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UNIQUE INELASTIC PHASE SHIFTS
FROM GENERAL STRUCTURE IN TWO VARIABLES

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A B S T R A C T

We look for valid general properties of the spinless scattering amplitude, which are sufficient to determine it uniquely from a complete set of perfect data, making the minimum use of elastic unitarity. In addition to analytic properties in both kinematic variables, the positivity of the imaginary parts in the real positive momentum transfer strips is essential. Even then a small amount of extra information is still needed. The difficulty of using these results in a practical analysis is discussed. They do throw light on other attempts to use analytic properties to remove phase shift ambiguities, and suggest that fixed t dispersion relations may provide a sound basis.

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1. - INTRODUCTION

It is now recognized that there is a continuum of scattering amplitudes giving exactly the same observables in the inelastic region, which satisfy the general requirements of Lorentz invariance, unitarity and finite range ^{1),2)}. Various suggestions for restoring uniqueness by the inclusion of extra theoretical information ³⁾ require analytic continuation of numerical information which involves model dependent assumptions if significant results are to be achieved with data of reasonable accuracy.

This explicit analytic continuation could be avoided if a suitable space of functions were found in which the observable modulus of the amplitude uniquely determined the phase (apart, possibly, from discrete alternatives). At fixed energy, the space of polynomials in z of finite order, L , is such a space but unfortunately we know that the physical amplitude does not lie in it; its popularity for phase shift analysis stems from the mistaken belief that the truncation at L is an innocent approximation to the full infinite series ¹⁾. One may hope that the imposition of some correct general analytic structure may be sufficient to define a space of functions which gives a unique solution.

It has already been shown that analytic structure in momentum transfer alone is not enough ³⁾. Here we hope to explore how general properties in both kinematic variables may be used to remove in principle the continuum ambiguity. There is a wide range of constraints on the amplitude which follow from analyticity, crossing and unitarity and we do not need them all (indeed, it has been widely believed in the past that together they may determine the amplitude without any experimental data ⁴⁾); here we shall aim to use analyticity to the utmost, rather than unitarity, which has usually been stressed ^{*}). The discussion will be confined to spinless particles.

We find that analyticity alone is not enough. The remaining freedom is associated with possible zeros of the amplitudes, which prevent the phase being given in terms of the logarithm of the modulus by a dispersion relation. Positivity of the imaginary part provides the main further constraint.

^{*}) It is well known ⁵⁾ that elastic unitarity gives a unique set of partial wave amplitudes at energies near threshold which can then be continued in energy; however, we know of no sound effective way in practice of using this as a constraint to remove the continuum ambiguity at higher energies, because of the inherent instability of numerical analytic continuation.

In the next sections, we examine in detail the effect of each constraint on the ambiguity, and then discuss the implications of there results for practical analyses.

2. - ANALYTIC RESTRICTIONS ON THE AMPLITUDE

We shall assume that we have two amplitudes, $F(s,t,u)$ and $G(s,t,u)$, for the scattering process $A+B \rightarrow A+B$ and its crossed channels, and we shall aim to find realistic conditions on the ratio

$$R(s,t,u) = F/G \quad (2.1)$$

which have $R \equiv 1$ as the only solution.

The experimental data give us the modulus of the amplitude in some or all parts of the physical regions, so that there we have

$$|R| = 1 \quad (2.2)$$

This is our first kind of condition.

Next we shall impose analyticity in both independent variables, which gives us Theorem I, and, in Eq. (2.3), the most general form for R , consistent with this and the modulus condition (2.2). It contains an arbitrary meromorphic function, and so still has a great deal of freedom. More precisely, the statement is as follows.

Theorem I

Let two two-body amplitudes F and G be analytic in the following domain : m is the common mass of the two initial and final particles. $s+t+u=4m^2$. The amplitudes are analytic in $|t| < 4m^2$ minus the cuts s real $> 4m^2$, u real $> 4m^2$ and for s real $+i\epsilon$ inside an ellipse containing the physical region $4m^2 - s \leq t \leq 0$, plus union over circular permutations of s , t and u .

For instance, this domain is contained in the axiomatic domain of analyticity of the pion-pion scattering amplitude for pions of mass m (these pions need not be neutral).

