A COMPLETE BPHZ VERSION OF THE SUPERSYMMETRY
AND BROKEN CHIRAL SYMMETRY MODEL

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ABSTRACT

It is shown, using BPHZ renormalization techniques, that the Wess and Zumino supersymmetric scalar model can be renormalized such that supersymmetry and broken chiral invariance are preserved. In particular, the broken chiral Ward identities for Green's functions turn out to be free of anomalies. This fact is directly related to properties of the Callan-Symanzik functions discovered by Ferrara, Iliopoulos and Zumino. We use first a notation in which the components of the superfields are displayed explicitly, whereas a supersymmetric notation is employed at the end in order to exhibit the results in a more compact form.

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1. INTRODUCTION

In this paper, we deal with the renormalization of the field theory model introduced first by Wess and Zumino^{1,2)}, and studied further by Ferrara, Iliopoulos and Zumino (FIZ)^{3,4)}. This model is the simplest one which is invariant under supersymmetry transformations, previously called restricted supergauge transformations by Wess and Zumino. It also exhibits chiral symmetry, broken by the mass term of the Lagrangian. Using an argument based on an invariant regulator treatment of this theory, Tsao⁵⁾ concluded the absence of anomalies for the Ward identity associated with broken chiral symmetry. From this, he showed easily the main result of Refs. 3 and 4, namely the fact that only a wave function renormalization is needed or, equivalently, the proportionality of the Callan-Symanzik β and γ functions.

The renormalization of the Wess-Zumino model will be achieved here by means of the BPHZ regulator free method designed by Zimmermann^{6,7)}. It has the nice feature of allowing for the construction of Green's functions involving composite fields ("normal products"), and its power resides in the algebraic identities fulfilled by such Green's functions, like the effective equations of motion and the Zimmermann's identities, which are very useful in the search for anomalies, for instance.

Our aim is to shed some light on the link between FIZ's result mentioned above and the absence of anomalies in broken chiral Ward identities for the Green's functions of fundamental fields. We shall see that this link is nothing else than supersymmetry.

After having recalled the classical Wess-Zumino theory (which gives the tree approximation Green's functions) and some of its properties in Sections 2 and 3, we state in Section 4 the BPHZ subtraction scheme used throughout this paper, as well as the renormalized supersymmetry Ward identities. We work out the Callan-Symanzik equation (together with the renormalization group equation) using Lowenstein's method of differential vertex operators^{7,8)}. Ferrara, Iliopoulos and Zumino's result is then obtained from an identity derived in Section 6, where the absence of anomalies for broken chiral Ward identities is proved. This latter proof relies on Weinberg's power counting theorem⁹⁾.

Deeper insight into the link between broken scale and chiral symmetries is attempted in the last section, where we use an explicitly supersymmetric notation in terms of superfields, introduced elsewhere^{10,11)}. As a by-product, we obtain without further calculation the anomaly for the Ward identity associated with the broken general supersymmetry transformations defined in Ref. 1. This unifying picture is to be put in correspondence with (and in fact was suggested by) the discovery of Ferrara and Zumino¹²⁾ that the local currents connected to broken scale, chiral and general supersymmetry transformations belong to the same supermultiplet.

2. SUPERSYMMETRY IN THE TREE APPROXIMATION

The supersymmetric model of Wess and Zumino^{1,2)} in its most simplified form involves a scalar field A, a pseudoscalar field B, an anticommuting Majorana spinor ψ , and two auxiliary scalar and pseudoscalar fields F and G. First let us summarize the classical theory, namely the tree approximation, ignoring radiative corrections. In such a case, our investigations are guided by the classical Lagrangian²⁾,

$$\mathcal{L}_{cl.} = \mathcal{L}_k + m \mathcal{L}_m + g \mathcal{L}_g \quad (1)$$

where $\mathcal{L}_k = -\frac{1}{2} (\partial A)^2 - \frac{1}{2} (\partial B)^2 - \frac{i}{2} \bar{\psi} \gamma \cdot \partial \psi + \frac{1}{2} F^2 + \frac{1}{2} G^2$

$$\mathcal{L}_m = FA + GB - \frac{i}{2} \bar{\psi} \psi \quad (2)$$

$$\mathcal{L}_g = F(A^2 - B^2) + 2GAB - i\bar{\psi} \psi A + i\bar{\psi} \gamma_5 \psi B$$

Since the transformation properties of the relevant fields under an infinitesimal (restricted) supergauge transformation are given by

$$\begin{aligned} \delta A &= [\bar{\alpha} Q, A] = i \bar{\alpha} \psi \\ \delta B &= [\bar{\alpha} Q, B] = i \bar{\alpha} \gamma_5 \psi \\ \delta \psi &= [\bar{\alpha} Q, \psi] = \partial_\mu (A - \gamma_5 B) \gamma^\mu \alpha + (F + \gamma_5 G) \alpha \\ \delta F &= [\bar{\alpha} Q, F] = i \bar{\alpha} \gamma^\mu \partial_\mu \psi \\ \delta G &= [\bar{\alpha} Q, G] = i \bar{\alpha} \gamma_5 \gamma^\mu \partial_\mu \psi \end{aligned} \quad (3)$$

(α is a constant totally anticommuting Majorana spinor^{*)}), one easily verifies that the three parts of the Lagrangian transform like total derivatives, so that the action built up from $\mathcal{L}_{cl}$ is invariant. In constructing a supergauge invariant Lagrangian we have ignored the possible occurrence of a term proportional to F such as

$$\mathcal{L}_F = -\lambda F \quad (4)$$

*) The canonical dimensions of the various fields are $\dim(A) = \dim(B) = 1$, $\dim(F) = \dim(G) = 2$, $\dim(\psi) = \frac{3}{2}$ and $\dim(\alpha) = -\frac{1}{2}$. The metric is $\eta_{\mu\nu} = (-1, 1, 1, 1)$. It is convenient to work in Majorana representation with real γ^μ 's, $(\gamma_0)^2 = (\gamma_5)^2 = -1$. The anticommutation relations read $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. In reality ψ and $\bar{\psi}$ are not independent.

because one can eliminate it by a shift in the field A^{*}). Such a shifting changes only the mass term²⁾.

To derive Ward identities we proceed as follows. One considers the Lagrangian in the presence of external sources, i.e. we replace $\mathcal{L}_{cl}$ by^{**)}

$$\mathcal{L}_{cl} \rightarrow \mathcal{L}_{cl} + J_A A + J_B B + J_F F + J_G G - i \bar{\psi} \eta \quad (5)$$

where $J \equiv \{J_A, J_B, J_F, J_G, \eta\}$ is the set of sources associated with the set of fields $\phi \equiv \{A, B, F, G, \bar{\psi}\}$. It is well known that the connected Green's functions, in the tree approximation, are generated by functional differentiation of the functional $\phi(J)$, which is a solution of the classical field equations in the presence of sources^{13,14)}:

$$\frac{\delta^E \mathcal{L}_{cl}}{\delta \phi} + J = 0 \quad (6)$$

(the exponent E means the Euler derivative) or, equivalently, of the classical action

$$G^c\{J\} \equiv I\{J\} = i \int d^4x [\mathcal{L}_{cl} + J\phi] \quad (7)$$

evaluated for ϕ satisfying the equation of motion (6).

Applying Noether's theorem to the supergauge transformation (2), one finds the following integrated Ward identity

$$\frac{\delta}{\delta \bar{\alpha}} G^c\{J\} = i \int d^4x J(x) \frac{\delta \phi(x)}{\delta \bar{\alpha}} = 0 \quad (8)$$

This is the well-known Ward identity characterizing supersymmetry. Equation (8) reads in detail

$$\int d^4x \left[- (J_A + \gamma_5 J_B) \psi + \gamma^\mu \partial_\mu (J_F - \gamma_5 J_G) \psi - \gamma^\mu \partial_\mu (A - \gamma_5 B) \eta + (F + \gamma_5 G) \eta \right] = 0 \quad (9)$$

Equation (9) is more conveniently expressed in terms of the generating functional for connected Green's functions. From Eq. (7) we get

*) Let $A' = A + a$ be the shifted quantity, then one can compensate the term $\mathcal{L}_F$ provided a satisfies $ga^2 + ma - \lambda = 0$. This gives only real solutions if $m^2 + 4\lambda g \geq 0$.

**) Since ψ is real, ψ and $\bar{\psi}$ are not independent and only one spinor source is needed.

$$A(x) = -i \frac{\delta G\{J\}}{\delta J_A(x)}, \quad B(x) = -i \frac{\delta G\{J\}}{\delta J_B(x)}, \quad F(x) = -i \frac{\delta G\{J\}}{\delta J_F(x)},$$

$$G(x) = -i \frac{\delta G\{J\}}{\delta J_G(x)}, \quad \Psi(x) = \frac{\delta G\{J\}}{\delta \eta(x)} \gamma^0 \quad (10)$$

and therefore the relation (9) becomes

$$\int dx \left[-(J_A + \gamma_5 J_B) \left(\frac{\delta G\{J\}}{\delta \eta} \gamma^0 \right) + \gamma (J_F - \gamma_5 J_G) \left(\frac{\delta G\{J\}}{\delta \eta} \gamma^0 \right) \right. \\ \left. + i \gamma \left(\frac{\delta}{\delta J_A} - \gamma_5 \frac{\delta}{\delta J_B} \right) \eta G\{J\} - i \left(\frac{\delta}{\delta J_F} + \gamma_5 \frac{\delta}{\delta J_G} \right) \eta G\{J\} \right] = 0 \quad (11)$$

To be complete, we also write explicitly the classical equations of motion (6) for the various fields components:

$$\begin{aligned} \partial^2 A + m F + 2g S_A + J_A &= 0 \\ \partial^2 B + m G + 2g S_B + J_B &= 0 \\ -\frac{i}{2} (\gamma + m) \Psi - ig S_\Psi - i\eta &= 0 \\ F + m A + 2g S_F + J_F &= 0 \\ G + m B + 2g S_G + J_G &= 0 \end{aligned} \quad (12)$$

where

$$\begin{aligned} S_A &= AF + BG - \frac{i}{2} \bar{\Psi} \Psi = \mathcal{L}_m \\ S_B &= AG - BF + \frac{i}{2} \bar{\Psi} \gamma_5 \Psi \\ S_\Psi &= (A - \gamma_5 B) \Psi \\ S_F &= \frac{1}{2} (A^2 - B^2) \\ S_G &= AB \end{aligned} \quad (13)$$

It is always possible to eliminate the auxiliary fields F and G from the Lagrangian (1) by using their equations of motion (12). However, we believe it is more convenient to treat F and G as fundamental fields, in order to avoid in the later approach the occurrence of anisotropies.

