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ASYMPTOTIC CONSTRAINTS ON $\sigma_{e^+e^-}^{\text{total}}$
IN ASYMPTOTICALLY FREE THEORIES

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A B S T R A C T

Using the analyticity properties of the vacuum polarization function we obtain conditions on the e^-e^+ total cross-section from the behaviour of this function for spacelike directions, where it is known in asymptotically free field theories. This work constitutes a generalization of the results of J.B. Healy.

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I. INTRODUCTION

Recently, Healy ¹⁾ has discussed the connection between the behaviour of the vacuum polarization function in asymptotically free theories ²⁾ and the behaviour of the total cross-section $e^+e^- \rightarrow \text{hadrons}$. Here we want to re-examine the problem and show that most of the results on the e^+e^- cross-section can be obtained from the spacelike behaviour of the vacuum polarization function $\Pi(s)$, i.e., from the behaviour for $s \rightarrow -\infty$. Indeed, strictly speaking, it is only for $s \rightarrow -\infty$ that the renormalization group techniques allow to make predictions for the vacuum polarization functions. Our contribution, as compared to Healy's very interesting paper, consists essentially in reducing still further (and sometimes eliminating them completely) the assumptions about the timelike region.

Let us summarize now the basic ingredients of the problem. The vacuum polarization function $\Pi(s)$ can be written

$$\Pi(s) = \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dx}{x} \frac{\rho(x)}{(x-s+i\epsilon)} \quad (1)$$

and the positive weight function ρ , in the one photon approximation, is related to the $e^+e^- \rightarrow \text{hadrons}$ cross-section by

$$\rho(s) = \frac{s}{32\pi\alpha^2} \sigma(s) \quad e^+e^- \rightarrow \text{hadrons} \quad (2)$$

The problem of obtaining information on $\rho(x)$ from the behaviour of $\Pi(s)$ for $s \rightarrow -\infty$ is very reminiscent of other problems in dispersion theory. The fact that information about the behaviour of $\Pi(s)$ for $s \rightarrow +\infty$ is not necessary may be linked to some results previously obtained by the authors ³⁾. If a function is analytic in a cut plane and has a bounded phase for $s \rightarrow +\infty$ and $s \rightarrow -\infty$ then in an averaged sense the ratio of its moduli for s and $-s$ approaches unity for $s \rightarrow +\infty$. Here the phase on the right-hand side lies between 0 and π and stays equal to π on the left-hand side ³⁾. Hence, in an averaged sense $\lim_{s \rightarrow \infty} |\Pi(s)/\Pi(-s)| = 1$.

II. LEADING BEHAVIOUR

1) Our first general result is the following :

Theorem I If for spacelike asymptotic values the solution of Eq. (1) is such that

$$i) \quad |\pi(-s) + a \log s| = o(\log s) \quad s \rightarrow +\infty \quad (3)$$

and if

ii) $\rho(x)$ is a continuous function

then either $\lim_{s \rightarrow \infty} \rho(s) = a\pi$ or there exists at least one sequence $\{s_n\}_{n \rightarrow \infty}$ such that $\rho(s_n) - a\pi \rightarrow 0$.

At the beginning, we consider only the assumption i), which is sufficient to show the following result : for any ϵ small but fixed, $0 < \epsilon < a$, and s large, either $(\rho(s)/\pi) > a + \epsilon$ or $(\rho(s)/\pi) < a - \epsilon$ are impossible. We define

$$A(s) = \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\rho(x) dx}{x(x+s)} = -\pi f(s) \quad (4)$$

Assuming $\rho(x) > \pi(a + \epsilon)$ and taking into account this inequality in Eq. (4) we get for $s > 0$ and s large,

$$A(s) - a \log s > \epsilon \log s \quad (5)$$

which contradicts our assumption i). It is easy to realize that the contradiction still remains if only for $x > x_0$, (x_0 finite) we have $\rho(x) > \pi(a + \epsilon)$.

Similarly if $\rho(x) < \pi(a - \epsilon)$, we get from Eq. (4)

$$a \log s - A(s) > \epsilon \log s \quad (6)$$

which again contradicts our assumption i).

If we now add the assumption ii), saying that ρ is continuous, then we get the result of Theorem I.

It is remarkable that in this type of proof we have not used the positivity of ρ and this is why it could still work for the non-leading term as we shall see in the next section.

