Ref.TH.1974-CERN

PHASE TRANSITIONS AND THE HIGH TEMPERATURE EXPANSION
FOR THE REGGEON CALCULUS

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A B S T R A C T

We study a recently proposed lattice version of the Reggeon field theory. First a simple one-dimensional system possessing many of the features of the Reggeon calculus (Ising model in an imaginary magnetic field) is solved. Surprisingly, the system is found to undergo a phase transition at a non-zero critical temperature, which, although of first order obeys the universality and scaling laws previously postulated for the Reggeon calculus. Returning to the full lattice theory, the machinery for performing a high temperature expansion is set up, and initial calculations carried out to order T^{-3} . In this order, best estimates for the critical indices η and ν in the asymptotic elastic amplitude

$$A(s,t) \sim s(\ln s)^\eta f(t(\ln s)^\nu)$$

yield $\eta = O(\frac{1}{2} \sim 1)$, $\nu = O(1\frac{1}{2} \sim 2)$. Checks on the method, including comparisons with known Ising model results, are also discussed.

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1. INTRODUCTION

A lattice formulation of the Reggeon field theory has recently been proposed¹⁾. As explained in Ref. 1 [hereafter referred to as BESZ], the formulation is expected to have the same infra-red behavior as the usual field theory, and also to possess smooth continuum and thermodynamic limits. It therefore presents itself as a possible tool for understanding the Reggeon calculus. Moreover, since it is a lattice theory, it has certain advantages over a continuum formulation in that: i) the nature of critical behavior is often more transparent in a lattice theory, and ii) the lattice theory offers techniques for the calculation of critical indices which are not applicable to a continuum theory.

Our lattice theory is described by the action:

$$A = \sum_{\underline{j}} E(\phi_{\underline{j}} \chi_{\underline{j}+\underline{e}} - \chi_{\underline{j}} \phi_{\underline{j}+\underline{e}}) + L(\phi_{\underline{j}} \phi_{\underline{j}+\underline{e}} + \chi_{\underline{j}} \chi_{\underline{j}+\underline{e}}) \\ + K[(\phi_{\underline{j}+\underline{e}} + \phi_{\underline{j}+\underline{g}})\phi_{\underline{j}} + (\chi_{\underline{j}+\underline{e}} + \chi_{\underline{j}+\underline{g}})\chi_{\underline{j}}] + iG\phi_{\underline{j}}(\phi_{\underline{j}}^* + \chi_{\underline{j}}^*) + M(\phi_{\underline{j}}^2 + \chi_{\underline{j}}^2) \quad (1.1)$$

where the sum extends over all three-component vectors $\underline{j}$ of the two-space and one-time dimensional lattice. The fields ϕ and χ are allowed to take on all values $-1 \leq \phi, \chi \leq 1$.

In this paper we will study this theory and try to answer two sets of basic questions:

- 1) Are there phase transitions associated with the action (1.1)? If so, what is their nature? Is universality applicable and is the analogy (1.1), therefore, meaningful *a posteriori*? Do the scaling relations described in BESZ hold?

Assuming favorable answers to these questions, we continue and ask:

- 2) Can we use the high-temperature expansion to calculate the critical indices of the Reggeon calculus? What are the results, and how much confidence do we have in them? How do they compare with the results of the ϵ -expansion²⁻⁴⁾?

To answer the first set of questions, we formulate and solve a one-dimensional system which has many of the properties of the Reggeon calculus. The system is a one-dimensional Ising model in an imaginary magnetic field. Contrary to usual expectations, we find that this system has a phase transition at a non-zero critical temperature, that scaling laws are obeyed, and that universality is evidently applicable -- all this despite the fact that it is a first order phase transition, in so far as the magnetization is infinite at the critical point.

Following this analysis, we turn to the high-temperature expansion. Long before the advent of renormalization group techniques, people calculated critical exponents simply by taking thermodynamic quantities such as the free energy

$$G = -\beta^{-1} \ln Z = -\beta^{-1} \ln [Tr \exp(-\beta H)]$$

expanding them in powers of $\beta \mathcal{H}$, ($\beta = 1/kT$), evaluating the resulting series, and examining them for possible divergences at some critical temperature $T = T_c$. This is the high-temperature expansion. (Note that it is an expansion in *all* the terms appearing in $\mathcal{H}$, including the terms which play the role of kinetic energy.) The critical exponents could be calculated very accurately if enough terms in the series were taken, and if they exhibited the universal behavior which we have described. Although this behavior was only understood later, around the time of the renormalization group, the most accurate estimates of critical exponents remain those calculated using the high-temperature series.

Since, in this paper, we only calculate the high-temperature series to $O(1/T^3)$, we cannot expect to get highly accurate estimates of the critical exponents. Rather, our purpose here is to understand at least qualitatively if the Reggeon calculus has a reasonable infra-red behavior, and parenthetically, to see if the ϵ -expansion is at all useful. We remind the reader that nothing rigorous can be said about the ϵ -expansion, and in the absence of rigor perhaps the only way to understand the nature of the problem is by comparing two non-rigorous methods.

If we write for the asymptotic behavior of the near-forward elastic scattering amplitude,

$$A(s,t) \sim i s \ln^{\eta} s f(t \ln^{\nu} s) \quad (1.2)$$

then our best high-temperature estimates for η and ν give

$$\eta \simeq O(\frac{1}{2} \sim 1) \quad \nu \simeq O(1\frac{1}{2} \sim 2)$$

If we follow, for example, Migdal, Polyakov, and Ter-Martirosyan²⁾ and describe the cross-sections for multiple (low mass) diffraction processes by

$$\frac{\sigma_n(s)}{\sigma_{el}(s)} \sim (\ln s)^{-\eta\beta}$$

then we find

$$\beta \approx O(\frac{1}{3} \sim \frac{2}{3})$$

We are able to check our approximations by using the same method to calculate critical indices of the Ising model in two and three dimensions. We find for the susceptibility index, γ_I : in three dimensions $\gamma_I = 0.8 \sim 1.5$ (cf. the well-accepted value $\gamma_I = 1.25$), and in two dimensions $\gamma_I = 1.4 \sim 1.6$ (cf. the exact value $\gamma_I = 1.75$). Estimates of the Ising index η_I are worse, and suggest that the above estimates for η and ν may be somewhat on the high side.

The arrangement of the rest of the paper is as follows: In Section 2 we solve the one-dimensional systems alluded to above, and discuss the nature of their phase transitions. In Section 3 we describe the general theory of the high-temperature expansion, and the particular formulation of it which we use. We also derive the perturbation theory which is necessary for performing the high-temperature expansion, and present general expressions for the thermodynamic averages which we compute. Section 4 is devoted to a complete presentation of our results, and a discussion of the checks of our method which we have carried out. The conclusion, Section 5, consists of a summary of what we have learned, comparisons of our results with the ϵ -expansion, and general comments. The paper ends with two technical Appendices, described in the text.

2. ONE-DIMENSIONAL SYSTEMS AND PHASE TRANSITIONS FOR THE REGGEON CALCULUS

The lattice analogue of the Reggeon field theory describes a rather complicated three-dimensional system. Since we have no rigorous theorems about the nature of the phase transition of this system (indeed, we do not really know if one exists), it may be useful to examine some soluble models which contain many of the features of the Reggeon system.