Note that the classical source terms J in Eqs. (6), (9), (11) and (12) represent contact terms, as it is readily seen from the following formal expression for the generating functional^{15,16)} of connected Green's functions^{*})

$$G^c\{J\} = \sum_{n=0}^{\infty} \dots \sum_{m=0}^{\infty} \dots \frac{i^n}{n!} \dots \frac{1}{m!} \dots \int dx_1 \dots dx_n \dots dy_1 \dots dy_m$$

$$J_A(x_1) \dots J_A(x_n) \dots$$
(14)

$$\langle T A(x_1) \dots A(x_n) \dots \bar{\Psi}_{\alpha_1}(y_1) \dots \bar{\Psi}_{\alpha_m}(y_m) \dots \rangle^c$$

$$\eta_{\alpha_1}(y_1) \dots \eta_{\alpha_m}(y_m)$$

It will later be useful to express Ward identities for 1PI Green's functions. The latter, namely the vertex functions, are generated by the following functional, obtained from G^c by a Legendre transform^{15,17)}

$$\Gamma\{R\} = G^c\{J\} - i \int dx (J_A A + J_B B$$

$$+ J_F F + J_G G - i \bar{\Psi} \eta)$$
(15)

Here, R stands for the set of "classical fields" $R = \{A, B, F, G, \Psi\}$, defined by [cf. Eqs. (10)]

$$A = -i \frac{\delta G^c}{\delta J_A}, \quad B = -i \frac{\delta G^c}{\delta J_B}, \quad F = -i \frac{\delta G^c}{\delta J_F}, \quad G = -i \frac{\delta G^c}{\delta J_G}$$

and

$$\Psi = \frac{\delta G^c}{\delta \bar{\eta}}, \quad \bar{\Psi} = - \frac{\delta G^c}{\delta \eta}$$
(16)

^{*}) The generating functional for disconnected Green's functions is $G\{J\} = \exp(G^c\{J\})$.

The converse formulae are given by

$$\frac{\delta \Gamma}{\delta A} = -i J_A, \quad \frac{\delta \Gamma}{\delta B} = -i J_B, \quad \frac{\delta \Gamma}{\delta F} = -i J_F, \quad \frac{\delta \Gamma}{\delta G} = -i J_G$$

and

$$\frac{\delta \Gamma}{\delta \Psi} = \bar{2}, \quad \frac{\delta \Gamma}{\delta \bar{\Psi}} = -2 \quad (17)$$

Of course, in the tree approximation, $\mathcal{A} = A$, etc., and $\Gamma\{\mathcal{R}\} = i \int d^4x \mathcal{L}_{cl}(\phi)$.

It is now straightforward to obtain from (11) the Ward identities for vertex functions:

$$\int d^4x \left\{ -i \left(\frac{\delta}{\delta A} + \gamma_5 \frac{\delta}{\delta B} \right) \Psi + i \bar{\Psi} \left(\frac{\delta}{\delta F} - \gamma_5 \frac{\delta}{\delta G} \right) \Psi - (A + \gamma_5 B) \bar{\Psi} \frac{\delta}{\delta \bar{\Psi}} - (F + \gamma_5 G) \frac{\delta}{\delta \bar{\Psi}} \right\} \Gamma\{\mathcal{R}\} = 0 \quad (18)$$

3. WARD IDENTITY FOR BROKEN CHIRAL SYMMETRY IN THE TREE APPROXIMATION

In our next step we consider first an idealized version of the model of Wess and Zumino^{1,2)}, namely, we take all fields as massless. For such a case there exists a larger supergroup which contains conformal and γ_5 transformations. Following Ref. 5 we study, therefore, the infinitesimal change of fields due to a supersymmetry transformation with a constant parameter α and a γ_5 transformation with constant parameter ε ⁵⁾:

$$\begin{aligned} \delta A &= i \bar{\alpha} \Psi - \varepsilon B \\ \delta B &= i \bar{\alpha} \gamma_5 \Psi + \varepsilon A \\ \delta \Psi &= \bar{\Psi} (A + \gamma_5 B) \alpha + (F + \gamma_5 G) \alpha + \frac{\varepsilon}{2} \gamma_5 \Psi \\ \delta F &= i \bar{\alpha} \bar{\Psi} \gamma_5 - 2 \varepsilon G \\ \delta G &= i \bar{\alpha} \gamma_5 \bar{\Psi} + 2 \varepsilon F \end{aligned} \quad (19)$$

Under these transformations the action from the following symmetric Lagrangian

$$\mathcal{L}_s = \mathcal{L}_k + g \mathcal{L}_g$$

is invariant; the formal expressions of $\mathcal{L}_k$ and $\mathcal{L}_g$ are given by Eq. (2).

To recover the model of Wess and Zumino and to discuss the problem of broken chiral symmetry, one introduces again mass terms. This is usually done by adding to $\mathcal{L}_S$ a term proportional to the F field so that a shift in the A field gives again the model (1). In the following we restrict ourselves only to the γ_5 part of the transformations (19), especially

$$\begin{aligned}\delta A &= -\varepsilon B \\ \delta B &= \varepsilon A \\ \delta \psi &= \frac{\varepsilon}{2} \gamma_5 \psi \\ \delta F &= -2\varepsilon G \\ \delta G &= 2\varepsilon F\end{aligned}\tag{20}$$

The transformation properties of the three pieces of the Lagrangian (1) under (20) are

$$\delta \mathcal{L}_k = \delta \mathcal{L}_g = 0$$

whereas the mass term changes to

$$\delta \mathcal{L}_m = \varepsilon m (FB - GA - \frac{i}{2} \bar{\psi} \gamma_5 \psi) = -\varepsilon m S_B\tag{21}$$

Note that S_B is the interaction term in the equation of motion (12) for the B field. This term will play a central role in our future investigations.

In the same way as we derived the Ward identity (9) for supersymmetry transformations we obtain, by varying the action (7) under an infinitesimal chiral transform (20) and by using (21),

$$\int dx \left[A J_B - B J_A - \frac{i}{2} \bar{\psi} \gamma_5 \psi + 2 F J_G - 2 G J_F - m S_B \right] = 0$$

Replacing the breaking term S_B by its expression deduced from the equation of motion (12) for the B field, this identity becomes

$$\begin{aligned}\int dx \left[\frac{m^2}{2g} G + \left(A + \frac{m}{2g} \right) J_B - B J_A - \frac{i}{2} \bar{\psi} \gamma_5 \psi \right. \\ \left. + 2 F J_G - 2 G J_F \right] = 0\end{aligned}\tag{22}$$

For later reference, we will call this Ward identity the "broken chiral Ward identity" in its classical form. We may write it as a functional equation for the generating functional $G^C\{J\}$:

$$\int dx \left\{ \left[-i \frac{m^2}{2g} \frac{\delta}{\delta J_G} - i \left(J_B \frac{\delta}{\delta J_A} - J_A \frac{\delta}{\delta J_B} \right) + \frac{i}{2} \frac{\delta}{\delta \eta} \gamma_5 \right] \right. \\ \left. - 2i \left(J_G \frac{\delta}{\delta J_F} - J_F \frac{\delta}{\delta J_G} \right) \right] G' \{ J \} + \frac{m}{2g} J_B \} = 0 \quad (23)$$

Its expression for vertex functions reads

$$\int dx \left\{ \frac{i m^2}{2g} G + \left[B \frac{\delta}{\delta A} - A \frac{\delta}{\delta B} - \frac{1}{2} \bar{\Psi} \gamma_5 \frac{\delta}{\delta \bar{\Psi}} \right. \right. \\ \left. \left. + 2 \left(G \frac{\delta}{\delta F} - F \frac{\delta}{\delta G} \right) - \frac{m}{2g} \frac{\delta}{\delta B} \right] \Gamma \{ R \} \right\} = 0 \quad (24)$$

4. RENORMALIZATION OF THE SUPERSYMMETRIC MODEL

In the BPHZ version of renormalizable Lagrangian perturbation theory there exists an unambiguous prescription for calculating the renormalized Green's functions via the finite part of the Gell-Mann-Low expansion^{6,7)}. To obtain renormalized finite Green's functions we adopt here the BPHZ subtraction procedure formulated by Zimmermann^{6,7)}. The two main ingredients for this renormalization program (besides the definition of the finite part of Green's functions) are an effective Lagrangian and power counting.

To obtain the naive degree of divergence of the proper diagrams of the model under consideration, we observe that our Lagrangian (1) in its classical form is in some sense a generalization of a super-renormalizable field theoretic model. This is due to the fact that in (1) only bilinear and trilinear interaction terms occur.

For this topology, straightforward power counting leads to³⁾

$$d_\gamma = d_\gamma(E) - n_{AF} - n_{BG}$$

with

$$d_\gamma(E) = 4 - N_A - N_B - 2 N_F - 2 N_G - \frac{3}{2} N_\Psi \equiv 4 - \mathcal{N} \quad (25)$$

for a proper diagram γ with non-trivial renormalization part.

In these formulae N_A , N_B , N_F , N_G and N_ψ are the numbers of external lines corresponding to the A, B, F, G and ψ fields emanating from a given diagram γ_n , while n_{AF} and n_{BG} count the number of the mixed propagators in γ .

At this stage we have only spoken about unrenormalized Green's function based on the Lagrangian (1) in its quantized form. To extend our model beyond the three approximation in the spirit of the BPHZ scheme we must introduce an effective Lagrangian preserving the super-symmetry of the model. In order to determine the finite Green's function with the aid of Zimmermann's "forest formula", one must assign to each one-particle irreducible subdiagram γ an integer at least as large as the superficial (degree of) divergence of γ . This means in our case to assign to the bilinear and trilinear "interaction terms" an integer δ greater than or equal to its dimension.

To generalize the Lagrangian (1) to an effective Lagrangian we have, in principle, two possibilities for the bilinear term: either we can treat it as a super-renormalizable vertex with $\delta = 3$, or as a mass term, with $\delta = 4$. We adopt the second possibility to avoid infrared problems by having masses and to get simpler bilinear equations of motion. The choice $\delta = 4$ for the mass terms is closely related to power counting; therefore we modify our power counting to

$$d_\gamma = 4 - N \quad (26)$$

This amounts to an oversubtraction scheme and tells us that this corresponds to the assignment of degree 4 to each interaction term.

Now we are able to state the effective Lagrangian hoping to preserve super-symmetry

$$\mathcal{L}_{eff} = z N_4[\mathcal{L}_k] + m_0 N_4[\mathcal{L}_m] + g_0 N_4[\mathcal{L}_g] \quad (27)$$

$\mathcal{L}_k$, $\mathcal{L}_m$ and $\mathcal{L}_g$ are specified through the formulae (2). The finite counter terms a , δm and δg in

$$z = 1 - a, \quad m_0 = m + \delta m, \quad g_0 = g + \delta g \quad (28)$$

are fixed by appropriate normalization conditions on the two- and three-point functions; we shall clarify this point at the end of this section. The corresponding Feynman rules are found in Appendix A.