Of course, if we assume that $\lim_{s \rightarrow \infty} \rho(s)$ exists, this limit is necessarily $a\pi$. However, the assumption that $\lim_{s \rightarrow \infty} \rho(s)$ exists is not necessary in order that $(\rho(s) - a\pi) \rightarrow 0$ for at least one sequence $(s_n) \rightarrow \infty$. Theorem I shows also that no explicit assumptions about the existence of the principal part of Eq. (1) when $s \rightarrow \infty$ is really necessary. Finally it shows that no assumption about the derivative $\partial\rho/\partial x$ is necessary in order that $\rho \rightarrow a\pi$.

2) Secondly, in addition to the above result, we would like to give sum rules which, from finite up to higher or infinite energy values, constrain the behaviour of $\Pi(s)$ for spacelike regions and that of $\rho(s)$ for $s > 0$. The proof is given in Appendix A.

Theorem II Let us assume :

i) $\rho > 0$, $\rho < \text{const} (\log s)^q$, q positive, arbitrarily large.

ii) $A(s) = |\Pi(-s)| < \text{const} \log s$ for $s > s_0$.

If i) and ii) hold, then

$$\left| \int_{\text{const.}}^R \frac{ds}{s \log s \log \log s} \left(\frac{\rho(s)}{\pi} - \frac{A(s)}{\log s} \right) \right| < \text{const}, \quad (7)$$

R arbitrarily large;

$$\int_{\text{const.}}^{\infty} \frac{ds}{s \log s (\log \log s) \gamma(s)} \left(\frac{\rho(s)}{\pi} - \frac{A(s)}{\log s} \right) \quad (8)$$

is convergent, where $\gamma(s)$ is any increasing function tending to ∞ .

Inequalities (7) and (8) imply Theorem I. They constitute, however, a more accurate statement. The technical assumption $\rho(x) < (\log x)^q$, where q may be very large, seems to us rather harmless.

III. NON-LEADING BEHAVIOUR

In this section and in the following we always assume that the dominant term in the spacelike region for $\Pi(s)$ satisfies the condition (3).

$|\Pi(-s) + a \log s| = o(\log s)$, $s > 0$ and s large. From Eq. (1), defining $\tilde{\rho}$ by $\rho = a\pi + \tilde{\rho}$, we get

$$\tilde{\Pi}(s) = \Pi(s) + a \log\left(\frac{4-s}{4}\right) = \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{\tilde{\rho}(x) dx}{x(s-x+i\epsilon)} \quad (1')$$

We are seeking the supplementary information that one can get for ρ (or $\tilde{\rho}$) when the non-leading term for $\Pi(s)$ [or leading one for $\tilde{\Pi}(s)$] is known in the spacelike region. If we know the dominant term for $\tilde{\Pi}(s)$ when $s \rightarrow -\infty$ or at least a bound of its modulus, we are led for $\tilde{\rho}$ to a problem similar to the previous section for ρ but where $\tilde{\rho}$ is not necessarily positive.

1) Firstly we would like to get a result similar to Theorem I, where we recall that the positivity of ρ was not taken into account.

Theorem I' Let us assume :

$$i) \quad |\tilde{\Pi}(-s) + b \log \log s| = o(\log \log s), \quad s \rightarrow +\infty \quad (3')$$

ii) $\rho(s)$ is a continuous function, then either $\lim_{s \rightarrow \infty} \log s \tilde{\rho}(s) = b\pi$ or there exists at least a sequence $\{s_n\}_{n \rightarrow \infty}$ such that $\tilde{\rho}(s_n) \log s_n - b\pi \rightarrow 0$.

The proof is the same as that of Theorem I. Let us define

$$\tilde{A}(s) = \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{\tilde{\rho}(x) dx}{x(x+s)} \quad (4')$$

$$\text{If } \tilde{\rho}(x) > \frac{\pi(b+\epsilon)}{\log x}, \quad 0 < \epsilon < b,$$

one gets

$$\tilde{A}(s) - b \log \log s > \epsilon \log \log s \quad (5')$$

which contradicts the assumption (3'). Similarly if

$$\tilde{\rho}(x) < \pi \frac{(b-\epsilon)}{\log x}$$

then

$$b \log \log s - \tilde{A}(s) > \epsilon \log \log s \quad (6')$$

which again contradicts assumption (3').

If we now add assumption ii) saying that ρ is continuous, then we get Theorem I'.

Let us remark that here also if in addition to the assumptions i) and ii) of Theorem I' we add that $\lim_{s \rightarrow \infty} \tilde{\rho}(s) \log s$ exists then we get that the limit is necessarily $b\pi$.