Consider, then, a one-dimensional Ising model of $2N$ spins with two kinds of nearest neighbor interactions, in an imaginary external magnetic field. The Hamiltonian of this system is

$$-\beta \mathcal{H} = \sum_{j=1}^{2N} K S_j S_{j+1} + E (-1)^j S_j S_{j+1} + i H S_j \quad (2.1)$$

where $S_j = \pm 1$. Notice that the couplings K , E , and H are defined with a factor T^{-1} . The interaction K is the usual nearest neighbor interaction, ferromagnetic for $K > 0$, and antiferromagnetic for $K < 0$. We choose periodic boundary conditions so that $S_{2N+1} = S_1$. Since $S_j^3 = S_j$, the imaginary magnetic field plays the role of the imaginary cubic coupling in the usual Reggeon field theory.

As we have already discussed, in Section 3 of BESZ, the presence of the term $i\psi^+ \vec{\sigma}_t \psi$ in the Reggeon field theory leads to a term in the lattice analogue which prefers to have neighboring spins in the time direction oppositely aligned in pairs. The interaction E in (2.1) mocks the effect of this term. Finally, K in (2.1) plays the role of K or L in (1.1).

The system (2.1) is easily solved using the transfer matrix method⁵⁾. We write the partition function as

$$\begin{aligned} Z_{2N} &= \sum_{S_j = \pm 1} \prod_j \exp \left\{ [K + (-1)^j E] S_j S_{j+1} + \frac{iH}{2} (S_j + S_{j+1}) \right\} \\ &= \sum_{S_j = \pm 1} \prod_{j=2,4,\dots}^{2N} \exp \left\{ (K+E) S_j S_{j+1} + (K-E) S_{j+1} S_{j+2} + \frac{iH}{2} (S_j + 2S_{j+1} + S_{j+2}) \right\} \quad (2.2) \\ &= \sum_{S_{2j} = \pm 1} T(S_2, S_4) T(S_4, S_6) \cdots T(S_{2N}, S_2) = \text{Tr } T^N = \lambda_+^N + \lambda_-^N \end{aligned}$$

where T is the 2×2 matrix:

$$T = \begin{pmatrix} e^{2K+2iH} + e^{-2K} & e^{2E+iH} - 2E - iH \\ e^{-2E+iH} + e^{2E-iH} & e^{-2K} + e^{2K-2iH} \end{pmatrix} \quad (2.3)$$

and $\lambda_{\pm}$ are its eigenvalues. Notice that in the first line of (2.2) we have replaced iHS_j by $(iH/2)(S_j + S_{j+1})$.

The eigenvalues of T are easily found to be

$$\lambda_{\pm} = e^{2K} \cos 2H + e^{-2K} \pm 2^{\frac{1}{2}} \left\{ 2 \cos^2 H (1 - e^{4K} \sin^2 H) + \cosh 4E - 1 \right\}^{\frac{1}{2}} \quad (2.4)$$

and a phase transition occurs when $\lambda_+ = \lambda_-$.

Now, suppose $\lambda_+ > \lambda_-$. Then, in the thermodynamic limit, the free energy per spin $\mathcal{G}$ becomes:

$$\mathcal{G} = -T \lim_{N \rightarrow \infty} \frac{1}{2N} \ln Z_{2N} = \frac{1}{2} \ln \lambda_+ \quad (2.5)$$

and the magnetization is

$$\begin{aligned} i\mathcal{M} &= \frac{\partial}{\partial h} \mathcal{G} \\ &= \frac{e^{2K} \sin 2H [2e^{2K} + \Delta^{-1/2} (e^{4K} \cos 2H + 1)]}{e^{4K} \cos 2H + 1 + e^{2K} \Delta^{1/2}} \end{aligned} \quad (2.6)$$

where $H = h/T$, and

$$\Delta = 4 \cos^2 H (1 - e^{4K} \sin^2 H) + 2 \cosh 4E - 2 \quad (2.7)$$

is a quantity which becomes small as we approach the critical point. Now, it is clear that in general (with the exception of some special values for E and K to be discussed below), the magnetization near the critical point behaves like

$$\mathcal{M} \propto \Delta^{-1/2} ; \Delta \rightarrow 0^+ \quad (2.8)$$

so it will be sufficient for us to discuss in detail the somewhat more transparent case $E = 0$. The nearly universal behavior of (2.8) may seem somewhat surprising in view of the discussion in Section 3 of BESZ, wherein we argued that the E -interaction was more infra-red divergent than the L (or K) interaction, so that the theory with $E = 0$ should have a qualitatively different behavior than the theory with $E \neq 0$. The answer is that the E interaction of (2.1), while it is an interaction which prefers to have the spins oppositely aligned in pairs in the ground state, is not more infra-red divergent than the ferromagnetic interaction, K . This is easy to see by trying to construct a simple continuum analogue of the E interaction which involves only one derivative.

There is another important lesson to be learned from (2.4). We argued in BESZ that it was necessary to add the L -interaction [see Eq. (1.1)] in order that

the lattice theory have a smooth continuum limit. In our model (2.1) a ferromagnetic interaction, which means $K > 0$, plays this role, and indeed, from (2.4) we see that if $K \leq 0$, there is no phase transition at all for real, non-zero values of E and H .

Let us return now to Eqs. (2.4)-(2.7), and set $E = 0$. In that case we have

$$G = k + \ln[\cosh H + \delta^{\frac{1}{2}}] \quad (2.9)$$

and

$$iM = -\delta^{\frac{1}{2}} \sinh H \quad (2.10)$$

with $\delta = e^{-4K} - \sin^2 H$.

We work in units where $K = k/T = 1/T$, and we expand about the critical point $\delta = 0$. Explicitly displaying the dependence on the temperature and the magnetic field, we have for

$$\varepsilon = \frac{1}{T_c} - \frac{1}{T} = \beta_c - \beta \simeq 0 \quad \text{and} \quad \xi = h_c - h \simeq 0$$

$$G(\varepsilon, \xi) = \beta_c + \ln \cos \beta_c h_c + (A\varepsilon + B\xi)^{\frac{1}{2}} \quad (2.11)$$

and

$$iM(\varepsilon, \xi) = \frac{-B}{2(A\varepsilon + B\xi)^{\frac{1}{2}}} \quad (2.12)$$

where

$$A = 4 \tan^2 h_c \beta_c + 2 h_c \tan \beta_c h_c$$

$$B = 2 \beta_c \tan \beta_c h_c$$

and the critical temperature and magnetic field satisfy

$$e^{-4\beta_c} = \sin^2 \beta_c h_c \quad (2.13)$$

The existence of a phase transition at $T_c \neq 0$ in this one-dimensional model may at first sight seem surprising, but it is perfectly consistent with the Lee-Yang theorems⁶⁾. Notice that in the limit $h \rightarrow 0$, we recover the usual phase transition at $T_c = 0$ of the usual one-dimensional Ising model.

What can we say about the nature of this phase transition? First, if $\xi = 0$ and $T > T_c$, the magnetization is purely imaginary. This is intuitively appealing, since a real magnetization would seem to imply that the spins tended to point in the same direction, and such an effect is not expected in the disordered phase of a system. On the other hand, $\mathcal{M}$ does become real for $T < T_c$ ($\epsilon < 0$).

Next we note that this phase transition has at least some of the characteristics of a first-order phase transition, in that the magnetization becomes infinite at $T \rightarrow T_c$. Since universality arguments are not generally applicable to first-order phase transitions, one might be tempted to take this observation as evidence against the applicability of universality to the Reggeon field theory. However, we showed earlier that the nature of the phase transition is largely independent of the exact values of the coupling constants, which is exactly what universality implies, despite the fact that the magnetization is infinite. Moreover, it is well known, for example in the usual Ising model, that as the lattice dimensionality increases, many critical exponents, in particular, the exponent of the magnetization, become more positive⁵⁾. Hence it is possible that in two-space and one-time dimension the magnetization will go to zero at the critical point, as expected for a higher-order phase transition.