The further procedure is guided by the BPHZ subtraction scheme as formulated by Zimmermann^{6,7)}. Having once assigned a number $\delta_k \geq d_k$ to each vertex of a certain operator product $D^{(k)}$ with canonical dimension d_k , we are able to define normal products^{6,7)}:

$$\langle T N_{s_k} [D^{(k)}(x)] \chi \rangle = \text{F.P.} \langle T : D^{(k)}_{(0)}(x) : \chi^{(0)} \exp i \int \mathcal{L}^{(0)}_{\text{int.}} dx \rangle_{(0)} \quad (29)$$

The finite part (F.P.) of such an object is obtained with the aid of the forest formula^{6,7)} with the degree function

$$\delta_\gamma = 4 - N - \sum_{V_k \in \gamma} (4 - \delta_k) \quad (30)$$

χ is an arbitrary product of the fields of the model. Our next step is devoted to deriving the effective equations of motion in their bilinear form. This is mainly achieved by the analysis of Lowenstein^{7,18)}, i.e. by purely diagrammatic considerations. We introduce the following abbreviations for the Euler variation

$$\frac{\delta^E \mathcal{L}_{\text{eff}}}{\delta \phi_i} \equiv \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}_{\text{eff}}}{\partial (\partial_\mu \phi_i)} \quad (31)$$

with $\mathcal{L}_{\text{eff}}$ given by (27), while ϕ_i is any fundamental field of the supermultiplet, $\phi_i = \{A, B, F, G, \psi\}$. Since we restrict ourselves only to bilinear equations of motion we avoid completely the anisotropic subtraction scheme contained in Lam's principle¹⁹⁾. A detailed analysis, closely following Lowenstein's, shows the "classical" results:

$$\left[N_{4-d_i} \left[\frac{\delta^E \mathcal{L}_{\text{eff}}}{\delta \phi_i} \right] + \mathcal{J}_i \right] G\{\mathcal{J}\} = 0$$

and

$$\left[N_{4+d_j-d_i} \left[\frac{\delta^E \mathcal{L}_{\text{eff}}}{\delta \phi_i} \phi_j \right] + \phi_j \mathcal{J}_i \right] G\{\mathcal{J}\} = 0 \quad (32)$$

Equations (32) are understood to be applied on the generating functional $G\{\mathcal{J}\}$ for disconnected Green's functions. For any monomial $\mathcal{O}(x)$ of fundamental fields,

$$N_s[\mathcal{O}](x) G\{\mathcal{J}\}$$

is the generating functional for disconnected Green's functions with this normal product inserted; moreover,

$$\mathcal{J}_i(x) G\{\mathcal{J}\} \quad , \quad \phi_j(x) \mathcal{J}_i(x) G\{\mathcal{J}\} \equiv -i \frac{\delta}{\delta \mathcal{J}_j(x)} \mathcal{J}_i(x) G\{\mathcal{J}\}$$

are contact terms, as already noted at the end of Section 2. The Euler derivatives (31) are the same as the classical one which gave Eqs. (12), except for

the coefficients of $\mathcal{L}_{\text{eff}}$; they read, for $\phi_i = A, B, \bar{\psi}, F$ and G , respectively:

$$\begin{aligned}
 \delta^E \mathcal{L}_{\text{eff}} / \delta A &= \square^2 A + m_0 F + 2 g_0 S_A \\
 \delta^E \mathcal{L}_{\text{eff}} / \delta B &= \square^2 B + m_0 G + 2 g_0 S_B \\
 \delta^E \mathcal{L}_{\text{eff}} / \delta \bar{\psi} &= -\frac{i}{2} \square \not{\partial} \psi - \frac{i}{2} m_0 \psi - i g_0 S_\psi \\
 \delta^E \mathcal{L}_{\text{eff}} / \delta F &= \square F + m_0 A + 2 g_0 S_F \\
 \delta^E \mathcal{L}_{\text{eff}} / \delta G &= \square G + m_0 B + 2 g_0 S_G
 \end{aligned} \tag{33}$$

S_A, S_B , etc., are given by Eqs. (13).

It is now straightforward to show that, with the effective Lagrangian (27), the supersymmetry Ward identities (11) or (18) hold at any order of the renormalized perturbation series. Indeed, the infinitesimal variations $\delta\phi_i$ [Eqs. (3)] under a supersymmetry operation are linear and homogeneous in the fields ϕ_i . Thus

$$\begin{aligned}
 \sum_i \int dx \, \delta\phi_i G\{J\} &= - \sum_i \int dx \, N_+ \left[\frac{\delta^E \mathcal{L}_{\text{eff}}}{\delta\phi_i} \delta\phi_i \right] G\{J\} \\
 &\equiv - \delta \int dx \, \mathcal{L}_{\text{eff}} G\{J\} = 0
 \end{aligned} \tag{34}$$

The first equality comes out from the second of the equations of motion (32), whereas the last one is a consequence of the formal invariance of the action built up from the effective Lagrangian (27). The left-hand side of Eq. (34) is nothing else than the Ward identity (9) or, equivalently, the Ward identities (11) and (18)*).

We give in Appendix B some explicit examples of the supersymmetry Ward identities, which will be useful later on.

At last, we turn our attention to the normalization conditions, which specify recursively the three finite counterterms a , δm and δg in the Lagrangian (27).

Because of the supersymmetry of the model, it is sufficient to discuss only the normalization conditions for the two-point function $\Gamma_{\psi\bar{\psi}}(p, -p)$ (two conditions) and for one three-point function, e.g. $\Gamma_{AAF}(q_1, q_2, s_1)$ (one condition). Following the notations of Appendix B, one writes

*) We note that the supersymmetry Ward identities for connected or disconnected Green's functions are identical.

$$\Gamma_{\psi\bar{\psi}}(p, -p) = i \left[\not{p} S_1(p^2) - i m S_2(p^2) \right] \quad (35)$$

We propose two alternative sets of normalization conditions:

$$\begin{aligned} S_1(-m^2) &= S_2(-m^2) \\ S_1(-\mu^2) &= 1 \end{aligned} \quad (36a)$$

or

$$\Gamma_{AAF}(q_1, q_2, s_1) \Big|_{\text{sym. pt. } (\mu^2)} = 2ig$$

$$\begin{aligned} S_1(-\mu^2) &= S_2(-\mu^2) = 1 \\ \Gamma_{AAF}(q_1, q_2, s_1) \Big|_{\text{sym. pt. } (\mu^2)} &= 2ig \end{aligned} \quad (36b)$$

The first set ensures the presence of a pole at the physical mass for all the propagators [see Eq. (B.10) of Appendix B], whereas the second set reduces to the normalization conditions at the origin of momentum space, considered by the authors of Refs. 3 and 4, when $\mu^2 = 0$. In our BPHZ subtraction scheme, this latter case means simply that $a = \delta m = \delta g = 0$, or $z = 1$, $m_0 = m$, $g_0 = g$.

5. CALLAN-SYMANZIK EQUATION

In this section, we shall show the proportionality of the Callan-Symanzik functions β and γ , a result first obtained by the authors of Ref. 4. In order to do so, we shall need the following identity, proved in the next section and related to the absence of chiral anomalies for renormalized Green's functions:

$$\Gamma_{AAAF}(0, 0, 0, 0) = 0 \quad (37)$$

Our first task, to derive the Callan-Symanzik equation by Lowenstein's method^{7,8)}, is to construct all the supersymmetric differential vertex operations (SDVO). By this, we mean the following. Let us write the supersymmetry Ward identity for vertex Green's functions, in a shorthand notation,

$$W_s \Gamma\{\mathcal{R}\} \equiv \int dx \sum_{ij} a_{ij} \mathcal{R}_i \frac{\delta}{\delta \mathcal{R}_j} \Gamma\{\mathcal{R}\} = 0 \quad (38)$$

where W_s is the functional differential operator acting in the left-hand side of Eq.(18). Let D be a differential vertex operation, which may be a vertex insertion,

a derivative with respect to a parameter of the theory, or a functional differential operator. By definition, D is an SDVO of it commutes with the Ward identity, namely if

$$W, D \Gamma\{R\} = [W, D] \Gamma\{R\} = 0 \quad (39)$$

For the case where D is a vertex insertion of the form

$$\Delta_s[\sigma] = \int dx N_s[\sigma](x) \quad (40)$$

one can extract from Lam's work¹⁹⁾ and Eq. (38) that

$$W, \Delta_s[\sigma] \Gamma\{R\} = i \Delta_{s+\frac{1}{2}}[\delta\sigma] \Gamma\{R\} \quad (41)$$

where $\delta\sigma(x)$ is the formal variation of $\sigma(x)$ under an infinitesimal supersymmetry transformation (3), up to a factor $i\bar{\alpha}$. [Such a simple result stems from the linearity of transformations (3)]. Hence, up to degree 4, the supersymmetric vertex insertions are the vertices of the Lagrangian, possibly with different degrees:

$$\begin{aligned} \Delta_m^0 &= i \Delta_3[\mathcal{L}_m] \quad , \quad \Delta_m = i \Delta_4[\mathcal{L}_m] \\ \Delta_k &= i \Delta_4[\mathcal{L}_k] \quad , \quad \Delta_g = i \Delta_4[\mathcal{L}_g] \end{aligned} \quad (42)$$

with $\mathcal{L}_k, \mathcal{L}_m$ and $\mathcal{L}_g$ given in Eq. (2).

The functional operator W_s in (38), (39) being independent of the parameters of the theory, it is clear that the derivatives

$$\partial_m \quad , \quad \partial_\mu \quad , \quad \partial_g \quad (43)$$

are SDVO's, too.

At last, the supersymmetric functional operators are

$$N = - \int dx \left\{ A \frac{\delta}{\delta A} + B \frac{\delta}{\delta B} + \sum_\alpha \bar{\Psi}_\alpha \frac{\delta}{\delta \bar{\Psi}_\alpha} + F \frac{\delta}{\delta F} + G \frac{\delta}{\delta G} \right\} \quad (44)$$

(which counts the number of external legs) and

$$\Delta_F, \text{ defined by } \Delta_F \Gamma\{R\} = i \int dx \mathcal{F}(x)$$

$$\Delta_A = \int dx \frac{\delta}{\delta A} \quad (45)$$

One easily shows that N is an SDVO: the bilinear equations of motion [the second of Eqs. (32)] indeed imply

$$N = 2z \Delta_k + 2m_0 \Delta_m + 3g_0 \Delta_g \quad (46)$$

which is obviously supersymmetric. The proof for Δ_F is easier when it is applied to connected Green's functions:

$$\Delta_F G^c\{J\} = \int dx \frac{\delta G^c\{J\}}{\delta J_F(x)}$$

Thus Δ_F is the insertion of the F field, and we deduce from its formal invariance that it is an SDVO, as we did for insertions (42). Δ_A is related to other SDVO's by the equation of motion for the A field

$$z \partial^2 \frac{\delta G^c}{i \delta J_A} + m_0 \frac{\delta G^c}{i \delta J_F} + 2g_0 N_3[l_m] G^c + J_A = 0$$

which, for vertex functions, reads

$$z \partial^2 A + m_0 F + [2g_0 N_3[l_m] + i \frac{\delta}{\delta A}] \Gamma\{R\} = 0 \quad (47)$$

thus

$$\Delta_A = m_0 \Delta_F + 2g_0 \Delta_m^0 \quad (48)$$

is an SDVO.

