ϵ -expansion have contributed greatly to our understanding of critical phenomena, the most accurate determinations of critical indices remain those of the high temperature expansion.

In this paper we lay the groundwork for subsequent investigations of these points¹⁰⁾. The paper is organized as follows: In Section 2 we discuss the nature of universality, and, in particular, what it means for the Reggeon field theory. In Section 3 we set up a lattice analogue for the Lagrangian of the Reggeon field theory. An important point, discussed there, is that the lattice theory have a smooth continuum limit. Section 4 is concerned with the derivation of relations among the critical exponents of the theory. These relations should be independent of the particular formulation of the lattice theory which we choose, and are derived by assuming the validity of scaling laws for the thermodynamic averages of the theory. A short discussion and summary are presented in Section 5.

2. REGGEON FIELD THEORY AND UNIVERSALITY

In this paper our discussion starts from the Reggeon Field Theory developed by Gribov and collaborators³⁾. They showed that the t-channel unitarity equations for Reggeons took the same form as field theory unitarity equations if the complex angular momentum J was replaced by an "energy" variable $E = 1 - J$. A Regge pole and its associated cuts could then be combined by solving the analogue field theory. A Pomeron, whose bare propagator is a j -plane pole would be represented by the introduction of a complex field, $\psi(\underline{b}, \tau)$, where $\underline{b}$ is the impact parameter and τ is the Fourier conjugate of E . A trajectory $\alpha_p(t)$ would correspond to an energy-momentum relation

$$E = 1 - \alpha_p(-\underline{k}_T^2)$$

where $t = -\underline{k}_T^2$ in the high energy diffraction limit ($s \rightarrow \infty$), and $\underline{k}_T$ is Fourier conjugate to $\underline{b}$. From the inverse Mellin transform, we see that $\tau = i \ln s$. We will be interested in $\alpha_p(0) \approx 1$ and hence the infra-red limit $E \rightarrow 0$, $\underline{k}_T \rightarrow 0$ of the analogue field theory. It remains to choose the trajectory function α_p and the interactions of the Reggeons. Motivated by field theory and simplicity (and, perhaps, dual resonance models) it is common to propose a trajectory linear in $\underline{k}_T^2$, at least for $\underline{k}_T \approx 0$:

$$E = [1 - \alpha_p(0)] + \alpha'_p \underline{k}_T^2 + \dots \quad (2.1)$$

The corresponding action is that of a non-relativistic free field theory:

$$A = \int d\tau d\underline{b} \left[\frac{i}{2} \psi^\dagger \overleftrightarrow{\partial}_\tau \psi - \alpha'_p \nabla \psi^\dagger \cdot \nabla \psi - (1 - \alpha_p(0)) \psi^\dagger \psi \right] \quad (2.2)$$

The case of physical interest is $D = 2$, but it is useful to regard the transverse dimensionality D as variable, in principle. Motivated by models and experiment it is also common to study a trilinear interaction

$$\frac{ig}{2} [\psi^+ \psi^2 + \psi^{\dagger 2} \psi] \quad (2.3)$$

The imaginary coupling constant is required to give the correct absorptive sign to the two-Reggeon cut.

Recent workers have studied the Reggeon field theory (2.3) using renormalization group techniques^{4,5,6}). They generally fix the renormalized intercept $\alpha_p(0) = 1$ and introduce renormalized parameters α'_R and g_R . As in ordinary field theory, the infra-red asymptotic behavior of the Reggeon model is governed by the infra-red stable zeroes of the Callan-Symanzik functions corresponding to g_R and α'_R . It is usual to introduce effective parameters $\tilde{g}(\sigma)$, $\tilde{\alpha}'(\sigma)$ which are solutions of the equations

$$\begin{aligned} \frac{d\tilde{g}}{d\sigma} &= -\beta(\tilde{g}) \\ \frac{d\tilde{\alpha}'}{d\sigma} &= \tilde{\alpha}' - \zeta(\tilde{\alpha}', \tilde{g}) \end{aligned} \quad (2.4)$$

An infra-red stable zero of β , $\beta(g^*) = 0$, would then yield asymptotically an elastic amplitude

$$A(s, t) \sim i s (\ln s)^{-\gamma(g^*)} f(t (\ln s)^{z(g^*)})$$

where $\gamma(g^*)$ is the anomalous dimension of the field ψ , and

$$z(g^*) = 1 - \frac{\zeta(\tilde{\alpha}', g^*)}{\tilde{\alpha}'}$$

so that $\eta = -\gamma(g^*)$, $\nu = z(g^*)$. Calculations in the theory defined by Eqs. (2.2) and (2.3) yielded a non-trivial zero of β whose position

$$g^* \propto \epsilon^{1/2}$$

where $\epsilon = 4 - D = 2$, physically. Critical exponents η , ν could then be expanded in ϵ , giving the results quoted in Section I.