It is important to note that even though the magnetization $\mathcal{M}(\epsilon, 0)$ has a singularity when $\epsilon \rightarrow 0$, it is a simple power law singularity. Consequently, we may expect that the scaling hypotheses which were used extensively in Section 4 of BESZ are, in fact, obeyed. As an example of how this comes about, consider the two series (2.11) and (2.12) near the critical point, and, in addition, the susceptibility:

$$\begin{aligned} \chi_T &= -\left(\frac{\partial \mathcal{M}}{\partial h}\right)_T = \delta^{-3/2} (\delta + \sin \beta h) \cos \beta h \\ &\approx \frac{B^2}{4(A\epsilon + B\delta)^{3/2}} \end{aligned} \quad (2.14)$$

where the last form gives the leading behavior for $\epsilon, \xi \rightarrow 0$. Now, at the critical point, $\mathcal{G}$ [Eq. (2.11)] goes to a constant. The physics near the critical point (i.e. the behavior of $\mathcal{M}$, χ_T , etc.) depending only on derivatives of $\mathcal{G}$, is independent of this constant, so we are free to define a new free energy,

$$\mathcal{G}'(\epsilon, \delta) = \mathcal{G} - \beta_c - \ln \cos \beta_c h_c = (A\epsilon + B\delta)^{1/2} \quad (2.15)$$

which is a generalized homogeneous function of ε and ξ , and to which we shall apply our scaling hypotheses.

Now, from Section 4 of BESZ, we have the following results:

$$\mathcal{G}'(\lambda^\varepsilon_\varepsilon, \lambda^{a_H}_\xi) = \lambda \mathcal{G}'(\varepsilon, \xi) \quad (2.16)$$

$$\lambda^{a_H} \mathcal{M}(\lambda^\varepsilon_\varepsilon, \lambda^{a_H}_\xi) = \lambda \mathcal{M}(\varepsilon, \xi) \quad (2.17)$$

$$\mathcal{G}'(\varepsilon, 0) = \varepsilon^{1/a_\varepsilon} \mathcal{G}'(1, 0) \quad (2.18)$$

$$\mathcal{M}(\varepsilon, 0) = \varepsilon^{\frac{1-a_H}{a_\varepsilon}} \mathcal{M}(1, 0) \quad (2.19)$$

$$\chi_T(\varepsilon, 0) = \varepsilon^{\frac{1-2a_H}{a_\varepsilon}} \chi_T(1, 0) \quad (2.20)$$

Comparing (2.15) and (2.18), we see that $a_\varepsilon = 2$. Moreover, from (2.16) and (2.15) it is clear that $a_H = 2$, also. We must now show that the scaling arguments applied to the other generalized homogeneous functions (2.12) and (2.14), give the same values for a_ε and a_H . Comparing (2.19) and (2.12), we find

$$\frac{1-a_H}{a_\varepsilon} = -\frac{1}{2} \quad (2.21a)$$

whereas setting $\lambda = \xi^{-1/a_H}$ in (2.17) and comparing with (2.12), we have

$$\frac{1-a_H}{a_H} = -\frac{1}{2} \quad (2.21b)$$

and we see that the scaling hypotheses applied to $\mathcal{M}$ also imply $a_\varepsilon = a_H = 2$.

Similar arguments applied to χ_T produce the equations

$$\frac{1-2a_H}{a_\varepsilon} = -\frac{3}{2} = \frac{1-2a_H}{a_H} \quad (2.22)$$

whose solutions are $a_\varepsilon = a_H = 2$. Hence, the relevant thermodynamic quantities obey the scaling hypotheses, even though the first derivative of the free energy

is infinite at the phase transition. This is an important observation and, along with the observations on universality discussed earlier in this section, encourages us to believe that we are not just whistling in the dark.

3. THE HIGH-TEMPERATURE EXPANSION: THEORY AND METHOD

According to the scaling hypotheses discussed in BESZ, certain statistical quantities associated with a system near a phase transition diverge or approach zero like $(T - T_c)^{-p}$. The idea of the high-temperature expansion consists of expanding these thermodynamic quantities in powers of T^{-1} (or, which is the same thing, in powers of the action), and trying to identify the expansion as the first terms of a function like $(T - T_c)^{-p}$. Consider, for example, a function like

$$\langle U \rangle = \frac{\sum_{\text{states}} U e^{-\beta H}}{\sum_{\text{states}} e^{-\beta H}} = C \left(\frac{T - T_c}{T_c} \right)^{-p} + R(T) \quad (3.1)$$

$R(T)$ is a function which becomes small compared to the first term for $T \approx T_c$. Ignoring $R(T)$, we can expand both sides of (3.1) in powers of T^{-1} :

$$\sum_{\text{states}} \sum_{l=0}^{\infty} U_l \left(\frac{-\beta H}{T} \right)^l \frac{1}{l!} \equiv \sum_{l=0}^{\infty} a_l T^{-l} = C T^{-p} \sum_{l=0}^{\infty} \frac{\Gamma'(p+l)}{\Gamma'(p)\Gamma'(l)} \left(\frac{T_c}{T} \right)^l \quad (3.2)$$

Taking ratios of successive terms in the expansions, we have

$$\frac{a_l}{a_{l-1}} = T_c \left(1 + \frac{p-1}{l} \right) \quad (3.3)$$

We see that the ratio of successive terms plotted as a function of $1/l$ should be a straight line with intercept T_c , and slope $T_c(p-1)$. In deriving (3.3), we have neglected the effects of correction terms to the scaling behavior. Such terms are expected to produce deviations from a straight line fit to the ratios a_l/a_{l-1} , especially for small values of l , but the hope of this simple application of the high-temperature expansion is that such deviations will not be large, and will not persist through many orders of the expansion. If, on the other hand, the left-hand side of (3.3) is found not to rapidly approach a straight line, other, more sophisticated, methods (e.g. Padé approximants) may be used to extract the leading behavior of the series for $T \approx T_c$ ⁵⁾. Alternatively, extra terms may be added to the Hamiltonian -- irrelevant variables -- which will not affect the values of the

critical exponents, but which may result in a faster convergence of the series (3.3) to a straight line. It is for this reason that we have included the M interactions [Eq. (1.1)] in our Lagrangian.

Let us turn now to our problem defined by the action (1.1). Using (1.1) and the definitions of the thermodynamic averages contained in Section 4 of BESZ, we are able to calculate a number of series of the form (3.3). The ones which we have chosen to calculate are $\mathcal{G}$, $\mathcal{M}$, $\mathcal{F}$, $\mathcal{F}_s$, $\mathcal{F}_t$, and $\mathcal{F}_{st}$, defined by

$$\begin{aligned}\mathcal{G} &= -T \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_N \\ i\mathcal{M} &= \langle \phi_i \rangle \\ \mathcal{F} &= \Gamma(\underline{q}, 0) \\ \mathcal{F}_s &= \int d^2 \underline{x} \Gamma(\underline{x}, 0) \\ \mathcal{F}_t &= \int dt \Gamma(\underline{q}, t) \\ \mathcal{F}_{st} &= \int d^2 \underline{x} dt \Gamma(\underline{x}, t)\end{aligned}\tag{3.4}$$

where

$$\Gamma(\underline{x}, t) = \langle \phi_i \phi_j \rangle - \langle \phi_i \rangle \langle \phi_j \rangle \quad ; \quad (\underline{i} - \underline{j}) = (\underline{x}, t)$$

and

$$Z_N = \sum_{\text{STATES}} e^{-\mathcal{H}}$$

is the partition function for a lattice with N sites.

The thermodynamic averages are to be understood in the sense of (3.7) in the limit $N \rightarrow \infty$.