A more systematic study shows that the insertions (42) are the only ones up to degree 4 and that the functional operators (44), (45) are the only linear and bilinear ones which are supersymmetric.

Now, we note that, of the eight SDVO's (42) to (44), only three are independent, which we may choose to be the three vertex insertions of degree 4. Indeed, by Lowenstein's effective action theorem^{7,8)}, we have first

$$\begin{aligned}\partial_m \Gamma &= (\partial_m z \Delta_k + \partial_m m_0 \Delta_m + \partial_m g_0 \Delta_g) \Gamma \\ \partial_\mu \Gamma &= (\partial_\mu z \Delta_k + \partial_\mu m_0 \Delta_m + \partial_\mu g_0 \Delta_g) \Gamma \\ \partial_g \Gamma &= (\partial_g z \Delta_k + \partial_g m_0 \Delta_m + \partial_g g_0 \Delta_g) \Gamma\end{aligned}\quad (49)$$

We have also Eq. (46) for N , and the Zimmermann's identity, proved in Appendix C, for Δ_m^0 :

$$\Delta_m^0 = c_k \Delta_k + \Delta_m + c_g \Delta_g \quad (50)$$

[Expressions for c_k and c_g are found in Eq. (C.7), Appendix C]. Hence, the five SDVO's ∂_m , ∂_μ , ∂_g , N and Δ_m^0 are all expressed in terms of the three independent insertions Δ_k , Δ_m and Δ_g , so that they must be related by two linear relations, namely the Callan-Symanzik equation and the renormalization-group equation. We write the Callan-Symanzik equation replacing Δ_m^0 by its expression in terms of Δ_A and Δ_F deduced from (48):

$$\begin{aligned}& \left[m \partial_m + \mu \partial_\mu + \beta(g, m/\mu) \partial_g - \gamma(g, m/\mu) N \right] \Gamma\{\mathcal{R}\} \\ &= \alpha(g, m/\mu) \frac{m}{2g} (\Delta_A - m_0 \Delta_F) \Gamma\{\mathcal{R}\}\end{aligned}\quad (51)$$

The renormalization-group equation reads^{*)}

$$\left[\mu \partial_\mu + \beta'_m(g, m/\mu) m \partial_m + \beta'_g(g, m/\mu) \partial_g - \gamma'(g, m/\mu) N \right] \Gamma\{\mathcal{R}\} = 0 \quad (52)$$

*) $\beta'_m = 0$ if normalization conditions (36a) are used, as one can see by applying the first of these to Eq. (52).

To prove Eqs. (51) and (52), we have to show that the elimination of Δ_k , Δ_m and Δ_g between the five Eqs. (49), (50) and (46) is indeed possible. This is the case if both the following systems, ensuring the eliminations required for Eqs. (51) and (52), respectively, admit solutions for α , β , γ and β'_m , β'_g , γ' :

$$\begin{aligned} \partial_g z \beta - 2 z \gamma - m \frac{g_0}{g} c_k \alpha &= 0 \\ \partial_g m_0 \beta - 2 m_0 \gamma - m \frac{g_0}{g} \alpha &= -m_0 \\ \partial_g g_0 \beta - 3 g_0 \gamma - m \frac{g_0}{g} c_g \alpha &= 0 \end{aligned} \quad (53)$$

and

$$\begin{aligned} m \partial_m z \beta'_m + \partial_g z \beta'_g - 2 z \gamma' &= -\mu \partial_\mu z \\ m \partial_m m_0 \beta'_m + \partial_g m_0 \beta'_g - 2 m_0 \gamma' &= -\mu \partial_\mu m_0 \\ m \partial_m g_0 \beta'_m + \partial_g g_0 \beta'_g - 3 g_0 \gamma' &= -\mu \partial_\mu g_0 \end{aligned} \quad (54)$$

In Eqs. (53), we have used the dimensional analysis identities $(\mu \partial_\mu + m \partial_m) z = (\mu \partial_\mu + m \partial_m) g = 0$ and $(\mu \partial_\mu + m \partial_m) m_0 = m_0$.

The determinants of both systems are non-vanishing at zeroth order of perturbation series ($z = 1$, $m_0 = m$, $g_0 = g = c_k = c_g = 0$, $\partial_g g_0 = 1$); thus Eqs. (53) and (54) have unique solutions as formal power series in g .

We confine ourselves now to the Callan-Symanzik equation (51). Using result (37) and Eq. (C.7) of Appendix C, we conclude that

$$c_g = 0 \quad (55)$$

This result is equivalent to the one found by the authors of Ref. 4. Indeed, using their normalization conditions [Eqs. (36b) with $\mu = 0$], which imply $z = 1$, $m_0 = m$ and $g_0 = g$, the system (53) solves to

$$\begin{aligned} \alpha &= \frac{1}{1 - m c_k} \\ \beta &= 3 g \gamma = -\frac{3}{2} \frac{g m c_k}{1 - m c_k} \end{aligned} \quad (56)$$

Recall that c_k is given in Eq. (C.7) of Appendix C. Thus we see that the three functions α , β and γ depend on one single independent function, as was found in Ref. 4.

We get the proportionality of β and γ even for more general normalization conditions: the third of Eqs. (53), Eq. (55) being taken into account, reads

$$\gamma/\beta = \frac{1}{3} \partial_g \ln(g_0) \quad (57)$$

One may also note that the result

$$\beta = 3g\gamma$$

holds whenever the three-point function is normalized at the origin, in which case $g_0 = g$, due to our subtraction scheme.

6. RENORMALIZED WARD IDENTITY FOR BROKEN CHIRAL SYMMETRY

We want to show that the broken chiral Ward identity (BCWI) for the Green's functions of the fundamental fields does not suffer from anomalies when radiative corrections are included^{*)}. In doing this, we shall prove the important identity (37), used in the preceding section to derive the special properties of the Callan-Symanzik functions first shown by FIZ⁴⁾. This makes a firm link between these latter properties and the absence of anomalies for BCWI, a fact already advanced by Tsao⁵⁾.

The BCWI can be derived in the same way as we did it for the supersymmetry Ward identity in Section 4. We start from Eq. (34), taking for the variation $\delta\phi_i$ the infinitesimal chiral transformations (20). The right-hand side is now non-vanishing, due to the non-invariance of the effective Lagrangian (27); thus one obtains

$$\begin{aligned} & \int dx \left\{ -J_A \frac{\delta}{i\delta J_B} + J_B \frac{\delta}{i\delta J_A} + \frac{i}{2} \frac{\delta}{\delta_2} \gamma_5 \eta - 2J_F \frac{\delta}{i\delta J_G} + 2J_G \frac{\delta}{i\delta J_F} \right\} G\{J\} \\ & = -\frac{\delta}{\delta \varepsilon} \int dx \mathcal{L}_{eff} G\{J\} = m_0 \int dx N_+ [S_B] G\{J\} = m_0 \Delta_+ [S_B] G\{J\} \end{aligned} \quad (58)$$

*) We do not say anything about Green's functions involving axial currents. Our BCWI is only the integrated form of the Ward identity for the divergence of the axial current.

[We used definition (40) for the last equality; S_B is given in Eq. (13)]. Degree 4 assignment for S_B comes from the use of bilinear equations of motion [second form of Eqs. (32)]. On the other hand, it is $\Delta_3[S_B]$ which is related, by a linear equation of motion [first form of Eqs. (32)], to insertions of fundamental fields: the integrated linear equation of motion for B reads [(32), (33)] indeed

$$\int dx \left\{ m_0 \frac{\delta}{i \delta J_B} + J_B \right\} G\{J\} + 2 g_0 \Delta_3[S_B] G\{J\} = 0 \quad (59)$$

This difference of effective degrees may be a source of anomalies. Indeed, we show in Appendix C the Zimmermann's identity

$$\begin{aligned} \Delta_3[S_B] = \Delta_4[S_B] + \\ + c_g \Delta_4 \left[(A^2 - B^2) G - 2 A B F + i \bar{\Psi} (B + \gamma_5 A) \Upsilon \right] \end{aligned} \quad (60)$$

where c_g , given by Eq. (C.7) of Appendix C, turns out to be the same as the c_g appearing in the Zimmermann's identity (50) for $\mathcal{L}_m$ in the preceding section, and whose (not yet proved) vanishing was so crucial.

If we succeed in proving that $c_g = 0$, $\Delta_3[S_B]$ as given by Eq. (59) may be substituted in Eq. (58), and the tree approximation form (23) would be recovered (up to the coefficients of the first and last terms).