How general are these results? It is well known that g^* and *a fortiori* η and ν are independent of the value of the renormalized constants g_R and α'_R . Furthermore, the Wilson method⁷⁾ has been applied to the Reggeon field theory showing that these results are unaffected if quadrilinear or multilinear or derivative multi-Pomeron interactions are included, or if the trajectory function (2.1) has higher order terms in an expansion in k_T^{2-11} . In general, the infra-red asymptotic behavior is determined by the most infra-red singular portion of the theory. This situation is well known in statistical mechanics and solid-state physics, where all second-order phase transitions obey the principle of "universality". According to this principle, the critical exponents characterizing such a system depend only on:

- a) the space-time dimension of the system;
- b) the number of interacting fields or spins;
- c) basic symmetry properties of the interactions.

The exponents are supposed to be independent of such trivia as

- d) the detailed properties of the interaction -- e.g. nearest or non-nearest neighbor spin interactions, derivative field theory interactions -- as long as they are basically short range;
- e) whether the field variable values are continuous (bounded or unbounded), or even discrete;
- f) whether the sites in the theory are continuous (ordinary field theory) or discrete (magnetic systems).

It is intuitively reasonable that the infra-red behavior of the theory is determined by properties a), b) and c): more complicated or derivative interactions are less singular in the infra-red. Even the independence of discretization is reasonable -- putting a theory on a lattice imposes an ultra-violet cut-off, but the long-range behavior should be unaltered.

The universality principle was prompted by experimental studies of second-order phase transitions and high temperature series calculations⁸⁾. It has been supported by formal arguments, and in fact underlies the application of ϕ^4 field theory and the ϵ -expansion to calculate Ising model critical exponents⁷⁾. Wilson observed that these very different models had similar properties a), b) and c), and hence was able to discard the differences d), e) and f) as irrelevant. We now draw some consequences of the universality principle for the Reggeon field theory:

A) Arbitrariness

Many people have objected that the trajectory and interaction [Eqs. (2.1) to (2.3)] were oversimplified and unreasonable. Universality tells us that the results

of the simple ϕ^3 theory would remain valid even in the presence of ϕ^4 or derivative ϕ^3 interactions, or if the trajectory was non-linear away from $\underline{k}_T^2 = 0$. Long investigations of "enhanced" and "non-enhanced" graphs³⁾, can be understood through universality, and the notion of relevant and irrelevant variables. In particular, multi-Pomeron vertices could be given exponential cut-offs without changing the results. The exceptions are that if all multi-Pomeron interactions involved even numbers of fields, the action would have a symmetry $\psi \rightarrow -\psi$, fall into another universality class, and have the same behavior as the ϕ^4 Reggeon field theory¹²⁾.

B) Weak coupling solutions

The universality principle indicates that while these solutions exist, they are exceptional¹¹⁾. They correspond to critical subspaces in the parameter space of possible interacting Pomeron theories, and given the absence of any symmetry principle to pick them out, must seem unlikely to be physically relevant.

C) s-channel unitarity

According to universality, only the most singular infra-red behavior is relevant, and so s-channel unitarity only affects critical exponents through its consequences for the multi-Pomeron sector. But we have seen the critical exponents are essentially independent of details of the multi-Pomeron interactions. Hence it seems possible that a Reggeon theory with s-channel unitarity would have the same critical exponents as the simple ϕ^3 theory.

D) Massless particles

One proviso for the multi-Pomeron interactions was that they be "short range". This means that in momentum space the interactions should be expandable as power series in $\underline{k}^2$ around $\underline{k}^2 = 0$. Hence the theory is not allowed to have massless physical particles. Exchange of a simple scalar particle in the t-channel could give a pole in the four-Pomeron interaction at zero momentum transfer:

$$\psi(\underline{k}_1) \psi(\underline{k}_2) \psi^\dagger(\underline{k}_3) \psi^\dagger(\underline{k}_1 + \underline{k}_2 - \underline{k}_3) \frac{1}{(\underline{k}_1 + \underline{k}_2)^2} \quad (2.4)$$

Even two particle exchange would give an illegal logarithmic singularity. Hence massless particles are excluded if we use the Reggeon calculus -- but does it obey the Froissart bound?