If we calculate enough ratios of the form (3.3), we will be able to express T_c and all the relevant critical exponents as functions of the coupling constants which appear in (1.1). What we need, then, is some criterion for choosing the values of the coupling constants so as to get the best approximations for the critical indices. There are in principle many such criteria, but we have chosen the following: We have calculated the series listed above to order T^{-3} . Depending on the series we then have two or three ratios of the form (3.3). We then use the ratios with $\ell = 3, 2$, and sometimes 1 to write T_c and the critical exponent as a function of the coupling constants. We choose the optimum values of the coupling constants by requiring that the T_c which appears in each of the series be the same.

If the number of series exceeds the number of independent coupling constants, it is not, in general, possible to make all the T_c 's have identical values. Hence a standard minimization program is used to search for the set of coupling constants which minimizes the function

$$\rho = \frac{\sum_i (T_{ci} - \bar{T}_c)^2}{\bar{T}_c^2} \quad (3.5)$$

where the i run over all series which were calculated, and $\bar{T}_c$ is the average of all the T_{ci} . This set of coupling constants is then used to calculate the critical exponents. Notice that despite the low order of the calculations it may be possible to get reasonable results for some exponents. Examination of the figures in chapter 9 of ref. 5 shows, for example, that the susceptibility series for Ising models give fairly accurate results even at very low orders in $1/T$.

We turn now to a discussion of the method of calculating the terms on the right-hand side of (3.2). As we explained earlier, the utility of the high-temperature expansion for lattice field theories is based on the formal analogy with statistical mechanics in which $e^{-\beta \mathcal{H}} \leftrightarrow e^{\mathcal{A}}$. For example, the propagator or correlation function for the field, ϕ , in coordinate space is

$$r(x, t) = \langle \phi_i \phi_j \rangle - \langle \phi_i \rangle \langle \phi_j \rangle = \frac{\sum_{\text{states}} \phi_i \phi_j e^{\mathcal{A}_T}}{Z_N} - \frac{\sum_{\text{states}} \phi_i e^{\mathcal{A}_T}}{Z_N} \frac{\sum_{\text{states}} \phi_j e^{\mathcal{A}_T}}{Z_N} \quad (3.6)$$

We are interested in general in expanding thermodynamic averages of the form (3.6) in powers of T^{-1} , so we will want to discuss such an expansion for a general operator $\mathcal{O}$:

$$\langle \mathcal{O} \rangle = \frac{\sum_{\text{states}} \mathcal{O} e^{\mathcal{A}_T}}{\sum_{\text{states}} e^{\mathcal{A}_T}} \quad (3.7)$$

However, before describing the general rules for calculating quantities such as (3.7), it is useful to establish some notation.

Expansions of averages such as $\langle \mathcal{O} \rangle$ will involve terms containing various powers of the action. These terms can be conveniently represented in graphical form. Our action contains five kinds of interaction, two of which, G and M , are

local, while the remaining three connect nearest neighbor points on a lattice. The notation for these interactions is shown in Fig. 1. Extra insertions of the ϕ field are denoted by a circle in which the number of extra fields at that point is given. The time direction is vertical, while the two space directions are horizontal.

Consider, for instance, a term contributing to $\langle \phi_{\underline{j}} \rangle$ coming from the second-order expansion of the numerator of (3.7):

$$\frac{1}{2!} \frac{1}{T^2} \int d[\text{states}] \sum_{\underline{j}} \phi_{\underline{j}} A_{\underline{j}} A_{\underline{j}} \quad (3.8)$$

where

$$\begin{aligned} A_{\underline{j}} = & E(\phi_{\underline{j}} \chi_{\underline{j}+\underline{i}} - \chi_{\underline{j}} \phi_{\underline{j}+\underline{i}}) + L(\phi_{\underline{j}} \phi_{\underline{j}+\underline{i}} + \chi_{\underline{j}} \chi_{\underline{j}+\underline{i}}) \\ & + K[(\phi_{\underline{j}+\underline{i}} + \phi_{\underline{j}+\underline{j}})\phi_{\underline{j}} + (\chi_{\underline{j}+\underline{i}} + \chi_{\underline{j}+\underline{j}})\chi_{\underline{j}}] \\ & + iG\phi_{\underline{j}}(\phi_{\underline{j}}^2 + \chi_{\underline{j}}^2) + M(\phi_{\underline{j}}^2 + \chi_{\underline{j}}^2) \end{aligned}$$

and

$$\int d[\text{states}] = \prod_{\underline{k}} \int_{-1}^1 \frac{d\phi_{\underline{k}}}{2} \frac{d\chi_{\underline{k}}}{2}$$

Some graphs arising from this term are shown in Fig. 2. The graphs 2a and 2e are disconnected, while 2b, c, and d are connected. Notice also that graphs 2d and 2e are zero because, from (3.8), they contain a factor like

$$\sim \int_{-1}^1 d\phi_{\underline{j}} \phi_{\underline{j}}^m$$

with m odd.

When calculating the average of a local operator, it is fairly clear what is meant by a connected graph. If we were interested in calculating the free energy per lattice site,

$$G = \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_N$$

we would choose some arbitrary point in the lattice and call all the graphs which were attached to that point, and otherwise connected, connected.

Consider now some bilocal operator, for example $\langle \phi_j \phi_{j+\hat{x}} \rangle$, which forms part of the correlation function (3.6). Some graphs contributing in second order to this object are shown in Fig. 3.

The graphs 3c and 3f are disconnected, graphs 3b and 3d are connected, and graphs 3a and 3e we call strongly connected. The importance of these distinctions will become clear when we discuss the full correlation function. Notice that graphs 3b, c, and e are identically zero.

Let us now turn to the expansion of the expression (3.7). In carrying out the expansion it is important to keep careful track of cancellations among various disconnected graphs. To order T^{-3} , the expression (3.7) may be written as follows:

$$\begin{aligned} \langle \phi \rangle = & \phi + \widetilde{\phi A} - \phi \otimes A + \frac{1}{2} [\widetilde{\phi A^2} - \phi \otimes A^2] + [\phi \otimes A - \widetilde{\phi A}] \otimes A \\ & + \frac{1}{6} [\widetilde{\phi A^3} - \phi \otimes A^3] + \frac{1}{2} [\phi \otimes A^2 - \widetilde{\phi A}] \otimes A \\ & + [\widetilde{\phi A} - \phi \otimes A] \otimes A \otimes A + \frac{1}{2} [\phi \otimes A - \widetilde{\phi A}] \otimes A^2 \end{aligned} \quad (3.9)$$

where a factor of T^{-1} is understood to accompany each factor of A . The derivation of the formula is described in more detail in Appendix A. Each separate factor in a given term is understood to be evaluated by itself. Thus

$$A^2 = \int d[\text{states}] \sum_{i,j} A_i A_j$$

and all terms, connected and disconnected, are to be included. Objects with a tilde, such as $\widetilde{\phi A^2}$, include only the connected graphs. Thus, for example $\widetilde{\phi_j \phi_{j+\hat{x}} A^2}$ contains graphs 3a, b, d, and e, but not 3c or 3f. The operation $\otimes$ is called superposition, and means that, in the product of several terms, only those terms are to be kept such that if the terms were superimposed upon each other the entire graph would look like a connected graph. For example, the term $\widetilde{\phi_j \phi_{j+\hat{x}} A} \otimes A^2$ contains the term shown in Fig. 4a, but not the term of Fig. 4b. Note that in 4a, although the factor A^2 contributes a piece which is itself disconnected, the entire superimposed structure looks connected (in fact, strongly connected).