Before doing this, let us transform Eq. (60). By a repeated use of integrated bilinear equations of motion inside the normal products in its last term, (60) is modified to

$$\begin{aligned} \Delta_3[S_B] = \Delta_4[S_B] + c_g \left\{ - \frac{m_0}{2 g_0} \Delta_4[S_B] \right. \\ \left. + \frac{1}{2 g_0} \int dx \left(J_A \frac{\delta}{i \delta J_B} - J_B \frac{\delta}{i \delta J_A} + \frac{i}{2} \frac{\delta}{\delta \eta} \gamma_5 \eta \right) \right\} \end{aligned} \quad (61)$$

Solving for $\Delta_4[S_B]$ and substituting it into (58) gives, after $\Delta_3[S_B]$ is replaced by its expression in terms of insertions of fundamental fields and classical sources as deduced from (59) [the whole thing is still multiplied by $1 - m_0 c_g / (2 g_0)$]:

$$\begin{aligned}
 & \int dx \left\{ -i \frac{m_0^2}{2g_0} \frac{\delta}{\delta J_G} - i \left(J_B \frac{\delta}{\delta J_A} - J_A \frac{\delta}{\delta J_B} \right) + \frac{i}{2} \frac{\delta}{\delta \eta} \gamma_5 \eta \right. \\
 & \left. - 2i \left(J_G \frac{\delta}{\delta J_F} - J_F \frac{\delta}{\delta J_G} \right) + \frac{m_0}{2g_0} J_B \right\} G\{J\} \\
 & + i \frac{m_0}{g_0} c_g \int dx \left\{ J_B \frac{\delta}{\delta J_A} - J_A \frac{\delta}{\delta J_B} + J_G \frac{\delta}{\delta J_F} - J_F \frac{\delta}{\delta J_G} \right\} G\{J\} \\
 & = 0
 \end{aligned} \tag{62}$$

The corresponding Ward identity for vertex functions is obtained by writing Eq. (62) first for connected Green's functions and next by application of the Legendre transform (15). One obtains

$$\left[W_{BC} + \frac{m_0}{g_0} c_g V \right] \Gamma\{R\} = 0 \tag{63}$$

where

$$\begin{aligned}
 W_{BC} \Gamma\{R\} = & \int dx \left\{ \frac{i}{2} \frac{m_0^2}{g_0} g + \left[B \frac{\delta}{\delta A} - A \frac{\delta}{\delta B} - \frac{1}{2} \bar{\Psi} \gamma_5 \frac{\delta}{\delta \bar{\Psi}} \right. \right. \\
 & \left. \left. + 2 \left(g \frac{\delta}{\delta f} - f \frac{\delta}{\delta g} \right) - \frac{m_0}{2g_0} \frac{\delta}{\delta B} \right] \Gamma\{R\} \right\}
 \end{aligned} \tag{64}$$

and

$$V \Gamma\{R\} = \int dx \left\{ A \frac{\delta}{\delta B} - B \frac{\delta}{\delta A} + f \frac{\delta}{\delta g} - g \frac{\delta}{\delta f} \right\} \Gamma\{R\} \tag{65}$$

Note that the right-hand side of Eq. (64) is just the classical BCWI (24).

It remains now to show that $c_g = 0$, thus proving the absence of anomalies. Let us apply the sum of functional derivatives

$$- \frac{\delta^3}{\delta A \delta A \delta g} + \frac{\delta^3}{\delta B \delta B \delta g} + \frac{\delta^3}{\delta A \delta B \delta f} + \frac{\delta^3}{\delta B \delta A \delta f}$$

to the left-hand side of Eq. (67). The result reads, in momentum space:

$$\begin{aligned} & \frac{1}{2} \left(\Gamma_{AAGB} - \Gamma_{BBGB} - \Gamma_{ABFB} - \Gamma_{BAFB} \right) (\lambda p_1, \lambda p_2, \lambda p_3, 0) \\ & = c_g \left(\Gamma_{AAF} - \Gamma_{BBF} + \Gamma_{ABG} + \Gamma_{BAG} \right) (\lambda p_1, \lambda p_2, \lambda p_3) \end{aligned} \quad (66)$$

where λ is a scale parameter. When $\lambda \rightarrow \infty$, Euclidian momenta p_1, p_2, p_3 being kept fixed, Weinberg's power counting theorem⁹⁾ states that the right-hand side of (66) is dominating. More precisely:

$$\text{L.H.S. (66)} \sim \lambda^{-1} (\ln \lambda)^x ; \text{R.H.S. (66)} \sim (\ln \lambda)^y \quad (67)$$

The behaviours (67) are the maximal possible ones, at any order of perturbation series. That the behaviour (67) for the right-hand-side is effectively reached is checked for the tree approximation:

$$\left(\Gamma_{AAF} - \Gamma_{BBF} + \Gamma_{ABG} + \Gamma_{BAG} \right)_{\text{tree}} = 4 i g$$

Hence the right-hand side of Eq. (66) is asymptotically non-vanishing at any order, in the sense of formal power series, whereas the left-hand side is vanishing. Thus, letting λ go to infinity in Eq. (66) yields the desired result

$$c_g = 0 \quad (68)$$

That is, using Eq. (C.7) of Appendix C,

$$\Gamma_{AAAF}(0,0,0,0) = 0 \quad (68')$$

7. SUPERFIELD NOTATION; ANOMALY FOR BROKEN GENERAL SUPERSYMMETRY

All results of preceding paragraphs can be stated in a very unified way, if one uses the notation in terms of superfield introduced by Ferrara, Wess and Zumino¹⁰⁾, and Salam and Strathdee²¹⁾. We shall apply the notations of Ref. 10, summarized in Appendix D.

If $\phi^+(x, \theta)$ is the superfield (D.1), whose components are related to the fields A, B, ψ, F and G by the relations (D.2), one can write the action for our model as^{10,20,22)}

$$S = \frac{1}{2} \int dx \int d^2\theta \left[z L_k(x, \theta) + m_0 L_m(x, \theta) + g_0 L_g(x, \theta) \right] + h.c. \quad (69)$$

The composite superfields L_k , L_m and L_g are given in Eqs. (D.9) of Appendix D.

The θ integration^{20,22)} is defined by

$$\int d\theta^\alpha d\theta^\beta = \delta^{\alpha\beta}, \quad \int d\theta^\alpha = 0$$

It follows that, using anticommuting properties of θ and $d\theta$,

$$\int d^2\theta (\theta\theta) = 2$$

Hence, this operation, when taking the real part, selects the F component [defined as in (D.1), (D.2)] of the integrand^{*)}. The F components of the different terms of (69) are displayed in Eqs. (D.10) of Appendix D; thus, we see that the action (69) is nothing else than the space-time integral of the Lagrangian density (1).

If we add to the action (69) a source term

$$S_{\text{source}} = \int dx d^2\theta J^+(x, \theta) \phi^+(x, \theta) + h.c. \quad (70)$$

where the external source superfield is written as

$$J^+(x, \theta) = \frac{1}{2} (J_F - i J_G) + \theta \eta^+ + \theta\theta \frac{1}{2} (J_A + i J_B) \quad (70')$$

the equations of motion in the presence of this external source are

$$\begin{aligned} & -\frac{z}{4} \frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \bar{\theta}_i} \phi^-(x - 2i\theta\sigma\bar{\theta}, \bar{\theta}) + m_0 \phi^+(x, \theta) \\ & + 2g_0 [\phi^+(x, \theta)]^2 + J^+(x, \theta) \\ & \equiv \frac{\delta S}{\delta \phi^+} + J^+ = 0 \end{aligned} \quad (71)$$

The A, B, ψ , F, and G components of Eq. (71) are the equations of motion (12) for the fields F, G, ψ , A, and B, respectively.

*) Since the F component of any superfield of the type (D.1) transforms under the supersymmetry group like a total derivative, this ensures the invariance of the action.

We obtain the effective equations of motion (32) for the renormalized Green's functions with the proper subtraction prescriptions if we write them as

$$\begin{aligned} N_2 \left[\frac{\delta S}{\delta \phi^+} \right] + J^+ &= 0 \\ N_3 \left[\phi^+ \frac{\delta S}{\delta \phi^+} \right] + \phi^+ J^+ &= 0 \end{aligned} \quad (72)$$

for the linear or bilinear case, respectively, adopting the rule^{*)}

$$N_\delta [\theta \sigma(x, \theta)] = \theta N_{\delta + \frac{1}{2}} [\sigma(x, \theta)] \quad (73)$$

for any composite superfield $\sigma(x, \theta)$ of dimension less than or equal to δ .

This latter rule enables us to write, in a supersymmetric way, the following Zimmermann's identity for the "mass term" of the Lagrangian (69):

$$\begin{aligned} N_2 [L_m] - N_3 [L_m] &= N_3 [c_g L_g + c_k L_k + c'_k L'_k] \\ &+ c_0 \partial^2 \phi^+ \end{aligned} \quad (74)$$

$L_m(x, \theta)$ being a chiral superfield, that is a superfield of the type (D.1) and with covariance (D.7), we have to put, in the right-hand side of (74), all independent composite chiral superfields of dimension 3. These are L_g , L_k , L'_k as given by Eqs. (D.9) of Appendix D, and $\partial^2 \phi^+$.

We may already note that the F and G components of the identity (74), when integrated over space-time points, are just the Zimmermann's identities (50) and (60) which gave the scale and chiral anomalies. This can be readily seen by inspecting the F and G components, respectively, of L_m , L_g , L_k and L'_k written in Appendix D, Eq. (D.10); in particular, L'_k does not contribute to identities (50) and (60), since its F and G components are total derivatives, nor does $\partial^2 \phi^+$, for the same reason.

The reduction coefficients c_g , c_k , c'_k and c_0 may be computed by considering a particular component of identity (74). Thus, by the remark above, c_g and c_k are the coefficients appearing in Zimmermann's identities (50) and (60); they are

*) We keep the subtraction procedure defined in Section 4, with the same power counting (26).

given by Eqs. (C.7) of Appendix C. To compute c'_k , we may consider, say, the F component of Eq. (74). In the same way as for the calculations of Appendix C, we find

$$\begin{aligned} c'_k &= \frac{i}{16g_0} \frac{\partial^2}{\partial p \partial q} \Gamma_{AAA}(-p-q, p, q) \Big|_{p=q=0} \\ c_0 &= \frac{i}{16g_0} \frac{\partial^2}{\partial p^2} \Gamma_{AF}(-p, p) \Big|_{p=0} \end{aligned} \quad (75)$$

Note that a Bose-Einstein symmetry argument²³⁾ implies

$$\frac{\partial^2}{\partial p^2} \Gamma_{AAA}(0, p, -p) \Big|_{p=0} = 2 \frac{\partial^2}{\partial p \partial q} \Gamma_{AAA}(-p-q, p, q) \Big|_{p=q=0}$$

so that

$$c'_k = \frac{1}{2} c_k \quad (76)$$

Thus, we have seen that the Zimmermann's identities, which give the scale and chiral anomalies through the reduction coefficients c_g and c_k , both are the F and G components of the same supersymmetric Zimmermann's identity (74). To show, in Appendix C, that the coefficients of the former two identities were identical and that two of them only were independent, we had used supersymmetry Ward identities; we learn from the present section that this result is straightforward if we use a supersymmetric notation. We recall that the effective absence of anomaly for the integrated broken chiral Ward identity comes from the vanishing of c_g proved in Section 6, and from the fact that the G component of L_k [Eqs. (D.10c) of Appendix D] is a total derivative. Results for the Callan-Symanzik functions in Section 5 are the consequence of the vanishing of c_g , too.

Zimmermann's identity (74) enables us to give directly the anomalies for the Ward identity associated with the broken supersymmetry transformations of the second kind (infinitesimal parameter linear in x_μ) defined in Ref. 1. These transformations read, for component fields (we put $\alpha = x\beta$, $x \equiv x_\mu \gamma^\mu$, β is a constant anticommuting Majorana spinor):

$$\begin{aligned} \delta A &= -i \bar{\beta} x \psi \\ \delta B &= -i \bar{\beta} x \gamma_5 \psi \\ \delta \bar{\psi} &= \bar{\beta} x \not{\partial} (A - \gamma_5 B) - \bar{\beta} x (F + \gamma_5 G) + 2 \bar{\beta} (A - \gamma_5 B) \\ \delta F &= -i \bar{\beta} x \not{\partial} \psi \\ \delta G &= -i \bar{\beta} x \gamma_5 \not{\partial} \psi \end{aligned} \quad (77)$$

The formal variation of the Lagrangian (1) or (27) under these infinitesimal transformations is

$$\delta \mathcal{L}_k = \delta \mathcal{L}_g = 0, \quad \delta \mathcal{L}_m = 2i\bar{\beta} S_\psi + \text{total derivative} \quad (78)$$

where $\mathcal{L}_k$, $\mathcal{L}_g$ and $\mathcal{L}_m$ are given by Eqs. (2) and S_ψ by Eqs. (12), (13).