2) Secondly we would like to give a theorem similar to Theorem II, but now because $\tilde{\rho}$ is not necessarily positive, the assumption about $\tilde{\rho}$ (or ρ) is stronger than the corresponding one of Theorem II. The proof is again established in Appendix B.

Theorem II' Let us assume :

- i) $\rho(x) < \text{const} \log \log x$ or equivalently. $-\pi < \tilde{\rho} < \text{const} \log \log s$;
- ii) $\tilde{A}(s) = |\tilde{\Pi}(-s)| < \text{const} \log \log s$ for $s > 0$ and large.

If i) and ii) hold, then

$$\left| \int_{\text{const.}}^R \frac{dx}{x \log \log x} \left(\frac{\tilde{\rho}(x)}{\pi} + \frac{\tilde{\Pi}(-x)}{\log x \log \log x} \right) dx \right| < \text{const.} \quad (7')$$

$$\int_{\text{const.}}^{\infty} \frac{dx}{x \log \log x \delta(x)} \left(\frac{\tilde{\rho}(x)}{\pi} + \frac{\tilde{\Pi}(-x)}{\log x \log \log x} \right) dx \quad (8')$$

is convergent.

where again $\gamma(x)$ is any increasing function increasing to ∞ . We still get that either

$$\frac{\tilde{\rho}(x)}{\pi} + \frac{\tilde{\Pi}(-x)}{\log x \log \log x} \rightarrow 0 \quad x \rightarrow \infty$$

or there exists at least one sequence $(x_n)_{n \rightarrow \infty}$ such that :

$$\frac{\tilde{\rho}(x_n)}{\pi} + \frac{\tilde{\Pi}(-x_n)}{\log x_n \log \log x_n} \rightarrow 0$$

IV. DERIVATIVE $[\partial \rho(s)/\partial s]$ OF THE SPECTRAL FUNCTION

In this section we would like to obtain for $\partial \rho/\partial x$ properties similar to those obtained in Theorems I and I' for ρ and $\tilde{\rho}$. We assume that in the spacelike region, we know the leading and the non-leading terms of $d\Pi/ds$. Furthermore, we assume that in the spacelike region, the leading term of $d\Pi/ds$ is the derivative of the leading term of $\Pi(s)$. We have also to assume that $\lim_{s \rightarrow \infty} \rho(s)$ exists (which is necessarily ∞ from Theorem I).

Let us define

$$\frac{\partial A}{\partial s} = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\rho}{(x+s)^2} dx \quad (9)$$

Theorem I'' Let us assume

- i) $|\Pi(-s) + a \log s| \underset{s \rightarrow \infty}{=} o(\log s)$, ρ is continuous and $\lim_{s \rightarrow \infty} \rho(s)$ exists;
- ii) $\partial \rho/\partial s$ exists,

$$\left| \frac{\partial A}{\partial s} - \frac{a}{s} - \frac{b}{s \log s} \right| = o\left(\frac{1}{s \log s}\right) \quad (10)$$

- iii) $\partial \rho/\partial s$ is continuous, then either

$$\lim_{s \rightarrow \infty} s (\log s)^2 \frac{\partial \rho}{\partial s} = -b\pi$$

or there exists at least one sequence $(s_n)_{n \rightarrow \infty}$ such that

$$s_n (\log s_n)^2 \left(\frac{\partial \rho}{\partial s} \right)_{s=s_n} + \pi b \xrightarrow{n \rightarrow \infty} 0$$

From i) and Theorem I, we see that $\lim_{s \rightarrow \infty} \rho(s) = a\pi$. Integrating by part Eq. (9) and taking into account (10), we get

$$\left| \frac{1}{\pi} \int_{4m_n^2}^{\infty} \frac{s}{x+s} \left(\frac{\partial \rho}{\partial x} \right) dx - \left(a - \frac{\rho(4)}{\pi} \right) - \frac{b}{\log s} \right| = o\left(\frac{1}{\log s}\right) \quad (11)$$

Let us define

$$B(s) = \frac{1}{\pi} \int_{4m_n^2}^{\infty} x \frac{\partial \rho}{\partial x} \frac{dx}{x+s} \quad (12)$$

then the first term in the left-hand side of Eq. (11) can be written

$$\frac{1}{\pi} \int_{4m_n^2}^{\infty} \frac{\partial \rho}{\partial x} dx = B(x)$$

from which follows that Eq. (11) is equivalent to

$$\left| B(s) + \frac{b}{\log s} \right| = o\left(\frac{1}{\log s}\right) \quad (3'')$$

We are thus led for $\partial \rho / \partial x$ [see Eq. (12)] to condition (3'') similar to condition (3) [or (3')] for ρ (or $\tilde{\rho}$).