E) Lattice analogues

The universality principle suggests that it should be possible to set up lattice analogues with the same critical behavior as the Reggeon field theory. In the next section we try to do just that.

3. REGGEON FIELD THEORY ON A LATTICE

Having introduced the idea of universality, we will now formulate a lattice analogue for the Reggeon field theory. Our purpose is to use the lattice theory as a tool for learning something about the critical behavior of the Reggeon field theory. Therefore we use universality in our search for a lattice theory with the same infra-red properties as the continuum field theory.

Let us begin, then, by referring to Eqs. (2.2) and (2.3). We want to go from a continuous set of space-time points (x, τ) , to a discrete lattice, (x_k, τ_i) . For the moment, it is sufficient to consider the lattice theory in the physical number of space dimensions, i.e. $D = 2$. $\underline{k}$ is therefore a two-dimensional vector labelling the lattice points in the two space directions.

Consider now the second term of Eq. (2.2). The discrete analogue of this term is

$$\begin{aligned} -\alpha' \nabla \psi^\dagger \cdot \nabla \psi &\rightarrow K \left[-(\phi_{\underline{j}+\underline{2}} - \phi_{\underline{j}})^2 - (\phi_{\underline{j}+\underline{3}} - \phi_{\underline{j}})^2 - (\chi_{\underline{j}+\underline{2}} - \chi_{\underline{j}})^2 - (\chi_{\underline{j}+\underline{3}} - \chi_{\underline{j}})^2 \right] \\ &\longrightarrow K \left[\phi_{\underline{j}+\underline{2}} \phi_{\underline{j}} + \phi_{\underline{j}+\underline{3}} \phi_{\underline{j}} + \chi_{\underline{j}+\underline{2}} \chi_{\underline{j}} + \chi_{\underline{j}+\underline{3}} \chi_{\underline{j}} \right] \end{aligned} \quad (3.1)$$

Where we have used the fact that $\psi = \phi + i\chi$, and the notation $\psi(x_k, \tau_i) \equiv \psi_{\underline{j}}$, where $\underline{j}$ is now a three-dimensional vector labelling the space and time points, and K is the coupling constant for this term in the Lagrangian. Equation (3.1) expresses the usual relation between the quadratic terms in a field theory propagator, and the nearest neighbor Ising interaction. Notice that in the last expression we have dropped the direct terms like $\phi_{\underline{j}}^2$, etc. These terms look just like mass insertions, and while their inclusion may change the critical temperature, according to the usual dicta of universality, the critical indices of the theory should not depend on whether or not these terms are present. We shall return to the question of including mass terms below.

In a similar way we easily formulate the lattice analogue of the first term in Eq. (2.2):

$$\frac{i}{2} \psi^\dagger \overleftrightarrow{\partial}_\mu \psi \longrightarrow E(\phi_j \chi_{j+\hat{\mu}} - \chi_j \phi_{j+\hat{\mu}}) \quad (3.2)$$

Note the important relative minus sign between the two terms in Eq. (3.2). We shall say more about this later.

For the moment let us skip over the mass term in Eq. (2.2), and discuss the interaction term (2.3). The discrete counterpart of this term is clearly

$$\frac{ig}{2} [\psi^\dagger \psi^2 + \psi^{\dagger 2} \psi] \longrightarrow iG \phi_j (\phi_j^2 + \chi_j^2) \quad (3.3)$$

We now must choose the measure over which ϕ_j and χ_j have support. In the continuous theory, these real fields can range from $-\infty$ to $+\infty$, but this is evidently not a convenient choice for our lattice theory. Two somewhat more tractable options which come to mind are $\phi, \chi = \pm 1$ (Ising model), and $-1 \leq \phi, \chi \leq 1$. In principle, both choices should give theories with the same critical behavior (universality, again), but it may be that for some purposes one choice is preferable over the other. For instance, when performing the high temperature expansion one choice may be an expansion which converges more rapidly. Since the second option looks, at least superficially, more like the continuum theory, and since it is not much more difficult to calculate, this is the measure chosen for performing the high temperature expansion in Ref. 10^{*)}.

We now want to examine some qualitative characteristics of the discrete Lagrangian which we have constructed so far. For the moment let us consider Ising-like support for ϕ and χ . Then the interaction term (3.3) $\sim iG\phi_j$. We see then [modulo the minus sign in Eq. (3.2)], that our theory looks very much like an Ising model in an external magnetic field. Such a theory is generally thought to have no phase transitions at $T_c \neq 0$. But in fact, the theorem of Lee and Yang¹³⁾ only requires that the real part of the magnetic field be zero. Indeed, in Ref. (10) it is shown that one dimensional Ising systems in the presence of an imaginary magnetic field do have phase transitions to which universality arguments are evidently applicable¹⁰⁾.