In addition, F and G possess the "reality" property, namely $F(s, t, u) = F^*(s^*, t^*, u^*)$, which implies, when combined with the previously stated analyticity properties that $F(s, t, u)$ is real for $s < 4m^2$, $t < 4m^2$, $u < 4m^2$. Similarly $G(s, t, u)$ is real in the same domain.

Now we assume that F and G have the same modulus in the three physical regions :

$$t \leq 0 \quad u \leq 0 \quad s \geq 4m^2$$

$$t \leq 0 \quad s \leq 0 \quad u \geq 4m^2$$

$$u \leq 0 \quad s \leq 0 \quad t \geq 4m^2$$

The Theorem says that the most general form of F/G is

$$R = \frac{\sqrt{(4m^2-s)(4m^2-t)(4m^2-u)} + f(s, t, u)}{\sqrt{(4m^2-s)(4m^2-t)(4m^2-u)} - f(s, t, u)} \quad (2.3)$$

where $f(s, t, u)$ is an arbitrary meromorphic function which possesses the reality property. f can be written globally as the ratio of two entire functions of s , t and u (with the mass shell restriction)⁶⁾. In this theorem no use is made of unitarity or of positivity in any way. Also notice that, even though R is meromorphic everywhere except on the cuts, F and G need not separately satisfy the Mandelstam representation ; the general form (2.3) for R remains valid even if they do.

The proof proceeds as follows. First notice that

$$G G^* = F F^*$$

valid for s real $+i\epsilon$, $t < 0$, can be continued first of all for s real and t inside the complex domain of analyticity in t . If $G(s, t=0) \neq 0$, the zeros of G around $t=0$ are isolated so that R is properly defined by (2.1) ; then

$$R(s, t, u) R^*(s, t^*, u^*) = 1 \quad (2.4)$$

If F and G have no accumulation of zeros along the s cut, we can use this equation to define a second sheet continuation of R through the s cut

$$R_{II}(s, t, u) = [R_I^*(s^*, t^*, u^*)]^{-1}. \quad (2.5)$$

However, it is obvious that R possesses the reality property, so that $R_I(s, t, u) = R_I^*(s^*, t^*, u^*)$ and (2.5) becomes

$$R_{II}(s, t, u) = [R_I(s, t, u)]^{-1}. \quad (2.6)$$

Now we use Eq. (2.3) as a definition of f :

$$f(s, t, u) = \sqrt{(4m^2 - s)(4m^2 - t)(4m^2 - u)} \frac{R - 1}{R + 1}. \quad (2.7)$$

We see that the only possible singularities of f , except for poles, are cuts, inside the analyticity domain of F and G . f also possesses the reality property. It is indeed real inside the Mandelstam triangle.

Using (2.6), we can compute the continuation of f across the s cut. One readily finds that f_{II} and f_I coincide, because the change in sign of $(R-1)/(R+1)$ from (2.6) is compensated by the change in $\sqrt{4m^2 - s}$. Hence f has no s cut. It is meromorphic in the topological product of the s plane with $|t| < 4m^2$. But it is also meromorphic in the product of the t plane with $|s| < 4m^2$. The meromorphy envelope of the union of these two domains coincides with the holomorphy envelope of the union of these two domains which is known to be the topological product of the entire s plane with the entire t plane. This completes the proof of Theorem I ^{*}).

So we see that if one does not want to make use of unitarity or positivity, there is a lot of arbitrariness left in the ratio F/G , and the hope that general analytic structure might remove the continuum ambiguity is a false one. The unrealistic polynomial assumption, which does eliminate it is of course much more severe, in that it reduces the number of parameters to a finite number, $L+2$, for order L ; a neutral parametrization of the true amplitude, on the other hand, must clearly involve an infinite number of parameters.

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This arbitrariness is still worse for processes in which there are unphysical thresholds such as $NN \rightarrow NN$. For this process, the domain of reality of the amplitude is given by $s < 4M^2$, $t < 4m^2$, $u < 4m^2$ where M is the nucleon mass and m is the pion mass, while the physical t channel threshold and u channel threshold are at $4M^2$. If F and G have the same modulus in the physical regions only f is no longer a meromorphic function. If F and G satisfy Mandelstam representation, one can prove that f is equal to the product of a meromorphic function times a function satisfying Mandelstam representation with a double spectral function having its support in the finite region $4m^2 < t < 4M^2$, $4m^2 < u < 4M^2$.