Let us assume $-x(\log x)^2 \frac{\partial \rho}{\partial x} > (b+\epsilon)\pi$, $0 < \epsilon < b$ then one gets

$$-B(s) - \frac{b}{\log s} > \frac{\epsilon}{\log s}$$

which contradicts the condition (3''). Similarly, if $-x(\log x)^2 \frac{\partial \rho}{\partial x} < (b-\epsilon)\pi$ one gets

$$B(s) + \frac{b}{\log s} > \frac{\epsilon}{\log s}$$

which again contradicts Eq. (3'').

Finally, if we add condition iii) saying that $\partial\rho/\partial s$ is continuous, then we obtain the result of Theorem I".

Let us remark again, similarly to the previous cases for ρ and $\tilde{\rho}$, that here if in addition to the condition of Theorem I" we require that $\lim_{s \rightarrow \infty} s(\log s)^2 \frac{\partial\rho}{\partial s}$ exists, then the limit is necessarily $-\pi b$.

V. CONCLUSION

In this paper we have shown that even if we do not assume anything in the timelike region, then from known leading and non-leading terms of the vacuum polarization function in the spacelike region, one can deduce some general constraints on the behaviour of the spectral function, and therefore on the total $e^+e^- \rightarrow \text{hadron}$ cross-section, at least on one infinite sequence of values. Furthermore, at the price of a very weak technical assumption on the maximum possible growth of the spectral function, we derive sum rules which constraint spacelike and timelike terms from finite energy up to infinity. Finally, assuming that the spectral function has asymptotically a limit, general constraints on its derivative (if it exists) must hold at least on an infinite sequence. Obviously all these results improve those of Ref. 1).

However, as pointed out by Healy ¹⁾, one cannot exclude oscillations of ρ or $\tilde{\rho}$ according to the case. In this respect we have made essentially no progress with respect to Ref. 1), except for the fact that the oscillations, if any, must be built in such a way that they satisfy the sum rule. Apart from that, one can only exclude these oscillations by saying that "nature cannot be so unkind" (Goldberger's theorem).

This being said, should we share the pessimistic views of Healy ? It is a matter of personal opinion. However, we should realize that this difficulty is almost always present in problems of dispersion theory. The most famous example is the Pomeranchuk theorem where one proves, under certain assumptions, that among the limiting values of $\sigma_{\text{particle}} - \sigma_{\text{antiparticle}}$ is the value zero. However, one cannot exclude oscillations. Similarly, the theorem which says that for cross-sections approaching a non-zero limit at infinite energy, or tending slowly to infinity, the even signature amplitude has a small real part, holds only in an average sense. The real part may be large at isolated points. So we are encountering a "fact of life" in dispersion theory. Our personal opinion is that while we should be aware of this limitation, we should not consider the information obtained from dispersion theory as negligible for that reason.

APPENDIX A

We want to prove Theorem II but before that we would like to establish some general result. Let us consider

$$\pi(s) = \frac{s}{\pi} \int_a^{\infty} \frac{\rho(x) dx}{x(x-s)}, \quad \rho > 0, a > 0 \quad (\text{A.1})$$

and define for $s < 0$ values

$$A(R) = -\pi(-R) = \frac{R}{\pi} \int_a^{\infty} \frac{\rho(x) dx}{x(x+R)}, \quad R > 0 \quad (\text{A.2})$$

Our aim is to get asymptotic upper bounds in the complex $s = Re^{i\theta}$ plane outside the cut $\theta = 0$ when we have some information about $A(R)$ and $\sup \rho(R)$. From (A.1) we get

$$|\pi(Re^{i\theta})| < \frac{R\sqrt{2}}{\pi} \int_a^{\infty} \frac{\rho dx}{x(|x-u|+v)}, \quad u = R \cos \theta, v = R |\sin \theta| \quad (\text{A.3})$$