Let us now turn to a more fundamental problem associated with our Lagrangian. In order for the lattice theory to bear any relation at all to the continuum theory, a smooth limit should exist when the lattice spacings become very small. Now, if we examine the term (3.2), which contains all the interactions in the

*) A somewhat different formulation of the lattice theory in which $\phi, \chi = \pm 1$ is under active investigation, and will be discussed elsewhere.

time direction, we see that (assuming $\phi_j = \chi_j = \uparrow$) the configuration of lowest energy has a time dependence for the fields shown in Fig. 1. Consequently, the ground state has no smooth continuum limit when the lattice spacings go to zero. We can remedy this situation^{*)} by adding another term to the Lagrangian. The term we add should dominate Eq. (3.2) when one considers distances on the order of a few lattice spacings, but it should be unimportant compared with Eq. (3.2) in the infra-red region -- i.e. when the relevant distances are much larger than the lattice spacings. One interaction which possesses these properties has the form

$$L(\phi_j \phi_{j+\hat{t}} + \chi_j \chi_{j+\hat{t}}) \quad (3.4)$$

which is the ordinary nearest neighbor Ising interaction in the time direction. If we choose $F = -NE$, $N > 1$, then this term clearly forces the spins, ϕ and χ to be in the same direction in the ground state, and results in a smooth continuum limit. On the other hand, we know that Eq. (3.4) is an interaction, which in the continuum limit, has the properties of a second derivative [cf. Eq. (3.1)]. Since there is a non-zero single time derivative term in the Lagrangian, which is more infra-red divergent than Eq. (3.4), universality tells us that Eq. (3.4) will play no role in this kinematic domain -- in Wilson's terminology it is an "irrelevant variable".

Finally, we want to return to the question of mass insertion terms in the Lagrangian. Although, according to universality, the critical behavior of the theory should be independent of whether or not such terms explicitly appear, their inclusion may result in a more rapid divergence of the high temperature expansion¹⁰⁾. Indeed, as the results of Ref. 10 indicate, this seems to be the case. Hence, we will include an interaction of the form

$$M(\phi_j^2 + \chi_j^2) \quad (3.5)$$

Note that this is just a constant if $\phi, \chi = \pm 1$, but not if $-1 \leq \phi, \chi \leq 1$. The full action for our lattice theory is therefore

$$\begin{aligned} \mathcal{A} = \sum_j \bigg\{ & E(\phi_j \chi_{j+\hat{t}} - \chi_j \phi_{j+\hat{t}}) + L(\phi_j \phi_{j+\hat{t}} + \chi_j \chi_{j+\hat{t}}) \\ & + K[(\phi_{j+\hat{x}} + \phi_{j+\hat{y}})\phi_j + (\chi_{j+\hat{x}} + \chi_{j+\hat{y}})\chi_j] + iG\phi_j(\phi_j^2 + \chi_j^2) + M(\phi_j^2 + \chi_j^2) \bigg\} \quad (3.6) \end{aligned}$$

*) We are grateful to J. Zinn-Justin for suggesting this solution.

We now want to describe some general scaling properties for the thermodynamic quantities which should be valid near the critical point, assuming the validity of universality.

4. CRITICAL EXPONENTS AND SCALING HYPOTHESES

4.1 Introduction

In the previous section we outlined a lattice theory which should have a thermodynamic limit corresponding to the Reggeon field theory in the infra-red region. In this section we discuss properties of the thermodynamic quantities which should be valid if universality is applicable to this system. These quantities include the Gibbs potential

$$G = -T \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_N$$

where Z_N is the partition function of the system:

$$Z_N = \sum_{\text{states}} e^{\mathcal{A}/T}$$

$\mathcal{A}$ corresponds to $-\mathcal{H}$ in a statistical system and N is the number of lattice sites. We will also be interested in the "magnetization"

$$m = \frac{\partial G}{\partial H}$$

where H is an external source coupled to the spin, and the susceptibility

$$\chi_T = -\frac{\partial^2 G}{\partial H^2}$$

The magnetization, $\mathcal{M}$, is just the thermodynamic average of the spin, $\langle \phi \rangle$, where the average of any quantity B is

$$\langle B \rangle = \frac{\sum_{\text{states}} B e^{\mathcal{A}/T}}{\sum_{\text{states}} e^{\mathcal{A}/T}}$$

In statistical systems, these thermodynamic quantities characteristically have singularities at some critical temperature, T_c , $\propto (T - T_c)^{-p}$ where p is some power the critical exponent. Empirically, both approximate calculations (high temperature expansions) and experiment suggested various relations between the critical exponents in statistical problems. Before the advent of the renormalization group these were motivated by postulating or making plausible "scaling hypotheses" according to which thermodynamic quantities and correlation functions were generalized homogeneous functions in the neighborhood of the critical temperature.