Using again the bilinear effective equations of motion [the second kind of Eqs. (32)], in the same way as in the previous sections, we find

$$\begin{aligned} & \int dx \left\{ -i(\mathcal{I}_A - \gamma_5 \mathcal{I}_B) * \left[\frac{\delta G^{\{J\}}}{\delta \eta} \gamma^0 \right] - i(\mathcal{I}_F - \gamma_5 \mathcal{I}_G) * \gamma \left[\frac{\delta G^{\{J\}}}{\delta \eta} \gamma^0 \right] \right. \\ & - \left[* \gamma \left(\frac{\delta}{\delta \mathcal{I}_A} - \gamma_5 \frac{\delta}{\delta \mathcal{I}_B} \right) - * \left(\frac{\delta}{\delta \mathcal{I}_F} + \gamma_5 \frac{\delta}{\delta \mathcal{I}_G} \right) + 2 \left(\frac{\delta}{\delta \mathcal{I}_A} - \gamma_5 \frac{\delta}{\delta \mathcal{I}_B} \right) \right] \eta G^{\{J\}} \right. \\ & \left. + 2im_0 N_{7/2} [S_\psi] G^{\{J\}} \right\} = 0 \end{aligned} \quad (79)$$

It is at this point, in order to relate the over-subtracted vertex insertion

$$\Delta_{7/2} [S_\psi] \equiv \int dx N_{7/2} [S_\psi] \quad (80)$$

to the corresponding minimally-subtracted one [recall that $\dim(S_\psi) = 5/2$], that we use the integrated ψ component of the Zimmermann's identity (74). Taking into account the result $c_g = 0$, the latter reads, when the rule (73) is applied:

$$\Delta_{5/2} [S_\psi] = \Delta_{7/2} [S_\psi] + c_k \Delta_{7/2} [L_k^{(\psi)}] \quad (81)$$

where [see Eqs. (D.10) of Appendix D]

$$L_k^{(\psi)} = \frac{1}{2} (A - \gamma_5 B) \gamma \psi + \frac{1}{2} (F - \gamma_5 G) \psi \quad (82)$$

and

$$S_\psi = L_m^{(\psi)} = (A - \gamma_5 B) \psi \quad (83)$$

The integrated linear effective equation of motion for ψ is, in terms of vertex functions,

$$\frac{1}{2} m_0 \int dx \psi + \left[g_0 \Delta_{5/2} [S_\psi] - \int dx \frac{\delta}{\delta \bar{\psi}} \right] \Gamma\{\mathcal{R}\} = 0 \quad (84)$$

Now, we write Eq. (79) for vertex functions, and we substitute into it $\Delta_{7/2}[S_\psi]$ as given from Eq. (81), with $\Delta_{5/2}[S_\psi]$ replaced by its expression inferred from Eq. (84). This gives the following integrated Ward identity, whose last term is the anomalous term:

$$\begin{aligned}
 & - \frac{m_0^2}{2g_0} \int dx \, \Psi + \int dx \left\{ -i \left(\frac{\delta}{\delta A} - \gamma_5 \frac{\delta}{\delta B} \right) * \Psi - i \left(\frac{\delta}{\delta F} - \gamma_5 \frac{\delta}{\delta G} \right) * \Psi \right. \\
 & \left. + \left[* \gamma (A - \gamma_5 B) - * (F + \gamma_5 G) + 2(A - \gamma_5 B) + 2 \frac{m_0}{g_0} \right] \frac{\delta}{\delta \bar{\Psi}} \right\} \Gamma\{R\} \quad (85) \\
 & - 2 m_0 c_k \Delta_{7/2} [L_k^{(r)}] = 0
 \end{aligned}$$

Notation of Eqs. (14) to (17) has been used; c_k is given by Eq. (C.7) of Appendix C, and $L_k^{(\psi)}$ by Eq. (82).

We may conclude with the following remark. The proportionality of the Callan-Symanzik functions β and γ (see Section 5), as discussed in Ref. 4, implies that they have the same zeros. This means that, if $\beta(g)$ has a non-trivial zero $g = g^*$, $\gamma(g^*)$ also vanishes: there is no scale anomaly for $g = g^*$. From our discussion in Section 5, such a zero is a zero of the Zimmermann's coefficient c_k [see Eqs. (56)]. Thus we see from Eq. (85) that the anomaly attached to the supersymmetry of the second kind is also cancelled if $g = g^*$.

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APPENDIX A

FEYNMAN RULES

Feynman rules are read from the effective Lagrangian (27). We give first the vertices and inverse propagators, obtained by functional differentiation of

$$\Gamma^{(0)}\{A, B, \psi, F, G\} = i \int dx \mathcal{L}_{eff} \quad (A.1)$$

The result is^{*)}

$$\begin{aligned} \Gamma_{AAF}^{(0)} &= -\Gamma_{BBF}^{(0)} = \Gamma_{ABG}^{(0)} = 2ig_0 \\ \Gamma_{\psi\bar{\psi}A}^{(0)} &= 2g_0, \quad \Gamma_{\psi\bar{\psi}B}^{(0)} = -2g_0 \gamma_5 \end{aligned} \quad (A.2)$$

$$\begin{aligned} \Gamma_{\psi\bar{\psi}}^{(0)}(p, -p) &= i(\not{p} - im_0) \\ &= i(\not{p} - im) - ia\not{p} + \delta m \end{aligned} \quad (A.3)$$

$$\begin{bmatrix} \Gamma_{AA}^{(0)} & \Gamma_{AF}^{(0)} \\ \Gamma_{FA}^{(0)} & \Gamma_{FF}^{(0)} \end{bmatrix} = i \begin{bmatrix} -p^2 \mathbb{1} & m_0 \\ m_0 & \mathbb{1} \end{bmatrix} = i \begin{bmatrix} -p^2 & m \\ m & 1 \end{bmatrix} + i \begin{bmatrix} ap^2 & \delta m \\ \delta m & -a \end{bmatrix} \quad (A.4)$$

The two-point functions for the B and G fields are identical to those for the A and F fields. Each two-point function was separated into its free and interaction parts. Inversion of the free parts [with a minus sign for the scalar ones¹⁵⁾; moreover, note that we have a matrix inversion to perform] gives the free propagators

*) Upper index (0) means "trivial part" (i.e. no loop).

$$S_{\psi\bar{\psi}}^F(p, -p) = \frac{-i}{\not{p} - i m} = \frac{-i(\not{p} + i m)}{p^2 + m^2}$$

$$\begin{bmatrix} \Delta_{AA}^F & \Delta_{AF}^F \\ \Delta_{FA}^F & \Delta_{FF}^F \end{bmatrix} = \begin{bmatrix} \Delta_{BB}^F & \Delta_{BG}^F \\ \Delta_{GB}^F & \Delta_{GG}^F \end{bmatrix} = \frac{i}{p^2 + m^2} \begin{bmatrix} -1 & m \\ m & p^2 \end{bmatrix} \quad (\text{A.5})$$

In order to compute the coefficients of the Zimmermann's identities in Appendix C, we need the following contraction rules (amputated free Green's functions):

$$\langle T \phi_i(x) \bar{\phi}_j(p) \rangle_{\text{free}}^{\text{amp.}} \equiv (\Delta^F)^{-1}_{jk} \langle T \phi_i(x) \tilde{\phi}_k(p) \rangle_{\text{free}} = e^{-ipx} \delta_{ij}$$

where $\phi_i = A, B, F$ or G (A.6)

$$\langle T \psi_\alpha(x) \bar{\tilde{\psi}}_\beta(p) \rangle_{\text{free}}^{\text{amp.}} \equiv (S^F)^{-1}_{\beta\gamma} \langle T \psi_\alpha(x) \tilde{\bar{\psi}}_\gamma(p) \rangle_{\text{free}} = e^{-ipx} \delta_{\alpha\beta}$$

APPENDIX B

SUPERSYMMETRY WARD IDENTITIES

We use the notations, for the momenta and spinor indices:

$$\begin{aligned} P &= (p_1 \dots p_N) , \quad Q = (q_1 \dots q_I) , \quad R = (r_1 \dots r_J) \\ S &= (s_1 \dots s_K) , \quad T = (t_1 \dots t_L) , \quad \alpha = (\alpha_1 \dots \alpha_N) \\ P_{\hat{k}} &= (p_1 \dots \cancel{p_k} \dots p_N) , \quad \dots \end{aligned} \quad (B.1)$$

A general vertex function is written in momentum space as:

$$\Gamma_{\psi^N A^I B^J F^K G^L}^a (P; Q; R; S; T) \quad (B.2)$$

and the explicit form of the supersymmetry Ward identities (18) is

$$\begin{aligned} & -i \sum_{k=1}^N (-1)^{k+1} \left\{ \gamma_{\alpha\alpha_k}^0 \Gamma_{\psi^{N-1} A^{I+1} B^J F^K G^L}^{a_{\hat{k}}} (P_{\hat{k}}; p_k, Q; R; S; T) \right. \\ & \quad \left. + (\gamma_5 \gamma^0)_{\alpha\alpha_k} \Gamma_{\psi^{N-1} A^I B^{J+1} F^K G^L}^{a_{\hat{k}}} (P_{\hat{k}}; Q; p_k, R; S; T) \right\} \\ & + \sum_{k=1}^K \Gamma_{\psi^{N+1} A^I B^J F^{K-1} G^L}^{\alpha, a} (s_k, P; Q; R; S_{\hat{k}}; T) \\ & + \sum_{k=1}^L \gamma_{s\alpha\beta} \Gamma_{\psi^{N+1} A^I B^J F^K G^{L-1}}^{\beta, a} (t_k, P; Q; R; S; T_{\hat{k}}) \end{aligned}$$

$$\begin{aligned}
 & - \sum_{k=1}^N (-1)^{k+1} \left\{ (\not{x}_k \gamma^0)_{\alpha \alpha_k} \Gamma_{\psi^{N-1}}^{a_k} A^I B^J F^{K+1} G^L (P_k; Q; R; P_k, S; T) \right. \\
 & \quad \left. + (\gamma_5 \not{x}_k \gamma^0)_{\alpha \alpha_k} \Gamma_{\psi^{N-1}}^{a_k} A^I B^J F^K G^{L+1} (P_k; Q; R; S; P_k, T) \right\} \\
 & + i \sum_{k=1}^I \not{x}_{k\alpha\beta} \Gamma_{\psi^{N+1}}^{\beta, a} A^{I-1} B^J F^K G^L (q_k, P; Q_k; R; S; T) \\
 & - i \sum_{k=1}^J (\not{x}_k \gamma_5)_{\alpha\beta} \Gamma_{\psi^{N+1}}^{\beta, a} A^I B^{J-1} F^K G^L (r_k, P; Q; R_k; S; T) = 0
 \end{aligned} \tag{B.3}$$