1. First, we want to show

$$\text{i) } |\theta| > \epsilon, \quad |\pi(Re^{i\theta})| < \text{const } A(R), \quad \epsilon \text{ finite and small} \quad (\text{A.4})$$

$$\text{ii) } |\theta| < \epsilon, \quad |\pi(Re^{i\theta})| < \text{const. } \frac{A(R)}{|\sin \theta|} \quad (\text{A.5})$$

$$|\pi(Re^{i\theta})| < \text{const. } A(R) + \text{const. } \sup \rho(x) |\log |\sin \theta||, \quad 0 < \lambda < 1 \\ \times [R(1-\lambda), R(1+\lambda)] \quad (\text{A.6})$$

$$\int_0^{\vartheta_0(R)} |\pi(Re^{i\theta})| d\theta < \sup \rho(x) \text{const. } \vartheta_0(R) |\log \vartheta_0(R)| + \text{const } A(R) \vartheta_0(R) \\ \vartheta_0(R) \rightarrow 0 \quad \times [R(1-\lambda), R(1+\lambda)] \quad R \rightarrow \infty \quad (\text{A.7})$$

$$\int_{\vartheta_0(R)}^{\vartheta \text{ finite}} |\pi(Re^{i\theta})| d\theta < \text{const. } A(R) |\log |\vartheta_0(R)|| \quad (\text{A.8})$$

a) We consider $|\theta| > \epsilon$. From (A.3) one gets

$$|\pi(R, \theta)| < R \int_a^{\infty} \frac{\rho B(x, R, \theta) dx}{x(x+R)}$$

where B (when ϵ is fixed) is bounded by a finite constant independently of x and R . The result (A.4) follows.

b) We consider $|\theta| < \epsilon$. We separate the integration path in (A.3) in three parts $[a, u(1-\lambda)]$, $[u(1-\lambda), u(1+\lambda)]$, $[u(1+\lambda), \infty]$, $0 < \lambda < 1$ and we get :

$$|\pi| < B_1 + B_2 + B_3$$

$$B_1 = \frac{\sqrt{2}}{\pi} R \int_a^{u(1-\lambda)} dx \frac{\rho}{x(x+R)} \left(\frac{x+R}{u-x+v} \right) < \frac{\sqrt{2}}{\pi} \left(\frac{1+\cos\theta(1-\lambda)}{\lambda\cos\theta+|\sin\theta|} \right) \int_a^{u(1-\lambda)} \frac{R \rho dx}{x(x+R)}$$

$$B_1 < \text{const.} \frac{A(R)}{|\theta|}, \quad B_1 < \text{const.} \frac{A(R)}{\lambda} \quad (\text{A.9})$$

$$B_2 = \frac{\sqrt{2}}{\pi} R \int_{u(1+\lambda)}^{\infty} dx \frac{\rho}{x(x+R)} \left(\frac{x+R}{x-u+v} \right) < \frac{\sqrt{2}}{\pi} \left(\frac{1+\cos\theta(1+\lambda)}{\lambda\cos\theta+|\sin\theta|} \right) \int_{u(1+\lambda)}^{\infty} \frac{R \rho dx}{x(x+R)}$$

$$B_2 < \text{const.} \frac{A(R)}{|\theta|}, \quad B_2 < \text{const.} \frac{A(R)}{\lambda} \quad (\text{A.10})$$

$$B_3 = \frac{\sqrt{2}}{\pi} R \int_{u(1-\lambda)}^{u(1+\lambda)} \frac{\rho dx}{x(1u-x|+v)} < \text{const.} \sup_{\frac{x}{R} \in [1-\lambda, 1+\lambda]} R \frac{\rho(x)}{x} \int_{u(1-\lambda)}^{u(1+\lambda)} \frac{dx}{(1u-v|+x)}$$

$$B_3 < \text{const.} \sup_{\frac{x}{R} \in [1-\lambda, 1+\lambda]} \rho(x) \left| \log \left| \frac{\lambda+|\theta|}{|\theta|} \right| \right| \quad (\text{A.11})$$

$$B_3 < \frac{\text{const.}}{|\theta|} \int_{u(1-\lambda)}^{u(1+\lambda)} \frac{dx}{x} \rho < \text{const.} \frac{A(R)}{|\theta|}$$

From (A.9)-(A.11) we get the two bounds (A.5) and (A.6) and by integration, the two bounds (A.7) and (A.8), (λ being a finite constant).

Finally, let us define

$$T(R) = \int_0^{2\pi} |\Pi(R e^{i\theta})| d\theta \quad (A.12)$$

which is a very convenient quantity when we write a Cauchy relation and try to estimate a bound on the contribution along the great circle of radius R . From (A.4), (A.7) and (A.8) we get

$$T(R) < \text{const. } A(R) |\log |\vartheta_0(R)|| + \text{const. } |\vartheta_0(R)| \log |\vartheta_0| \sup \rho(x) \quad (A.13) \\ x \in [R(1-\lambda), R(1+\lambda)]$$

(A.13) is our main result, $\vartheta_0(R) \rightarrow 0$ can be chosen at our convenience in order to get the best bound and even can be taken as a constant.