In this section we first briefly review the scaling hypotheses of statistical mechanics, and then formulate the corresponding hypotheses for the analogue Reggeon model. We then show how these hypotheses enable the critical exponents η and ν of the Reggeon field theory to be calculated from a knowledge of the divergent behavior of the lattice correlation functions near the critical temperature.

4.2 Scaling in magnetic systems

For a magnetic (e.g. Ising) system the "static scaling hypothesis" ⁸⁾ asserts that the Gibbs potential, $\mathcal{G}$, is a generalized homogeneous function of the magnetic field, H , and the reduced temperature $\varepsilon = (T - T_c)/T_c$ in the neighborhood of the critical temperature. That is:

$$\mathcal{G}(\lambda^{a_\varepsilon} \varepsilon, \lambda^{a_H} H) = \lambda \mathcal{G}(\varepsilon, H) \quad (4.1a)$$

where a_ε and a_H are *a priori* unknown scaling parameters. Since $\partial \mathcal{G} / \partial H = \mathcal{M}$, differentiation of Eq. (4.1a) with respect to H yields

$$\lambda^{a_H} \mathcal{M}(\lambda^{a_\varepsilon} \varepsilon, \lambda^{a_H} H) = \lambda \mathcal{M}(\varepsilon, H) \quad (4.1b)$$

and hence if $H = 0$

$$\mathcal{M}(\varepsilon, 0) = \lambda^{a_H - 1} \mathcal{M}(\lambda^{a_\varepsilon} \varepsilon, 0)$$

Setting $\lambda = \varepsilon^{-1/a_\varepsilon}$ gives

$$\mathcal{M}(\varepsilon, 0) = \varepsilon^{(1-a_H)/a_\varepsilon} \mathcal{M}(1, 0) \quad (4.2)$$

Further, since

$$\frac{-\partial^2 G}{\partial H^2} = \chi_T$$

a similar analysis yields

$$\chi_T(\epsilon, 0) = \epsilon^{(1-2\alpha_H)/\alpha_\epsilon} \chi_T(1, 0) \quad (4.3)$$

and from Eq. (4.1) we deduce

$$G(\epsilon, 0) = \epsilon^{1/\alpha_\epsilon} G(1, 0) \quad (4.4)$$

From Eqs. (4.2) and (3.4) we see that if we denote the divergences of G , M and χ_T near the critical temperature by

$$G(\epsilon, 0) \approx \epsilon^{-x}, \quad M(\epsilon, 0) \approx \epsilon^{-y}, \quad \chi_T(\epsilon, 0) \approx \epsilon^{-z} \quad (4.5)$$

then the static scaling hypothesis implies

$$x + z = 2y \quad (4.6)$$

The parameters y and z can be related to the conventional critical exponents of magnetic systems:

$$y = -\beta, \quad z = \gamma$$

One of the arguments which supports the static scaling hypothesis is the Kadanoff construction¹⁴⁾. In this scheme, the sites are grouped together in blocks, L spins on a side: the resulting system has associated block spins $\tilde{s}$, temperature $\tilde{\epsilon}$ and magnetic field $\tilde{H}$. Kadanoff then hypothesized that the same free energy characterized the block system near the critical temperature, so that (we work on a d -dimensional lattice)

$$G(\tilde{\epsilon}, \tilde{H}) = L^d G(\epsilon, H)$$

and also that $\tilde{\epsilon} = L^Y \epsilon$, $\tilde{H} = L^X H$, for some powers X, Y near the critical point.

These hypotheses immediately give the homogeneity property (4.1a) with

$$a_\varepsilon = \frac{Y}{J} \quad , \quad a_H = \frac{X}{J}$$

The Kadanoff scheme also has implications for the correlation functions.