It should now be clear that the only topological structures that need to be considered in the expansion for $\langle O \rangle$ are those that have the appearance of connected graphs. In fact, in the calculation of the correlation functions such as (3.6), only graphs and superpositions of graphs which look *strongly* connected contribute. The reason is that terms in $\langle \phi_j \phi_k \rangle$ in which the two fields are not joined by an interaction are cancelled by pieces of the term $\langle \phi_j \rangle \langle \phi_k \rangle$. Note, however, that there are other contributions coming from $\langle \phi_j \rangle \langle \phi_k \rangle$ which must explicitly be included in the correlation function. An example of such a term is shown in Fig. 5 *).

To complete our discussion of the evaluation of $\langle O \rangle$, we want to give some examples of how the graphs are actually calculated. Consider then the strongly connected graphs of Fig. 6 which contribute to various of the $\langle \phi_j \phi_k \rangle$ in third order. Now, we want to calculate $\frac{1}{6} \widetilde{OA}^3$ for these graphs. First look at graphs 6a and 6b. In each of these graphs there are three *distinct* interactions. Although in graph 6b two of the interactions are of the K-type, they occur at different points and are thus distinct. Hence, in each of these graphs there is a factor of 6 from the six different ways in which the three factors of A can give rise to the three interactions. In graph 6a there is an additional factor of 2 from the two different configurations in space which this graph has (i.e. we can go sideways first, or up first). Graph 6a therefore has the value

$$\begin{aligned} \langle \phi_j \phi_{j+\hat{x}} \rangle &= \frac{1}{6} \times 6 \times 2 \text{MKL} \int \prod_{i=1}^3 \frac{d\phi_i}{2} \frac{d\chi_i}{2} \phi_1(\phi_1\phi_2+\chi_1\chi_2)(\phi_2^2+\chi_2^2)(\phi_2\phi_3+\chi_2\chi_3)\phi_3 \\ &= \text{MKL} \frac{2}{9} \left(\frac{1}{5} + \frac{1}{9} \right) \end{aligned}$$

Graph 6b is calculated similarly.

Now turn to graph 6c. (There is a similar graph with the dangle attached to the point $j + \hat{x}$, but we disregard that graph here.) In this graph there are only

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*) In formulating the rules for a perturbative lattice expansion, it is sometimes possible to find a factorization principle so that the thermodynamic averages can be calculated according to rules which associate some factor with each vertex of a connected graph, which automatically takes into account the various terms in (3.9). (See, for example, Ref. 7). Such a formulation of our theory, which also results in important calculational simplicity does not seem to exist. However, we have investigated a somewhat different theory belonging to the same universality class, in which such rules are applicable. The results of these investigations will be reported elsewhere.

three different ways in which A^3 can give rise to the interactions which are shown, since the two E interactions are identical. The rule then is to include a factor $1/n!$ for each n identical interactions at a vertex. This graph also has two spatial configurations -- the graph with the dangle up has the same value as the graph with the dangle down. Hence it makes the following contribution to $\langle \phi_j \phi_{j+\hat{x}} \rangle$:

$$\begin{aligned} \langle \phi_j \phi_{j+\hat{x}} \rangle &= \frac{1}{6} \times 6 \times \frac{1}{2!} \times 2 \times K E^2 \int_{-1}^1 \prod_{i=1}^3 \frac{d\phi_i}{z} \frac{d\chi_i}{z} \phi_1 (\phi_1 \chi_2 - \chi_1 \phi_2)^2 (\phi_1 \phi_3 + \chi_1 \chi_3) \phi_3 \\ &= K E^2 \frac{1}{q} \left(\frac{1}{s} + \frac{1}{q} \right) \end{aligned}$$

After having calculated the non-zero correlation functions, one must add them together properly to obtain $\mathcal{F}_s$, $\mathcal{F}_t$, and $\mathcal{F}_{st}$ as defined in Eqs. (3.4). We will not dwell on this operation, since it is clear how it is to be done. The only important point to remember is that all the relevant spatial orientations of a given correlation function must be included. To convey an idea of the nature of the problem, we show in Fig. 7 all the topologically distinct graphs which contribute in order T^{-3} to the correlation functions. The graphs which contribute to the other series are just as far out.

The expansions of the six series $\mathcal{M}$, $\mathcal{G}$, $\mathcal{F}$, $\mathcal{F}_s$, $\mathcal{F}_t$, and $\mathcal{F}_{st}$ to order T^{-3} are given in Appendix B.

4. THE HIGH-TEMPERATURE EXPANSION: RESULTS

4.1 The Reggeon system

We come now to the results of our calculations. The numerical expansions tabulated in Appendix B were used in various different ways to obtain different approximations to the critical exponents. These give us some feeling for the range in which the actual exponents of the Reggeon Field Theory are likely to lie.

We first calculated the critical exponents keeping all five coupling constants as variables (actually, there are only four independent variables, as only the ratios of the coupling constants are important) and searching for the minimum of expression (3.5) where i runs over all six functions $\mathcal{F}$, $\mathcal{F}_s$, $\mathcal{F}_t$, $\mathcal{F}_{st}$, $\mathcal{M}$, $\mathcal{G}$. The results are shown in the first column of Table 1. Using the entries of this column we see that the elastic scattering amplitude near the forward direction should behave asymptotically as

$$A(s,t) \sim i s (\ln s)^\eta f(t(\ln s)^\nu) ; \quad \eta \simeq 1.79, \quad \nu \simeq 2.63 \quad (4.1)$$

These exponents may appear large, but in view of the length of the theoretical and calculational path to these results, it is gratifying that they are of a reasonable order of magnitude. We can ask how well the consistency conditions derived in BESZ are satisfied. We find

$$N+W=1.46, \quad Z+U=2.38 \quad (4.2)$$

and

$$X+Z=2.32, \quad 2Y=5.54 \quad (4.3)$$

and

$$U=0.19, \quad 2Y=5.54 \quad (4.4)$$

while, in principle, we should have had

$$N+W=Z+U \quad (4.5)$$

and

$$X+Z=2Y \quad (4.6)$$

and

$$U=2Y \quad (4.7)$$

Thus two consistency conditions are violated by 0(50%) and one by essentially 100%. There are at least two possible sources for this. One is the low order, and hence crudity, of the calculation, and another is the fact that several of the series, namely $\mathcal{G}$, $\mathcal{M}$, and $\mathcal{F}$, while singular at the critical temperature, are not divergent if the phase transition is second order^{*)}. Later on we will discuss calculations avoiding these possibly troublesome series.

In a second calculation we tested for the effects of the M interaction by setting $M = 0$, and minimizing the function (3.5). This time we restricted i to run over $\mathcal{F}_{st}$, $\mathcal{F}_s$, $\mathcal{F}_t$ and $\mathcal{M}$ -- $\mathcal{F}$ and $\mathcal{G}$ were omitted because the first terms of their high temperature series are zero if $M = 0$. The results are shown in the

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*) Hence more sophisticated techniques such as Padé approximants may be necessary in order to extract the singularities of these series⁵⁾.

second column of Table 1. From this column we see that

$$\eta \simeq 0.22 \quad , \quad \nu \simeq 1.23 \quad (4.8)$$

While the difference between Eqs. (4.1) and (4.8) is quantitatively substantial, both estimates lie in a reasonable range, despite the low order of the calculation. We are unable to use the second column of Table 1 to check the conditions (4.5), (4.6) and (4.7), but notice that γ , the exponent of $\mathcal{M}$, is suspiciously large in both columns. If the transition were second, or higher order, then γ should be negative, corresponding to a singular but finite magnetization at the critical temperature.