2. We want to give some estimates of $\Gamma(R)$ when we have at our disposal upper bounds for $A(R)$ and $\sup \rho(R)$.

a) Let us assume for instance that the bound on $\sup \rho(R)$ grows faster than that of $A(R)$. Then we can choose

$$\left(\begin{array}{l} \vartheta_0 \approx \frac{1}{\sup \rho(R)} \\ T(R) < \text{const. } A(R) |\log |\vartheta_0|| \end{array} \right. \quad (A.14)$$

For instance if

$$\left(\begin{array}{l} \sup \rho(R) < \text{const. } R^p, \quad \vartheta_0 \approx R^{-p} \text{const.} \\ T < \text{const. } A(R) \log R \end{array} \right. \quad (A.15)$$

For instance if

$$\left(\begin{array}{l} \sup \rho(R) < \text{const. } (\log R)^p, \quad \vartheta_0 \approx \text{const. } (\log R)^{-p} \\ T < \text{const. } A(R) \log \log R \end{array} \right. \quad (A.16)$$

b) If the bound on $\sup \rho(R)$ does not grow faster than that of $A(R)$, we can choose

$$\left(\begin{array}{l} \vartheta_0 \approx \text{const.} \\ T < \text{const. } A(R) \end{array} \right. \quad (A.17)$$

3. We now come back to Theorem II where $A(R) < \text{const } \log R$:

$$\text{if } \rho(R) < R^P \rightarrow \Gamma < \text{const}(\log R)^2;$$

$$\text{if } \rho(R) < (\log R)^P \rightarrow \Gamma < \text{const}(\log R) \log \log R;$$

$$\text{if } \rho(R) < \log R \rightarrow \Gamma < \text{const } \log R.$$

a) We shall write sum rules when $A(R) < \text{const } \log R$, $\rho(R) < (\log R)^P$ and hence, $\Gamma < \text{const}(\log R) \log \log R$. In this case we consider the function $\psi(s) = [\Pi(s)/\log s \log \log s]$. We take for the logarithmic terms a cut along the negative real axis from 0 to $-\infty$ and we choose $\log s$, $\log \log s$ real for $s > 0$, whereas for $s = -|s| \pm i0$ they are $(\log |s| \pm i\pi)$, $\log(\log |s| \pm i\pi)$. We apply Cauchy's theorem to a contour integral along a path containing a circle of radius R large but finite and including the two cuts along the positive and the negative real axis. The discontinuity of the cut along the + real axis is proportional to $[\rho(s)/\log s \log \log s]$ whereas the discontinuity of the cut along the negative real axis is proportional to

$$A(s) \text{Im} \left(\frac{1}{(\log |s| + i\pi)(\log(\log |s| + i\pi))} \right)$$

From the above relations we see that the contribution along the circle is bounded and we get the sum rule given by Theorem II [Eq. (7)] :

$$\int_C^R \frac{ds}{s \log s \log \log s} \left[\frac{\rho(s)}{\pi} - \frac{A(s)}{\log s} \right] < \text{const},$$

R arbitrarily large. On the other hand, the following integral is convergent :

$$\int_C^\infty \frac{ds}{s \log s \log \log s \gamma(s)} \left[\frac{\rho(s)}{\pi} - \frac{A(s)}{\log s} \right]$$

where $\gamma(s)$ is an increasing function approaching infinity (arbitrarily slowly !). We see indeed from Eq. (7) that

$$\int_{R_1}^{R_2} \frac{ds}{s \log s \log \log s \gamma(s)} \left[\frac{\rho(s)}{\pi} - \frac{A(s)}{\log s} \right]$$

approaches zero for R_1 and R_2 big enough.

b) If $\rho(R) < \text{const } R^P$, (P arbitrarily large) and if $A(R) < \text{const log } R$, we consider $\psi(s) = \Pi(s)(\log s)^{-2}$. Following a reasoning similar to that of a) we apply Cauchy's theorem and get

$$\int_4^{\infty} \frac{\rho(s) ds}{s (\log s)^2 \psi(s)} < \infty \quad (\text{A.18})$$

with $\psi(s)$ of the same type as above. To obtain the constant [at the right-hand side of (A.18)] we have to take into account the integration of $A(s)/s(\log s)^3 \psi(s)$ when $s \rightarrow \infty$, this quantity being integrable under the assumptions stated here.