3. - THE CONSTRAINT OF POSITIVITY

In the present approach we have decided not to include the near threshold elastic unitarity condition in our constraints because, although it at once gives uniqueness in principle with perfect data ⁵⁾, the error build-up in its analytic extrapolation in practice swiftly destroys all its information as the energy rises. Instead we look for another general constraint, valid at all energies.

The positivity of the imaginary part of the amplitude in the strip $0 < t < 4m^2$, and its crossed equivalents provides such a constraint, and we shall now show that it reduces the ambiguity remarkably - almost, in fact, to the point of uniqueness, $R(s,t,u)=1$, in certain cases. Positivity is a consequence of the unitary bound on the partial wave amplitudes, though unitarity implies rather more than this, of course. We argue as follows.

First we shall disregard completely the t channel and exploit knowledge of the modulus of the amplitude in the s and u physical regions only. This allows us to treat the more general case of reactions $A+B \rightarrow A+B$ and $A+\bar{B} \rightarrow A+\bar{B}$. We assume that the cuts in the s and u channel begin at the physical thresholds $s=s_0=(M_A+M_B)^2$, $u=u_0=s_0$.

In the s channel the absorptive part is positive for $s > s_0$, $0 \leq t \leq 4m^2$, where m is the pion mass. A similar statement can be made about the absorptive part in the u channel.

It is known ^(7),8) that one can deduce from full exploitation of positivity and inelastic unitarity that the scattering amplitude satisfies a twice subtracted dispersion relation for $0 \leq t < 4m^2$. Then from the fact that the discontinuities across the left and right hand cuts have constant signs one can prove that the amplitude has at most two zeros in the complex s plane for a given value of t in this interval.

If the amplitude for t in this interval has no zero, it is completely fixed by the knowledge of the modulus in the s and u physical region. Indeed, the modulus is also fixed by analytic continuation of $F(s,t)F^*(s,t^*)$ for fixed s in $0 < t \leq 4m^2$. The same holds for the u channel. Then, after mapping of the twice cut plane on a circle, one can write Poisson integrals for the real and imaginary parts of $\log F$. In this way the phase of F is fixed up to a constant. This constant is itself fixed by two things: reality tells us that the phase is 0 or π between the two cuts; however, only one of the two choices, obviously, will give a positive discontinuity across the s cut. We can write

$$F(s,t,u) = \pm \exp P(s,t,u) \quad (3.1)$$

where P is a weighted integral of $\log|F|$ along the cuts.

If the amplitude has one or two zeros it will contain an extra factor which has modulus unity along the cuts and preserves the reality property between the cuts. By performing a certain number of conformal mappings to transform the domain into a half plane it is possible to find the most general form of this factor. Returning to the original variables one gets

$$F(s,t,u) = \pm \frac{A(t) + sB(t) + \sqrt{(s_0-s)(u_0-u)(t_0-t)}}{A(t) + sB(t) - \sqrt{(s_0-s)(u_0-u)(t_0-t)}} \times \exp P(s,t,u) \quad (3.2)$$

where, for the time being we only know that $A(t)$ and $B(t)$ are real functions of the real variable t . $A(t)$ and $B(t)$ are locally analytic if the zeros are inside the analyticity domain, i.e., do not approach the cuts or infinity.

Suppose now that we have two amplitudes with the same modulus in the s and u physical region, with the correct positivity properties. We have three cases.