We also examined the effects of setting $L = 0$ in the expansions, and minimizing Eq. (3.5) summing over all the series. We found $\eta \sim -80$, $\nu \sim -150$ indicating that the L interaction plays a vital role in our calculations (indeed, the solution of the first column of Table 1 had $L/K \simeq 0.95$). Even though the L interaction (as discussed in BESZ) should not affect the infra-red (long distance) behavior, it has a crucial role in smoothing the continuum limit. Hence it is reasonable that in low order calculations where the calculated graphs extend over very few lattice spacings, the presence of L is necessary in order to extrapolate smoothly the expected singular behavior of the series.

Motivated by the possibility that the series $\mathcal{G}$, $\mathcal{M}$, and $\mathcal{F}$ may not diverge, we tried various calculational schemes ignoring these series. These results are summarized in Table 2. Because of the small number of remaining series, leaving all the couplings free would have given a continuum of solutions with the expression (3.5) equal to zero. Hence we restricted the couplings in various ways:

- a) We set $M = 0$, $L = 1$, kept the other couplings free, and minimized expression (3.5) using the ratios a_2/a_1 , a_3/a_2 for each of the series $\mathcal{F}_{st}$, $\mathcal{F}_s$, and $\mathcal{F}_t$. The results are given in the first column of Table 2. It is not possible to check any of the consistency conditions (4.5), (4.6) and (4.7), but the various exponents look reasonable, and yield

$$\eta \simeq 0.93 \quad , \quad \nu \simeq 2.12 \quad (4.9)$$

- b) We set $M = 0$, kept the other couplings free, and calculated expression (3.5) using, as well as a_2/a_1 and a_3/a_2 , also a_1/a_0 for the series $\mathcal{F}_{st}$. The results are given in the second column of Table 2. We see that

$$\eta \simeq 0.42 \quad , \quad \nu \simeq 1.49 \quad (4.10)$$

- c) We set $L = 1$, kept the other couplings free, and calculated the same series as in (b) above, with the results given in the third column of Table 2. Thus

$$\eta \simeq 0.99, \quad \nu \simeq 1.49 \quad (4.11)$$

We notice that the exponents for all the approximations (a), (b) and (c) lie well within the range corresponding to a second order phase transition. For, if the consistency conditions (4.5), (4.6) and (4.7) are used to estimate u , y , and x , the exponents of $\mathcal{F}$, $\mathcal{M}$, and $\mathcal{G}$, they are all comfortably negative. The results (4.9), (4.10), and (4.11) all lie in the centre of the range bracketed by the earlier approximations (4.1) and (4.8). Hence our procedure of discarding putatively non-divergent series is not *a posteriori* absurd, and leads us to the estimates

$$\eta \simeq \frac{1}{2} \sim 1 \quad \nu \simeq 1\frac{1}{2} \sim 2 \quad (4.12)$$

with the approximations (4.1) and (4.8) as salutary warnings as to the reliability of this estimate.

We would now like to discuss the last rows of Tables 1 and 2. An analysis similar to that of Section 4 of BESZ shows that the critical index β characterizing the cross-section for the multiple low mass diffraction process of Fig. 8, and defined by

$$\sigma_{n+2}(s) \sim \frac{\sigma_2(s)}{(\ln s)^{n\beta}}$$

is related to the exponents u , v , w and z via

$$\beta = \frac{2z - w - \nu - 2}{z - \nu} \quad (4.13)$$

From the results of Table 1 we find $\beta \approx 0.27$ to 0.59 , suggesting

$$\beta \approx \frac{1}{3} \sim \frac{2}{3} \quad (4.14)$$

We should note two general features of the values of the couplings leading to the approximations (4.1), (4.8), (4.9), (4.10) and (4.11). The first is that all the solutions have non-trivial values of the ratios of the couplings which are $O(1)$. Hence all the couplings play an essential rôle in the approximation

scheme, and we have reason to hope that we may be approaching a genuine Reggeon theory critical point, and not a critical point of an Ising-type theory. The second feature worth remarking is that solutions do not in general have G imaginary. At first this seems to be a disconcerting result, since i) we have shown in Section 2 that in one-dimension, at least, an imaginary magnetic field is required for a phase transition and ii) an imaginary effective coupling constant is required, by renormalization group arguments, to obtain an infra-red stable fixed point for the Reggeon calculus²⁾. It is, however, a mistake to identify the values of the coupling appearing in (4.6) with any genuine field theoretic coupling constants, or effective (fixed-point) coupling constants. For instance, we know that the effective coupling constant corresponding to the L -interaction should vanish in the infra-red limit, but to any finite order in the high-temperature expansion L will be non-zero. An interesting example of this effect may be had by expanding the exact expression for the magnetization (2.10) in the one-dimensional system in a high-temperature expansion to order T^{-3} . Taking the ratios a_2/a_1 and a_3/a_2 of this expansion [see Eq. (3.3)], we can solve for the critical exponent of $\mathcal{M}$. We find

$$\frac{p+2}{p+1} = \frac{1}{8} H^2 + \frac{3}{4}$$

To obtain, in this order, a real value for iH means $H^2 < 0$. It is easy to see that, in this approximation, values of $H^2 < 0$ would give rise to an estimate for p satisfying

$$-1 > p > -5$$

whereas the actual value is $p = \frac{1}{2}$. This simple example illustrates the fact that to finite order in the high-temperature expansion it is possible to obtain estimates for the critical indices which are of roughly the right size, even though the couplings have an unexpected phase.

4.2 Checks using Ising-type models

We notice that if we set $G = E = 0$ in the Reggeon system, then it reduces to a more conventional anisotropic lattice theory^{*)}, which the standard arguments suggest belongs to the same universality class as the Ising model. Hence we can apply our approximation schemes to this truncated model, and compare the results with known Ising model values, using the scaling relations discussed in Section 4 of BESZ. The results for various different approximation schemes are:

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*) Three-dimensional if $L \neq 0$, two-dimensional if $L = 0$.

Three-dimensional model ($L \neq 0$)

With $M \neq 0$ and using all the series, we find for the susceptibility index, γ_I , and for the pair correlation function index, η_I :

$$\gamma_I \simeq 1.07 \quad \eta_I \simeq 0.69 \quad (4.15)$$

Setting $M = 0$, varying L and using just the $\mathcal{F}_{st}$, $\mathcal{F}_s$, $\mathcal{F}_t$ and $\mathcal{M}$ series we find

$$\gamma_I \simeq 0.83 \quad \eta_I \simeq 0.57 \quad (4.16)$$

Varying M and L , and using just the ratios a_3/a_2 and a_2/a_1 from the series $\mathcal{F}_{st}$, $\mathcal{F}_s$, and $\mathcal{F}_t$ we get

$$\gamma_I \simeq 1.53 \quad \eta_I \simeq 0.60 \quad (4.17)$$

Including also the a_1/a_0 ratio for the $\mathcal{F}_{st}$ series we find

$$\gamma_I \simeq 1.36 \quad \eta_I \simeq 0.61 \quad (4.18)$$

For comparison, the most accurate high-temperature expansion results for the Ising model yield

$$\gamma_I \simeq 1.25 \quad \eta_I \simeq 0.64 \quad (4.19)$$

Looking at the above results we see that while the susceptibility index is reasonably accurate, the pair correlation index is over-estimated. This lead us to believe that in the results of Tables 1 and 2, the estimates of z may be the most accurate, followed by v , then w , then u . In particular, the estimates of $z-v$ and $z-w$ may be too small, resulting perhaps in an over-estimate of η .