The convergence of the integral when $s \rightarrow \infty$ at the right-hand side of (A.18) shows that it is only on a set of asymptotically zero density that ρ can rise like a positive power of the energy.

APPENDIX B

Let us consider

$$\tilde{\pi}(s) = \frac{1}{\pi} \int_a^\infty \frac{s \tilde{\rho}^2(x) dx}{x(x-s)} \quad a > 0 \quad (\text{B.1})$$

where $\tilde{\rho}$ has not a well-defined sign. Let us define $\tilde{A}(R) = -\tilde{\pi}(-R)$ for $R > 0$. We assume

$$|\tilde{A}(R)| < \text{const.} \log \log R \quad (\text{B.2})$$

$$|\tilde{\rho}^2(x)| < \text{const.} \log \log x \quad (\text{B.3})$$

Investigating the behaviour of $|\tilde{\pi}(R e^{i\theta})|$ for R large, $\theta \neq 0$, outside the cut, we want to show

$$|\theta| > \eta \quad |\tilde{\pi}(R e^{i\theta})| < \text{const.} \log \log R \quad (\text{B.4})$$

$$|\theta| < \eta \quad |\tilde{\pi}(R e^{i\theta})| < \text{const.} |\log |\theta|| \log \log R \quad (\text{B.5})$$

$$\int_0^{2\pi} |\tilde{\pi}(R e^{i\theta})| d\theta = \tilde{T} < \text{const.} \log \log R \quad (\text{B.6})$$

where η is small but finite. Finally, we will apply these results to get sum rules connecting the asymptotic behaviour of $\tilde{A}(s)$ and $\tilde{\rho}(s)$.

1. Let us define $\epsilon > 0$, small, $\epsilon \ll 1$ and R large such that $\epsilon R > a$, $\epsilon R > 1$. We separate $\tilde{A}$ in two terms $\tilde{A} = \tilde{A}_1 + \tilde{A}_2$

$$\tilde{A}_1 = \int_{\epsilon R}^\infty \frac{R \tilde{\rho}^2}{x(x+R)} dx = \int_{\epsilon R}^{2R} + \int_{2R}^\infty$$

Applying (B.3), we get

$$|\tilde{A}_1| < \text{const.} |\log \epsilon| \log \log R \quad (\text{B.7})$$

From (B.2) it follows that

$$|\tilde{A}_2| = \left| \int_a^{\epsilon R} \frac{R \tilde{\rho}^2 dx}{x(x+R)} \right| < \text{const. } \log \log R \quad (\text{B.8})$$

2. We recall

$$\begin{cases} \operatorname{Re} \tilde{\Pi} = \frac{R}{\pi} \int_a^{\theta} \frac{x \cos \theta - R}{x D} \tilde{\rho}^2 dx & \operatorname{Im} \tilde{\Pi} = \frac{R \sin \theta}{\pi} \int_a^{\infty} \frac{\tilde{\rho}^2 dx}{x D} \\ D = (x-R)^2 + 2xR(1-\cos \theta) \end{cases} \quad (\text{B.9})$$

$$\begin{cases} |\tilde{\Pi}(R e^{i\theta})| < \frac{\sqrt{2} R}{\pi} \int_a^{\infty} \frac{R |\tilde{\rho}^2| dx}{x (|u-x|+v)} \\ u = R \cos \theta, \quad v = R |\sin \theta| \end{cases} \quad (\text{B.10})$$

We separate in (B.1) the integration path $[a, \omega]$ into two parts: $[a, \epsilon R]$ and $[\epsilon R, \omega]$. For the first interval, we get bounds valid for θ fixed and $\theta \rightarrow 0$ whereas in the second one we must distinguish between $|\theta| < \eta$ and $|\theta| > \eta$.