The consequence for the two-point functions is that they are expressed in terms of two independent scalar functions $S_1(p^2)$ and $S_2(p^2)$:

$$\Gamma_{\psi\bar{\psi}}(p, -p) = i (\not{x} S_1(p^2) - i m S_2(p^2)) \tag{B.4}$$

(this defines S_1 and S_2)

and

$$\begin{aligned}
 \Gamma_{AA}(p^2) &= \Gamma_{BB}(p^2) = -p^2 \Gamma_{FF}(p^2) = -p^2 \Gamma_{GG}(p^2) = -i p^2 S_1(p^2) \\
 \Gamma_{AF}(p^2) &= \Gamma_{BG}(p^2) = i m S_2(p^2)
 \end{aligned} \tag{B.5}$$

Three-point functions with one scalar particle at zero momentum are expressed in terms of three independent scalar functions $\Gamma_1(p^2)$, $\Gamma_2(p^2)$ and $\Gamma_1^5(p^2)$:

$$\begin{aligned}
 \Gamma_{A\psi\bar{\psi}}(0, p, -p) &= 2g (\Gamma_1 + \not{x} \Gamma_2) \\
 \Gamma_{B\psi\bar{\psi}}(0, p, -p) &= -2g \gamma_5 \Gamma_1^5
 \end{aligned} \tag{B.6}$$

$$\Gamma_{AAF}(0, p, -p) = \Gamma_{ABG}(0, p, -p) = 2ig\Gamma_1$$

$$\Gamma_{BBF}(0, p, -p) = -\Gamma_{ABG}(p, 0, -p) = -2ig\Gamma_1^5$$

$$\Gamma_{AFF}(0, p, -p) = \Gamma_{AGG}(0, p, -p) = 2g\Gamma_2$$

$$\Gamma_{AAA}(0, p, -p) = \Gamma_{ABB}(0, p, -p) = -2gp^2\Gamma_2$$

$$\Gamma_{BFG}(0, p, -p) = \Gamma_{ABB}(p, -p, 0) = 0$$

(B.6)

One notes also that

$$\Gamma_{A\psi\bar{\psi}}(p, -p, 0) = \gamma_5 \Gamma_{B\psi\bar{\psi}}(-p, p, 0)$$

(B.7)

From the four-point functions, we retain only the zero-momentum identities

$$\left[\Gamma_{AAAF} = \Gamma_{AABG} = -\Gamma_{ABBF} = -\Gamma_{BBBG} \right.$$

(B.8)

$$\left. = \frac{i}{4} \text{Tr} \Gamma_{\psi\bar{\psi}AA} = \frac{i}{4} \text{Tr} \gamma_5 \Gamma_{\psi\bar{\psi}AB} = -\frac{i}{4} \text{Tr} \Gamma_{\psi\bar{\psi}BB} \right]_{p=0}$$

We add the following zero-momentum identity, true for any integers I, J:

$$\Gamma_{A^I B^J}(p=0) = 0$$

(B.9)

We also need the expressions for the renormalized propagators, obtained by inversion of vertex functions (B.5):

$$G_{\psi\bar{\psi}}(p, -p) = \frac{-i}{\not{p} S_1 - im S_2} = -i \frac{\not{p} S_1 + im S_2}{p^2 S_1^2 + m^2 S_2^2}$$

$$\begin{bmatrix} G_{AA} & G_{AF} \\ G_{FA} & G_{FF} \end{bmatrix} = \begin{bmatrix} G_{BB} & G_{BG} \\ G_{GB} & G_{GG} \end{bmatrix} = \frac{i}{p^2 S_1^2 + m^2 S_2^2} \begin{bmatrix} -S_1 & m S_2 \\ m S_2 & p^2 S_1 \end{bmatrix} \quad (\text{B.10})$$

ZIMMERMANN'S IDENTITIES

i) Proof of identity (50)

We have to compute

$$N_3 [\mathcal{L}_m] - N_4 [\mathcal{L}_m] = ? \quad (C.1)$$

$\mathcal{L}_m$ being given by Eq. (2). The right-hand side of Eq. (C.1) will contain^{6,7)} a superposition of all composite fields of dimension 4 which are scalars, with subtraction degree 4. Since we are interested in the space-time integrated Zimmermann's identity, we shall neglect all total derivative contributions. Thus

$$\begin{aligned} N_3 [\mathcal{L}_m](x) - N_4 [\mathcal{L}_m](x) = & N_4 [r_1 A \partial^2 B + r_2 B \partial^2 B \\ & + r_3 \bar{\Psi} \not{x} \Psi + r_4 F^2 + r_5 G^2 + s_1 A^2 F + s_2 B^2 F \\ & + s_3 ABG + s_4 A \bar{\Psi} \Psi + s_5 B \bar{\Psi} \not{x}_5 \Psi + t_1 A^4 + t_2 A^2 B^2 \\ & + t_3 B^4] (x) \quad + \text{total derivative} \end{aligned} \quad (C.2)$$

The reduction coefficients r_1 to t_3 are computed by applying to (C.2) the zero-momentum normalization conditions for normal products^{6,7)} inserted into N-point vertex functions

$$\begin{aligned} D^M \langle T N_s [\mathcal{O}](\omega) \tilde{\Phi}_1(p_1) \dots \tilde{\Phi}_N(p_N) \rangle^{1PI} \Big|_{p_i=0} \\ = \text{trivial contribution (i.e. without loop's graphs)} \end{aligned} \quad (C.3)$$

where D^M is a differential operator in the q_i 's of degree $M \leq \delta - N$. To evaluate the right-hand side of (C.3), we use the contraction rules (A.6). Thus, inserting Eq. (C.2) into the vertex function of fields $\tilde{A}(p) \tilde{A}(-p)$, for instance, and using (C.3) with $M = N = 2$, $\delta = 4$, yields

$$\langle T N_3 [\mathcal{L}_m](\omega) \tilde{A}(p) \tilde{A}(-p) \rangle^{1PI} = -2 r_1 p^2 + \text{loop's contributions ,}$$

so that

$$r_1 = -\frac{1}{16} \frac{\partial^2}{\partial p^2} \langle T N_3 [L_m](0) \tilde{A}(p) \tilde{A}(-p) \rangle^{PI} \Big|_{p=0} \quad (C.4)$$

Now, the linear equation of motion (47) for the A field allows us to write (C.4) as

$$r_1 = \frac{i}{32 g_0} \frac{\partial^2}{\partial p^2} \Gamma_{AAA}(0, p, -p) \Big|_{p=0}$$

In the same way, we find

$$r_2 = \frac{i}{32 g_0} \frac{\partial^2}{\partial p^2} \Gamma_{ABB}(0, p, -p) \Big|_{p=0}$$

$$r_3 = -\frac{1}{64 g_0} \frac{\partial}{\partial p^\mu} \text{Tr} \gamma^\mu \Gamma_{A\psi\bar{\psi}}(0, p, -p) \Big|_{p=0}$$

$$r_4 = -\frac{i}{4 g_0} \Gamma_{AFF}(0) \quad ; \quad r_5 = -\frac{i}{4 g_0} \Gamma_{AGG}(0)$$

$$s_1 = -\frac{i}{4 g_0} \Gamma_{AAAF}(0) \quad ; \quad s_2 = -\frac{i}{4 g_0} \Gamma_{ABBF}(0)$$

$$s_3 = -\frac{i}{2 g_0} \Gamma_{AABG}(0)$$

$$s_4 = -\frac{i}{16 g_0} \text{Tr} \Gamma_{AA\psi\bar{\psi}}(0) \quad ; \quad s_5 = \frac{i}{16 g_0} \text{Tr} \gamma_5 \Gamma_{AB\psi\bar{\psi}}(0)$$

$$t_1 = -\frac{i}{48 g_0} \Gamma_{A^5}(0)$$

$$t_2 = -\frac{i}{8 g_0} \Gamma_{A^3 B^2}(0) \quad ; \quad t_3 = -\frac{i}{48 g_0} \Gamma_{AB^4}(0) \quad (C.5)$$

The supersymmetry Ward identities (B.6), (B.8) and (B.9) imply, respectively,

$$\begin{aligned} r_1 &= r_2 = i r_3 = r_4 = r_5 \equiv \frac{1}{2} c_k \\ s_1 &= -s_2 = \frac{1}{2} s_3 = i s_4 = -i s_5 \equiv c_g \\ t_1 &= t_2 = t_3 = 0 \end{aligned} \quad (C.6)$$

with

$$\begin{aligned} c_k &= \frac{i}{16 g_0} \left. \frac{\partial^2}{\partial p^2} \Gamma_{AAA} (0, p, -p) \right|_{p=0} \\ c_g &= - \frac{i}{4 g_0} \Gamma_{AAAF} (0) \end{aligned} \quad (C.7)$$

Hence, Eq. (C.2) becomes

$$N_3[\mathcal{L}_m] - N_4[\mathcal{L}_m] = N_4[c_k \mathcal{L}_k + c_g \mathcal{L}_g] + \text{total derivatives} \quad (C.8)$$

with $\mathcal{L}_m$ and $\mathcal{L}_g$ given by Eq. (2). Space-time integration yields the desired result (50).

ii) Proof of identity (60)

We have now to compute

$$N_3[S_B] - N_4[S_B] = ? \quad (C.9)$$

S_B being given by Eqs. (13). The right-hand side of (C.9) will contain all composite fields of dimension 4 and subtraction degree 4 which are pseudoscalars, as S_B is. Thus

$$\begin{aligned} N_3[S_B] - N_4[S_B] &= N_4 \left[u_1 A \partial^2 B + u_2 B \partial^2 A + u_3 \partial A \cdot \partial B \right. \\ &\quad \left. + u_4 FG + v_1 A^2 G + v_2 B^2 G + v_3 ABF + \right. \end{aligned} \quad (C.10)$$

$$+ v_4 B \bar{\Psi} \Psi + v_5 A \bar{\Psi} \gamma_5 \Psi + w_1 A^3 B + w_2 A B^3$$

+ total derivatives

(C.10)

($\bar{\Psi} \gamma_5 \not{\partial} \Psi$ has not been included, since it is a total derivative.)