Two-dimensional model ($L = 0$)

Various approximations give exponents in the ranges

$$\gamma_I \simeq 1.38 \text{ to } 1.57, \quad \eta_I \simeq 7.2 \text{ to } 7.9 \quad (4.20)$$

to be compared with the exact Ising model values

$$\gamma_I = 1.75 \quad \eta_I = 0.25 \quad (4.21)$$

Again γ_I is reasonably accurate but η_I is over-estimated, disastrously this time. We believe that the origin of this problem is the same as the similar over-estimate of η_I in the three-dimensional case, but is further enhanced by the small number of terms in the series which are left when L , G and E are all zero, and the fact that only one independent parameter M/K is left free to carry out the minimization. Our approximation scheme then starts to run out of gas (petrol).

5. SUMMARY AND CONCLUSIONS

Let us briefly review the major results of this paper and discuss their implications. Our purpose was to study the lattice version of the Reggeon field theory defined in Ref. 1. In particular, we hoped to learn something about the applicability of universality and the nature of critical behavior in the Reggeon calculus, and, if possible, to calculate the critical exponents.

To learn something more about the phase transitions of our theory near the critical point, we solved exactly a model possessing many of the features of the Reggeon field theory, namely a one-dimensional Ising model with the usual nearest neighbor interactions, as well as an additional nearest neighbor interaction which oscillates in sign -- positive between one pair, and negative between the next pair. When this system was coupled to an external imaginary magnetic field, we found, somewhat surprisingly, that the model in general possesses a phase transition at a non-zero critical temperature. However, if the normal Ising (ferromagnetic) interaction is turned off, no such phase transition exists, pointing up the importance of including the L -interaction when defining the lattice theory [see Eq. (1.1)]. After restoring the ferromagnetic interaction, the nature of the phase transition is independent of the particular values of the couplings, confirming the expectations of universality, at least in a limited domain. Moreover, although the transition is of first order, inasmuch as the magnetization diverges at the critical point, the thermodynamic functions are generalized homogeneous functions, whose exponents obey the relations derived in Ref. 1. All this encourages us to believe that the Reggeon field theory and our lattice analogue possess a phase transition to which universality arguments and the scaling hypotheses discussed in BESZ are applicable.

Fortified by this example, we proceeded in Sections 3 and 4 to describe the high-temperature expansion, derive the rules for perturbation theory, and calculate the first three terms for a number of the thermodynamic averages associated with our lattice theory. The general method used to obtain the optimum values of the exponents [namely, minimizing expression (3.5)] should be generally useful for people performing high-temperature expansions, especially when a large number

of coupling constants are involved. We first calculated using all the thermodynamic functions that we could, with $M \neq 0$ and with $M = 0$, getting results (4.1) and (4.8) which, while rather different, certainly correspond to fairly reasonable critical behavior. We then recalculated discarding the series G , M and $\mathcal{F}$ which may very well not be divergent at the critical point, and three different calculational schemes yielded qualitatively similar results with

$$\eta \simeq \frac{1}{2} \sim 1, \quad \nu \simeq \frac{1}{2} \sim 2, \quad \beta \simeq \frac{1}{2} \sim \frac{2}{3} \quad (5.1)$$

We also applied our methods to truncated theories resembling the Ising model in two and three dimensions, getting results correct to about 20% for the susceptibility index γ_I , but deviating markedly for the pair correlation function index, η_I . This leads us to suspect that the actual values of η and ν , particularly η , may be closer to the lower, rather than the upper end of the range (5.1).

It is interesting to compare our results with the estimates obtained from the first- and second-order ϵ -expansions^{2,3}). This is done in Table 3. We see that our preferred approximations yield somewhat larger exponents η and ν than the ϵ -expansion. In this context, it must be remembered that the ϵ -expansion is probably only asymptotic, and when $\epsilon = 2$ we do not know *a priori* where to truncate the series. At the present stage we are unable to say whether the high temperature expansion results differ significantly from the ϵ -expansion. However, our results do indicate that the Reggeon field theory probably has a sensible infrared behavior. More quantitative results -- especially a better approximation for η -- must await the outcome of higher order calculations which are now in progress. However, we hope that the initial progress reported here will encourage other workers to study this and other possible lattice analogues to the Reggeon field theory.

Acknowledgements

We are indebted to J. Zinn-Zustin for numerous invaluable discussions; without him this work would have taken an extra six months. One of us (R.S.) would like to thank the American-Swiss Foundation for Scientific Exchange for partial support, and D. Amati for the hospitality of the CERN Theory Division.

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Table 1

Function	Exponent	Value	
		Full theory	M = 0
$\mathcal{F}_{st}$	z	2.09	1.16
$\mathcal{F}_s$	v	1.34	0.21
$\mathcal{F}_t$	w	0.12	-0.01
$\mathcal{F}$	u	0.19	-
$\mathcal{M}$	y	2.77	1.39
$\mathcal{J}$	x	0.23	-
Cross-section	$\eta = \frac{v}{z - v}$	1.79	0.22
Slope	$v = \frac{z - w}{z - v}$	2.63	1.23
Multiple diffraction	$\beta = \frac{2z - w - v - 2}{z - v}$	0.94	0.12

Table 2

Function	Exponent	Value		
		Version (a)	Version (b)	Version (c)
$\mathcal{F}_{st}$	z	1.53	1.28	1.36
$\mathcal{F}_s$	v	0.73	0.38	0.68
$\mathcal{F}_t$	w	-0.15	-0.06	-0.17
Cross-section	$\eta = \frac{v}{z - v}$	0.93	0.42	0.99
Slope	$v = \frac{z - w}{z - v}$	2.12	1.49	2.43
Multiple Diffraction	$\beta = \frac{2z - w - v - 2}{z - v}$	0.59	0.27	0.30

Table 3

Exponent	High-temperature expansion preferred range	$O(\epsilon)$	$O(\epsilon^2)$
η	$\frac{1}{2} \sim 1$	0.17	0.38
v	$1\frac{1}{2} \sim 2$	1.08	1.18
β	$\frac{1}{3} \sim \frac{2}{3}$	0.50	-

APPENDIX A

In this appendix we will demonstrate how Eq. (3.9) is derived. For this purpose it is sufficient to consider the terms in the expansion of (3.7) which are of order T^{-2} . It is easy to see that the coefficient of T^{-2} is

$$\frac{1}{2} [\mathcal{O}A^2] - [\mathcal{O}A][A] - \frac{1}{2} [\mathcal{O}][A^2] + [\mathcal{O}][A][A] \quad (\text{A.1})$$

where

$$[N] \equiv N = \int_{-1}^1 \pi \frac{d\phi_i}{2} \frac{d\chi_i}{2} \eta$$

The brackets are included only for clarity, where necessary.

Now, terms of the form $[\mathcal{O}A^n]$ contain, in general, disconnected as well as connected contributions. Recalling the notation used in the text, it is clear, for instance, that

$$[\mathcal{O}A] - [\mathcal{O}][A] = \widetilde{\mathcal{O}A} - \mathcal{O} \otimes A \quad (\text{A.2})$$

since the disconnected pieces in $[\mathcal{O}A]$ are cancelled by the contributions from the product $[\mathcal{O}][A]$ which look disconnected.