3. Let us define

$$C_1 = R \int_a^{\epsilon R} \frac{\tilde{\rho}^2}{D} dx, \quad C_2 = \int_a^{\epsilon R} \frac{R^2 \tilde{\rho}^2}{x D} dx$$

we get

$$\begin{aligned} |C_1| &< \sup_{x \in [a, \epsilon R]} |\tilde{\rho}^2(x)| \int_a^{\epsilon R} \frac{R dx}{(x-R)^2} < \text{const. } |\log |\log \epsilon|| \log \log R \\ C_2 - A_2 &= \int_a^{\epsilon R} \frac{dx \tilde{\rho}^2 (R^2 - x^2 + 2R^2 \cos \theta)}{D(x+R)} \end{aligned} \quad (\text{B.11})$$

and

$$\begin{cases} |C_2| < |A_2| + \sup_{x \in [a, eR]} |\rho(x)| \int_0^e \frac{|1 - \lambda^2 + 2\cos\theta|}{(1+\lambda)(1-\lambda)^2} d\lambda \\ |C_2| < \text{const.} \epsilon |\log |\log \epsilon|| \log \log R \end{cases} \quad (\text{B.12})$$

4. $|\theta| > \eta$, η finite. Let us define

$$C_3 = R \int_{eR}^{\infty} \frac{\rho^2}{D} dx, \quad C_4 = R^2 \int_{eR}^{\infty} \frac{\tilde{\rho}}{x D} dx$$

$$|C_3| < \text{const.} \log \log R \left[\int_0^2 \frac{d\lambda}{(1+\lambda^2-2\lambda\cos\theta)} + \int_2^{\infty} \frac{\log \log \lambda}{(1+\lambda^2-2\lambda\cos\theta)} d\lambda \right]$$

$$|C_3| < \text{const.} \log \log R \quad (\text{B.13})$$

$$|C_4| < \text{const.} \log \log R \left[\int_0^2 \frac{d\lambda \lambda^{-1}}{(1+\lambda^2-2\lambda\cos\theta)} + \int_2^{\infty} \frac{\lambda^{-1} \log \log \lambda}{(1+\lambda^2-2\lambda\cos\theta)} d\lambda \right]$$

$$|C_4| < \text{const.} |\log |\log \epsilon|| \log \log R \quad (\text{B.14})$$

From (B.11)-(B.14) we get (B.4).

5. $|\theta| < \eta$. From (B.10) we see that it remains to consider

$$C_5 = \int_{eR}^{\infty} \frac{R |\rho^2| dx}{x(|v-x|+v)} = C_6 + C_7 + C_8$$

For θ small, $eR \simeq eu$, so that we get

$$C_6 = R \int_{\epsilon U}^U \frac{|\tilde{p}| dx}{x(u-x+v)} < \text{const.} \sup_{x \in [\epsilon U, U]} |\tilde{p}(x)| \log \left(\frac{x}{x-u-v} \right) \Big|_{\epsilon U}^U$$

$$C_6 < \text{const.} \log \log R |\log |\epsilon \Theta|| \quad (\text{B.15})$$

$$C_7 = R \int_U^{2U} \frac{|\tilde{p}| dx}{x(x-u+v)} < \sup_{x \in [U, 2U]} |\tilde{p}(x)| \log \left(\frac{x-u+v}{x} \right) \Big|_U^{2U}$$

$$C_7 < \text{const.} \log \log R \log |\sin \Theta| \quad (\text{B.16})$$

$$C_8 = R \int_{2U}^{\infty} \frac{dx |\tilde{p}|}{x(x-u+v)} < 2R \int_{2R}^{\infty} \frac{dx |\tilde{p}|}{x^2}$$

$$C_8 < \text{const.} \log \log R \int_2^{\infty} \frac{\log \log \lambda d\lambda}{\lambda^2}$$

$$C_8 < \text{const.} \log \log R \quad (\text{B.17})$$

From (B.11), (B.12), (B.15)-(B.17), we get (B.5) and, by integration, (B.6).

6. Similarly to what was done in Appendix A, we apply Cauchy's theorem for $\tilde{\Psi}(s) = (\tilde{H}(s)/\log \log s)$. We again take for the logarithmic terms a cut from 0 to $-\infty$ and consider the determinations $\log|s|$, $\log \log|s|$ for $s > 0$ whereas we have $\log|s| \pm i\pi$, $\log(\log|s| \pm i\pi)$, for $s = -|s| \pm i0$. The discontinuity for $s > 0$ is proportional to $(\tilde{p}(s)/\log \log s)$, and

$$\tilde{A}(s) \text{Im} \left(\frac{1}{\log(\log|s| + i\pi)} \right)$$

for $s < 0$. The contributions along the great circles of radius R are evaluated using our result (B.6) and we get the relations written down in Eq. (7'). Similarly to what was in Appendix A, introducing $\gamma(s)$, an increasing function approaching infinity arbitrarily slowly, we get the integral condition (8').

R E F E R E N C E S

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