If we assume

$$\tilde{s} = \mathcal{L}^{-1} L^{-d} \sum s_i \quad (4.7)$$

then the pair correlation function $\Gamma(\underline{r}, \varepsilon) = \langle (s_i - \langle s_i \rangle) (s_j - \langle s_j \rangle) \rangle$ where $\underline{r} = \underline{x}_i - \underline{x}_j$ obeys $\Gamma(\underline{r}, \varepsilon) = \mathcal{L}^2 \Gamma(\tilde{\underline{r}}, \tilde{\varepsilon})$ where $\tilde{\underline{r}} = 1/L \underline{r}$. Rewriting the magnetic interaction Hs_i in terms of $\tilde{H}, \tilde{s}$ yields $\tilde{H} = L^d \mathcal{L} M$, implying $\mathcal{L} = L^{X-d}$. Thus we get a homogeneous expression for $\Gamma(\underline{r}, \varepsilon)$, valid as $\varepsilon \rightarrow 0$.

$$\Gamma(\underline{r}, \varepsilon) = \varepsilon^A f(|\underline{r}|/\varepsilon^B) \quad (4.8)$$

with (as can easily be checked)

$$A = -2 \left(\frac{1 - a_H}{a_\varepsilon} \right) = -2\gamma \quad (4.9)$$

and B also related to a_H and a_ε .

Equation (4.8) has many consequences: for example

$$\Gamma(0, \varepsilon) \sim \varepsilon^A \quad ; \quad \varepsilon \rightarrow 0 \quad (4.10)$$

Suppose we now integrate the correlation function over Δ dimensions in the full d-dimensional space. Then

$$\begin{aligned} \int d^\Delta \underline{r} \Gamma(\underline{r}, \varepsilon) &= \int d^\Delta \left(\frac{\underline{r}}{\varepsilon^B} \right) f(|\underline{r}|/\varepsilon^B) \varepsilon^{\Delta B + A} \\ &\simeq \varepsilon^{\Delta B + A} \quad ; \quad \varepsilon \rightarrow 0 \end{aligned} \quad (4.11)$$

Comparing Eqs. (4.9), (4.10) and (4.11), we see there are many relations between critical exponents. In particular if we take a 3-dimensional model and denote the 1- and 2-dimensional integrals by Γ_1, Γ_2 respectively (the 3-dimensional integral is just the susceptibility χ_T), then

$$\Gamma^I(0, \epsilon) \sim \epsilon^A; \Gamma_1 \sim \epsilon^{A+B}; \Gamma_2 \sim \epsilon^{A+2B}; \chi_T \sim \epsilon^{A+3B} \quad (4.12)$$

4.3 Scaling for the Reggeon calculus

Thus the scaling hypotheses [Eqs. (4.1) and (4.8)] give useful relations between critical exponents, exemplified by Eqs. (4.6) and (4.12). In Ref. 10 some of these exponents and relations are used as checks that the methods described there for the Reggeon field theory work reasonably well when applied to magnetic systems resembling Ising models. We now formulate the corresponding scaling hypotheses and relations for the lattice versions of the Reggeon field theory, and derive the connections between the divergences of the pair correlation function, and the Reggeon field theory exponents η and ν . We emphasize that these relations should be independent of the particular formulation of the lattice analogue, as long as it is a member of the required universality class.

Since the Reggeon field theory analogue has two real "spins" ϕ and χ the Gibbs potential $\mathcal{G}$ is now a function of two sources H, J analogous to the external field, H , of the magnetic system^{*)}. The assumption corresponding to Eq. (4.1) is

$$\mathcal{G}(\lambda^{a_\epsilon} \epsilon, \lambda^{a_H} H, \lambda^{a_J} J) = \lambda \mathcal{G}(\epsilon, H, J) \quad (4.13)$$

and

$$\mathcal{G}(\epsilon, 0, 0) \approx \epsilon^{1/a_\epsilon} \quad (4.14)$$

Differentiating with respect to H (and J) to get the "magnetizations" $\mathcal{M}$ (and $\mathcal{N}$) we find

$$\mathcal{M}(\epsilon, 0, 0) \approx \epsilon^{(1-a_H)/a_\epsilon}, \quad \mathcal{N}(\epsilon, 0, 0) \approx \epsilon^{(1-a_J)/a_\epsilon} \quad (4.15)$$

*) For the purposes of this discussion, we imagine adding terms like $H\phi$ and $J\chi$ to our action (3.6). In fact, however, the term $\propto G$ plays a role similar to the term $H\phi$. This can be easily seen in the Ising model, where $\phi, \chi = \pm 1$. Hence, in that model the critical exponents associated with $\partial \mathcal{G} / \partial G$ and $\partial \mathcal{G} / \partial H$ are the same. By universality, therefore, they should also be the same in our system.