Now consider $[\mathcal{O}A^2]$. This object may be expanded as follows:

$$\begin{aligned} [\mathcal{O}A^2] = & \widetilde{\mathcal{O}A^2} + 2 \widetilde{\mathcal{O}A}[A] - 2 \widetilde{\mathcal{O}A} \otimes [A] + [\mathcal{O}][A^2] \\ & - 2[\mathcal{O}] \otimes [A][A] + 2[\mathcal{O}] \otimes [A] \otimes [A] - [\mathcal{O}] \otimes [A^2] \end{aligned} \quad (\text{A.3})$$

The first term on the right includes all the connected contributions. The next two terms, taken together, reproduce all the contributions in which one of the insertions of the action is connected to the operator, but the other one is disconnected from the resulting structure, $\widetilde{\mathcal{O}A}$ (partially connected graph). Finally, the last four terms take care of the graphs in which neither insertion of the action is connected to the operator $\mathcal{O}$. This last result comes about as follows: The term $[\mathcal{O}][A^2]$ contains all the graphs in which neither A is attached to $\mathcal{O}$, but it also contains contributions which look either connected or partially connected. The two terms $-2[\mathcal{O}] \otimes [A][A] + 2[\mathcal{O}] \otimes [A] \otimes [A]$ subtract off from $[\mathcal{O}][A^2]$ the terms which have a partially connected topology, while the term $-[\mathcal{O}] \otimes [A^2]$ removes from $[\mathcal{O}][A^2]$ the contributions with a connected topology.

Substituting the expansion (A.3) into the expression (A.1) and using the relation (A.2), it is easy to verify the second-order piece of the equation (3.9). The third-order terms in (3.9) can be derived by similar reasoning. It might also be remarked that the n^{th} order term in (3.9) has the form of an Ursell-Mayer semi-invariant in that all the splits (superpositions) of the operator $\mathcal{O}$ and the n -powers of A appear with their required coefficients.

APPENDIX B

The six series, $\mathcal{G}$, $\mathcal{M}$, $\mathcal{F}$, $\mathcal{F}_s$, $\mathcal{F}_t$, and $\mathcal{F}_{st}$ to $(1/T^3)$ are given by

$$\begin{aligned}\mathcal{G} = & M(.6667) + K^2(.2222) + E^2(.1111) - G^2(.1714) + M^2(.0889) + L^2(.1111) \\ & + MK^2(.1185) + ME^2(.0593) - MG^2(.0965) - G^2K(.1936) \\ & + M^3(.0057) + L^2M(.0593) - LG^2(.0968)\end{aligned}\quad (B.1)$$

$$\begin{aligned}\mathcal{M} = & G(.3111) + GK(.4148) + GM(.1355) + LG(.2074) \\ & + E^2G(-.0240) + K^2G(.5051) - G^3(.0006) + MKG(.2912) \\ & + M^2G(.0237) + LGM(.1457) + L^2G(.1143) + KLG(.5531)\end{aligned}\quad (B.2)$$

$$\begin{aligned}\mathcal{F} = & \frac{1}{3} + M(.0889) + G^2(.0307) + E^2(.0296) + K^2(.0593) + M^2(.0084) \\ & + L^2(.0297) + G^2K(.1264) + G^2M(.0556) + E^2M(.0629) \\ & + K^2M(.1260) + M^3(-.0012) + L^2M(.0629) + LG^2(.0631)\end{aligned}\quad (B.3)$$

$$\begin{aligned} \mathcal{F}_s = & \frac{1}{3} + M(.0889) + K(.4444) + G^2(-.0307) + E^2(-.0296) \\ & + K^2(.5033) + M^2(.0084) + MK(.2372) + L^2(.0297) + G^2K(-.2081) \\ & + G^2M(-.0556) + E^2M(.0629) + K^2M(.4824) + M^3(-.0012) \\ & + M^2K(-.0358) + E^2K(.1776) + K^3(.5152) + L^2M(.0629) + LG^2(-.0631) + L^2K(.1780) \end{aligned} \quad (B.4)$$

$$\begin{aligned} \mathcal{F}_t = & \frac{1}{3} + M(.0889) + L(.2222) + G^2(-.0307) + E^2(-.0444) + K^2(.0593) \\ & + M^2(.0084) + L^2(.1037) + LM(.1186) + G^2K(-.1264) + G^2M(-.0556) \\ & + E^2M(.0035) + K^2M(.1260) + M^3(-.0012) + L^2M(.1223) + LM^2(.0270) \\ & + LG^2(-.1041) + LK^2(.1776) + LE^2(-.0593) + L^3(.0454) \end{aligned} \quad (B.5)$$

$$\begin{aligned} \mathcal{F}_{st} = & \frac{1}{3} + M(.0889) + K(.4444) + L(.2222) + G^2(-.0307) + E^2(-.0444) \\ & + K^2(.5033) + M^2(.0084) + MK(.2372) + L^2(.1037) + LM(.1186) + LK(.5926) \\ & + G^2K(-.2081) + G^2M(-.0556) + E^2M(.0035) + K^2M(.4824) \\ & + M^3(-.0012) + M^2K(-.0358) + E^2K(-.1184) + K^3(.5152) + KLM(.4741) \\ & + L^2M(.1223) + LM^2(.0270) + LG^2(-.1041) + LK^2(.10665) \\ & + L^2K(.4740) + LE^2(-.0593) + L^3(.0454) \end{aligned} \quad (B.6)$$

Figure captions

- Fig. 1 : Graphical definition of the interactions appearing in the high-temperature expansion. The point $\underline{j}$ is indicated in the nearest-neighbor interactions.
- Fig. 2 : Some second-order graphs which arise in the calculation of $\langle \phi_{\underline{j}} \rangle$.
- Fig. 3 : Some second-order graphs which arise in the calculation of $\langle \phi_{\underline{j}} \phi_{\underline{j}+\hat{x}} \rangle$.
- Fig. 4 : a) A term contained in $\overbrace{\phi_{\underline{j}} \phi_{\underline{j}+\hat{x}}} A \otimes A^2$.
b) A term which does not contribute to $\overbrace{\phi_{\underline{j}} \phi_{\underline{j}+\hat{x}}} A \otimes A^2$.
- For clarity, the interactions are labelled according to their positions on the lattice.
- Fig. 5 : A term in the product $\langle \phi_{\underline{j}} \rangle \langle \phi_{\underline{j}+\hat{x}} \rangle$ which explicitly contributes to $\langle \phi_{\underline{j}} \phi_{\underline{j}+\hat{x}} \rangle - \langle \phi_{\underline{j}} \rangle \langle \phi_{\underline{j}+\hat{x}} \rangle$.
- Fig. 6 : Some strongly connected graphs contributing to $\langle \phi_{\underline{j}} \phi_{\underline{k}} \rangle$ for various values of $\underline{k}$.
- Fig. 7 : The mind-boggling plethora of graphs contributing to the correlation functions in $\mathcal{O}(1/T^3)$.
- Fig. 8 : The multiple diffraction cross-section σ_{n+2} described in the text.

$$\left. \vphantom{\int} \right\}_j = E (\phi_j X_{j+i} - X_j \phi_{j+i})$$

$$\left. \vphantom{\int} \right|_j = L (\phi_j \phi_{j+i} + X_j X_{j+i})$$

$$\text{---}_j = K (\phi_j \phi_{j+s} + X_j X_{j+s})$$

$$G = iG \phi_j (\phi_j^2 + X_j^2)$$

$$M = M (\phi_j^2 + X_j^2)$$

$$\textcircled{n} = \phi_j^n$$

FIG.1

FIG. 2

FIG. 3

FIG. 4

$$\langle \phi_{\underline{j}} \rangle \times \langle \phi_{\underline{j}+\hat{x}} \rangle = \left[\begin{array}{c} \textcircled{1} \\ \underline{j} \end{array} \text{---} \text{G} \text{---} \begin{array}{c} \textcircled{1} \\ \underline{j}+\hat{x} \end{array} \right] \times \left[\begin{array}{c} \textcircled{1}^G \\ \underline{j}+\hat{x} \end{array} \right]$$

FIG.5

FIG.6

FIG. 7

($n+2$) low-mass clusters of particles
($n+1$) large rapidity gaps

FIG. 8