The same analysis as before, including the equation of motion for the B field, yields

$$u_1 = \frac{i}{16 g_0} \frac{\partial^2}{\partial q^2} \Gamma_{ABB} (0, q, -q) \Big|_{q=0}$$

$$u_2 = \frac{i}{16 g_0} \frac{\partial^2}{\partial p^2} \Gamma_{ABB} (p, -p, 0) \Big|_{p=0}$$

$$u_3 = \frac{i}{8 g_0} \frac{\partial^2}{\partial p \partial q} \Gamma_{ABB} (p, q, -p-q) \Big|_{p=q=0}$$

$$u_4 = - \frac{i}{2 g_0} \Gamma_{BFG} (0) \quad ; \quad v_1 = - \frac{i}{4 g_0} \Gamma_{AABG} (0)$$

$$v_2 = - \frac{i}{4 g_0} \Gamma_{BBBBG} (0) \quad ; \quad v_3 = - \frac{i}{2 g_0} \Gamma_{ABBF} (0)$$

$$v_4 = - \frac{i}{16 g_0} \text{Tr} \Gamma_{BB\Psi\bar{\Psi}} (0) \quad ; \quad v_5 = \frac{i}{16 g_0} \text{Tr} \gamma_5 \Gamma_{AB\Psi\bar{\Psi}} (0)$$

$$w_1 = - \frac{i}{12 g_0} \Gamma_{A^3 B^2} (0) \quad ; \quad w_2 = - \frac{i}{12 g_0} \Gamma_{AB^4} (0)$$

(C.11)

Bose symmetry implies [see argument preceding Eq. (76)] $u_1 = u_3$. Supersymmetry Ward identities (B.6), (B.8) and (B.9) lead to the relations

$$\begin{aligned}
 u_1 &= u_3 = c_k, & u_2 &= u_4 = 0 \\
 v_1 &= -v_2 = -\frac{1}{2} v_3 = -i v_4 = -i v_5 = c_g \\
 w_1 &= w_2 = 0
 \end{aligned}
 \tag{C.12}$$

where c_g and c_k are given by Eqs. (C.7). Hence

$$\begin{aligned}
 N_3[S_B] - N_4[S_B] &= c_g N_4[(A^2 - B^2)G - 2ABF + i\bar{\Psi}(B + \gamma_5 A)\Psi] \\
 &\quad + \text{total derivatives}
 \end{aligned}
 \tag{C.13}$$

Integration gives the identity (60).

APPENDIX D

In the notation of Ref. 10, a superfield^{*)}

$$\phi^+(x, \theta) = A^+(x) + \theta^\alpha \psi_\alpha^+(x) + \theta^\alpha \theta_\alpha F^+(x) \quad (D.1)$$

is a function of the space-time point x and of a two-component complex spinor $\theta = (\theta^1, \theta^2)$ whose components are anticommuting numbers, as well as those of ψ^+ . The complex conjugate of (5.1) is

$$\phi^-(x, \bar{\theta}) = A^-(x) + \bar{\theta}_{\dot{\alpha}} \psi^{-\dot{\alpha}}(x) + \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} F^-(x) \quad (D.1')$$

where $\bar{\theta}_{\dot{\alpha}} = (\theta_\alpha)^*$, $\psi^{-\dot{\alpha}} = (\psi^+{}^\alpha)^*$

lowering and rising of spinor indices (dotted or undotted) are performed by multiplying (from left) by the antisymmetric tensors $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta}$ ($\epsilon^{12} = 1$); for instance, $(\sigma_u)_\alpha{}^{\dot{\alpha}} = \epsilon^{\alpha\dot{\beta}} (\sigma_u)_{\dot{\beta}\alpha}$, where $(\sigma_u)_{\alpha\dot{\beta}} = (1, \sigma_i)$, σ_i ($i = 1, 2, 3$) being the usual Hermitian Pauli matrices.

The fields appearing in the definitions (D.1) and (D.1') are related to the fields of the model studied in the first sections by the relations

$$A^\pm = \frac{1}{2} (A \mp iB), \quad F^\pm = \frac{1}{2} (F \pm iG)$$

$$\text{and} \quad \begin{bmatrix} \psi_\alpha^+ \\ \psi^{-\dot{\alpha}} \end{bmatrix} = \psi \quad (D.2)$$

The notation used is found in Van der Waerden²⁴⁾. The 4×4 Dirac matrices are

$$\gamma_\mu = \begin{bmatrix} 0 & i(\sigma_\mu)_{\alpha\dot{\alpha}} \\ i(\sigma_\mu^\tau)^{\dot{\beta}\beta} & 0 \end{bmatrix}, \quad \gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \quad (D.3)$$

and the following correspondences between two-component and four component Majorana spinors hold:

*) We restrict ourselves to the so-called "chiral" superfields.

$$\begin{aligned}
 \psi^+ &\longrightarrow \psi && [\text{as in (D.2)}] \\
 i \psi^+ &\longrightarrow \gamma_5 \psi \\
 i \sigma^\mu \psi^- &\longrightarrow \gamma^\mu \psi \\
 \sigma^\mu \psi^- &\longrightarrow -\gamma_5 \gamma^\mu \psi
 \end{aligned}
 \tag{D.4}$$

One can also check that

$$\begin{aligned}
 \text{Re}(\phi^+ \psi^+) &= \frac{i}{2} \bar{\phi} \psi \\
 \text{Im}(\phi^+ \psi^+) &= -\frac{i}{2} \bar{\phi} \gamma_5 \psi \\
 \text{Re}(i \phi^+ \sigma^\mu \psi^-) &= \frac{i}{2} \bar{\phi} \gamma^\mu \psi \\
 \text{Im}(i \phi^+ \sigma^\mu \psi^-) &= -\frac{i}{2} \bar{\phi} \gamma_5 \gamma^\mu \psi
 \end{aligned}
 \tag{D.5}$$

It may be useful to note the following identities, too:

$$\begin{aligned}
 \phi^+ \psi^+ &\equiv \phi^{+\alpha} \psi^+_{\alpha} = -\phi^+_{\alpha} \psi^{+\alpha} \\
 \phi^- \psi^- &\equiv \phi^{-\dot{\alpha}} \psi^{-\dot{\alpha}} = -\phi^{-\dot{\alpha}} \psi^{-\dot{\alpha}} = (\phi^+ \psi^+)^* \\
 (\theta \psi^+) (\theta \phi^+) &= -\frac{1}{2} (\theta\theta) (\psi^+ \phi^+) \\
 (\theta \sigma^\mu \bar{\theta}) (\theta \sigma^\nu \bar{\theta}) &= -\frac{1}{2} g^{\mu\nu} (\theta\theta) (\bar{\theta}\bar{\theta})
 \end{aligned}
 \tag{D.6}$$

The superfield (D.1), called a "chiral" superfield [there exist more general superfields $\phi(x, \theta, \bar{\theta})$], behaves under a supersymmetry transformation like

$$\delta \phi^+ = \left[\left\{ \frac{\partial}{\partial \theta} + 2i \theta \sigma^\mu \bar{\theta} \right\} \partial_\mu \right] \phi^+
 \tag{D.7}$$

The infinitesimal transformation (D.7) is identical, when projected into components, to the transformations (3); the two-component complex anticommuting infinitesimal parameter ξ corresponds to the Majorana four-component spinor α of Eq. (3).

The complex conjugate $\phi^-(x, \bar{\theta})$ is not chiral, but

$$\frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \bar{\theta}} \left[\phi^-(x - 2i\theta\sigma\bar{\theta}, \bar{\theta}) \right]^n \quad \text{for any } n, \quad (\text{D.8})$$

is chiral. Moreover, any product of chiral superfields is chiral.

There exists one chiral composite superfield of dimension 2, and three of dimension 3 ($\dim \phi = \dim A = 1$; $\dim \theta = -\frac{1}{2}$):

$$\begin{aligned} L_m &= \left[\phi^+(x, \theta) \right]^2 \\ L_g &= \frac{4}{3} \left[\phi^+(x, \theta) \right]^3 \\ L_k &= -\frac{1}{4} \phi^+(x, \theta) \frac{\partial}{\partial \bar{\theta}^\alpha} \frac{\partial}{\partial \bar{\theta}^\alpha} \phi^-(x - 2i\theta\sigma\bar{\theta}, \bar{\theta}) \\ L'_k &= -\frac{1}{4} \frac{\partial}{\partial \bar{\theta}^\alpha} \frac{\partial}{\partial \bar{\theta}^\alpha} \left[\phi^-(x - 2i\theta\sigma\bar{\theta}, \bar{\theta}) \right]^2 \end{aligned} \quad (\text{D.9})$$

Their A, B, ψ , F and G components, defined as in (D.1) and (D.2) are, respectively:

$$L_m = \begin{bmatrix} \frac{1}{2} (A^2 - B^2) \\ AB \\ (A - \gamma_5 B) \psi \\ AF + BG - \frac{i}{2} \bar{\psi} \psi \\ AG - BF + \frac{i}{2} \bar{\psi} \gamma_5 \psi \end{bmatrix} \quad (\text{D.10a})$$

$$L_g = \begin{bmatrix} \frac{1}{3} A (A^2 - 3B^2) \\ -\frac{1}{3} B (B^2 - 3A^2) \\ (A - \gamma_5 B)^2 \psi \\ (A^2 - B^2)F + 2ABG - i \bar{\psi} (A - \gamma_5 B) \psi \\ (A^2 - B^2)G - 2ABF + i \bar{\psi} (B + \gamma_5 A) \psi \end{bmatrix} \quad (D.10b)$$

$$L_k = \frac{1}{2} \begin{bmatrix} AF - BG \\ AG + BF \\ (A - \gamma_5 B) \not{\partial} \psi + (F - \gamma_5 G) \psi \\ A \not{\partial}^2 A + B \not{\partial}^2 B - i \bar{\psi} \not{\partial} \psi + F^2 + G^2 \\ \not{\partial}_\mu (A \not{\partial}^\mu B - B \not{\partial}^\mu A + \frac{i}{2} \bar{\psi} \gamma^5 \gamma^\mu \psi) \end{bmatrix} \quad (D.10c)$$

$$L'_k = \begin{bmatrix} AF + BG - \frac{i}{2} \bar{\psi} \psi \\ AG - BF + \frac{i}{2} \bar{\psi} \gamma_5 \psi \\ \not{\partial} [(A - \gamma_5 B) \psi] \\ \frac{1}{2} \not{\partial}^2 (A^2 - B^2) \\ \not{\partial}^2 (AB) \end{bmatrix} \quad (D.10d)$$

Note that the components of L_m are nothing else than the interaction terms S_F , S_G , S_ψ , S_A and S_B [Eq. (13)] of the equations of motion (12).

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