and the "isothermal susceptibilities" $\chi_{HH}, \chi_{HJ}, \chi_{JJ}$:

$$\chi_{HH} \approx \varepsilon^{(1-2a_H)/a_\varepsilon}, \quad \chi_{HJ} \approx \varepsilon^{(1-a_H-a_J)/a_\varepsilon}, \quad \chi_{JJ} \approx \varepsilon^{(1-2a_J)/a_\varepsilon} \quad (4.16)$$

But now we notice that corresponding to the vanishing of the Green's functions^{*)}

$$\langle 0 | T(\psi(x, t) \psi(0, 0)) | 0 \rangle, \quad \langle 0 | T(\psi^\dagger(x, t) \psi^\dagger(0, 0)) | 0 \rangle$$

we should have

$$\chi_{HH} \pm 2i\chi_{HJ} - \chi_{JJ} = 0$$

and hence

$$a_H = a_J \quad (4.17)$$

so that we only need to calculate the correlation functions between two ϕ fields to know the behavior of the full set of correlation functions. Introducing the parameters x, y, z as before we recover the relation (4.6) for the lattice analogue to the Reggeon field theory also.

We now move on to the pair correlation function for two ϕ "spins" to be denoted by $\Gamma(\underline{r}, t, \varepsilon)$. The relevant generalization of the Kadanoff construction is to group the spins into blocks of size L in the space directions, and in the time direction of size L^p , for some unknown power p . As in the magnetic case, we recover the homogeneity property (4.13) of the free energy, and a homogeneity law for the pair correlation function

$$\Gamma(\underline{r}, t, \varepsilon) = \varepsilon^A g(|\underline{r}|/\varepsilon^B, t/\varepsilon^C) \quad (4.18)$$

where $(\underline{r}, t)$ is the (space, time) separation of the two ϕ "spins", and $C/B = p$. As before, $A = -2y$ and

$$f \equiv \Gamma(0, 0, \varepsilon) \approx \varepsilon^A \quad ; \quad \varepsilon \rightarrow 0 \quad (4.19)$$

*) Recall that fields ψ and $\psi^\dagger$ are non-relativistic and so only the mixed propagators are non-zero.

Denoting the time integral of $\Gamma(\underline{r}, t, \epsilon)$ by $\Gamma_t(\underline{r}, \epsilon)$ we find

$$\Gamma_t^1(\underline{r}, \epsilon) = \epsilon^{A+C} \int d(\frac{t}{\epsilon^c}) g(|\underline{r}|/\epsilon^B, t/\epsilon^c)$$

so that

$$f_t \equiv \Gamma_t^1(0, \epsilon) \approx \epsilon^{A+C} \quad (4.20)$$

Denoting the space integral by $\Gamma_s(t, \epsilon)$ we similarly find

$$\Gamma_s^1(t, \epsilon) = \epsilon^{A+DB} \int d^D(\frac{\underline{r}}{\epsilon^B}) g(|\underline{r}|/\epsilon^B, t/\epsilon^c)$$

so that

$$f_s \equiv \Gamma_s^1(0, \epsilon) \approx \epsilon^{A+DB} \quad (4.21)$$

Denoting the susceptibility χ_{HH} , the space-time integral of $\Gamma(\underline{r}, t, \epsilon)$, by f_{st} :

$$f_{st} = \epsilon^{A+DB+C} \int d^D(\frac{\underline{r}}{\epsilon^B}) d(\frac{t}{\epsilon^c}) g(|\underline{r}|/\epsilon^B, t/\epsilon^c)$$

we see that

$$f_{st} \approx \epsilon^{A+DB+C} \quad (4.22)$$

Therefore if we denote the divergences in the series for f , f_s , f_t and f_{st} by

$$f \approx \epsilon^{-u}, f_s \approx \epsilon^{-\nu}, f_t \approx \epsilon^{-w}, f_{st} \approx \epsilon^{-z} \quad (4.23)$$

we see that Eqs. (4.19), (4.20), (4.21), (4.22) and (4.23) imply

$$u+z = \nu+w \quad (4.24)$$

This equation is the analogue of Eq. (4.12) and is useful as a check on the results of any calculational scheme, including the high temperature expansion¹⁰.

To derive the critical index for the total cross-section we note
[cf. Eq. (4.21)]

$$\begin{aligned}\Gamma_s(t, \epsilon) &= \epsilon^{A+DB} h(t/\epsilon^c) \\ &= t^{(A+DB)/c} \tilde{h}(t/\epsilon^c)\end{aligned}$$

and so

$$\Gamma_s(t, 0) \approx t^{\frac{A+DB}{c}} \quad (4.25)$$

But $\Gamma_s(t, 0)$ is proportional to the total cross-section and so

$$\begin{aligned}\eta &= \frac{A+DB}{c} \\ &= \frac{\mathcal{N}}{\mathcal{W}-u} = \frac{\mathcal{N}}{z-\mathcal{N}}\end{aligned} \quad (4.26)$$

The slope of the diffraction peak is determined by another critical index.
This index can be found by considering Eq. (4.18) which can be rewritten as

$$\Gamma(r, t, \epsilon) = t^{A/c} \tilde{g}(|r|/t^{B/c})$$

and hence

$$\begin{aligned}\nu &= \frac{2B}{c} \\ &= \frac{2}{D} \frac{\mathcal{N}-u}{\mathcal{W}-u} = \frac{2}{D} \frac{z-\mathcal{W}}{z-\mathcal{N}}, \text{ etc.}\end{aligned} \quad (4.27)$$

The Eqs. (4.23), (4.26) and (4.27) summarize the connections between the divergences of the pair correlation function and the critical exponents of the Reggeon field theory. A third critical exponent, γ , is related to the behaviour of the 3 spin correlation function $\Gamma^{(3)}(\underline{r}_1, \underline{r}_2, t_1, t_2, \epsilon)$ where $\underline{r}_1, \underline{r}_2$ (t_1, t_2) are relative space (time) separations of the three spins. The Kadanoff construction can be applied to $\Gamma^{(3)}$ and yields

$$\Gamma^{(3)}(r_1, r_2, t_1, t_2, \epsilon) = \epsilon^{\frac{3A}{2}} g^{(3)}(r/\epsilon^B, k/\epsilon^B, t_1/\epsilon^C, t_2/\epsilon^C) \quad (4.28)$$

Appropriate manipulations of expression (4.28) yield the result

$$\gamma = 1 + \frac{3A}{2C} + \frac{DB}{C} \quad (4.29)$$

which can be used with Eqs. (4.26) and (4.27) to obtain the relation

$$2\gamma = 3\eta - \frac{D\nu}{2} + 2 \quad (4.30)$$

which was previously obtained by Gribov et al.³⁾. Other relations which they obtained for higher order Green's functions can also be obtained from the Kadanoff construction.

5. SUMMARY AND DISCUSSION

Motivated by the hope that the ideas of universality are applicable to the Reggeon calculus, we have formulated a lattice version of the usual continuum field theory. The lattice analogue was constructed to i) have smooth continuum and thermodynamic limits and ii) hopefully possess the same infra-red behavior as the continuum field theory. The importance of defining a lattice theory with a smooth thermodynamic limit cannot be over-emphasized. Indeed, not only do the results of the high temperature expansion depend strongly on this property¹⁰⁾, but model studies imply that there may, in fact, be no phase transition at all without it¹⁰⁾.

Having formulated the lattice theory for the Reggeon calculus, and explained the importance of universality, we discussed the scaling hypotheses for the thermodynamic averages of the theory. These scaling hypotheses are expected to be valid in systems amenable to the dictates of universality, and they enable us to derive many relations among the critical indices of the theory. Such relations can serve as useful consistency checks of approximation schemes, such as the high temperature expansion which are designed to calculate the critical exponents of the theory.

In closing, let us reiterate what we finally hope to learn from the lattice theory:

- 1) Despite the successes of the renormalization group and the ϵ -expansion, the characteristics of the infra-red behavior of the Reggeon field theory are still somewhat cloudy. By setting up a discrete analogue for this theory, and studying the nature of its phase transitions in the thermodynamic limit, we may hope to learn something about the infra-red behavior of the continuum theory. Although universality was implicitly assumed to be applicable to the Reggeon calculus in defining the lattice analogue, a study of its phase transitions should tell us *a posteriori* if this assumption is warranted¹⁰⁾.
- 2) Given the validity of universality, we are free to use the high temperature expansion which is defined only for lattice theories to calculate the critical indices, (e.g. η , ν and γ) of the Reggeon calculus. Past experience has shown that high temperature estimates of critical exponents are generally quite reliable, and we may hope that this is also the case for the present theory.

Two of us (J.E. and R.S.) have recently made progress on these questions. The results of our investigations are discussed in Ref. 10.

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Figure caption

Fig. 1 : Ground state configuration of the fields, ϕ and χ , as a function of t in a theory without the L-interaction.

Fig. 1