- A) - Neither F nor G have zeros. Then $F/G = \pm 1$. However, the minus sign is excluded because one of the absorptive parts would be negative. The equality originally valid for complex s , t real in a certain interval inside the analyticity domain of F and G can be continued everywhere.
- B) - One of the two has one or two zeros, the other no zero. Then for $0 \leq t < 4m^2$ one has

$$R = \frac{+A(t) + sB(t) + \sqrt{(s_0 - s)(u_0 - u)(t_0 - t)}}{-A(t) + sB(t) - \sqrt{(s_0 - s)(u_0 - u)(t_0 - t)}} \quad (3.3)$$

Now, taking two different energies s_1 and s_2 we can write a system of two linear equations for A and B in terms of $F/G(s_1, t)$ and $F/G(s_2, t)$. In this way we prove that A and B are meromorphic functions of t inside $|t| < 4m^2$ and more generally, by taking s_1 and s_2 large enough, inside the parabola of focus $t=0$ and summit $t=4m^2$, which is known to be an analyticity domain of the amplitude for s large enough⁸⁾. Then Eq. (3.3) holds all through this domain. Of course, only one of the two signs is acceptable, but we do not know in advance which one.

- C) - Both F and G have zeros. Then for $0 < t \leq 4m^2$ one has

$$R = \pm \frac{A(t) + sB(t) + s^2 C(t) + [a(t) + sb(t)] \sqrt{(s_0 - s)(u_0 - u)(t_0 - t)}}{A(t) + sB(t) + s^2 C(t) - [a(t) + sb(t)] \sqrt{(s_0 - s)(u_0 - u)(t_0 - t)}} \quad (3.4)$$

Here again, solving for A/a , B/a , C/a , b/a by writing (3.4) for four different energies, we prove that these ratios are meromorphic inside the union over all s of the fixed s analyticity domain in the variable t of the amplitudes. By a convenient choice we can take A , B , C , a , b to be holomorphic inside this domain.

We notice that by putting some coefficients equal to zero we can cover all the previous cases. So we can now state :

Theorem II

If we have two amplitudes describing the process $A+B \rightarrow A+B$ and the crossed elastic process, which have :

- a) the same modulus in the two physical channels,
- b) the correct positivity properties,
- c) no unphysical thresholds in these two channels,
- d) the axiomatic high energy bounds,

then their ratio is given by Eq. (3.4).

Notice that positivity leaves as the arbitrary functions in the ratio a set of four functions of the single variable t , with a definite analyticity domain.

Now if we have the moduli of the amplitude in all three channels, no unphysical thresholds, and positivity in all three channels, we can combine the two theorems. The only process satisfying all these conditions is the elastic scattering of neutral pions. This case was considered by J.D. Bessis and one of us ⁹⁾ some years ago but we want to reconsider it in the light of our new results.

In this situation the meromorphic function appearing in Eq. (2.3) must be crossing symmetric and, at the same time, of the form given in Eq. (3.4). The only possible form for the ratio R is then

$$R = \pm \frac{a + b(s^2 + t^2 + u^2) + cstu + \sqrt{(4m^2 - s)(4m^2 - t)(4m^2 - u)}}{a + b(s^2 + t^2 + u^2) + cstu - \sqrt{(4m^2 - s)(4m^2 - t)(4m^2 - u)}} \quad (3.5)$$

This form is great progress in the direction of uniqueness, since there are no arbitrary functions and only three undetermined real parameters. These can surely be fixed in a variety of ways and we shall look now at one of them, which is sufficiently powerful to remove the ambiguity in the general case (3.4).

4. - THE OPTICAL THEOREM AND THE THRESHOLD CONDITION

The general requirements of positivity and analyticity have not been enough to remove the continuum ambiguity. We now look for some further reliable ingredients which will reduce the general form (3.4) to the unique form $R=1$. We find that we can do this with use of the optical theorem and the elastic unitarity condition on the s wave at threshold. Essentially, we show that the optical theorem determines the amplitude uniquely at $t=0$, which is not surprising; the result can be extended to all t because the unknown functions of t in (3.4) are independent of s and we know that at threshold the s wave dominates and there is no t dependence. The detailed argument is as follows.

If total cross-sections for AB and $\overline{A}\overline{B}$ collisions are given, in addition to the differential cross-section, then the forward amplitude is unique, i.e., $F=G$ at $t=0$. To prove this, notice that both F and G satisfy a forward dispersion relation which cannot need more than two subtractions; the cut integrals in the dispersion relations will be the same, so the difference

$$F - G = A + Bs. \quad (4.1)$$

But $|F| = |G|$ and $\text{Im} F = \text{Im} G$ on the cuts, so $\text{Re} F = \pm \text{Re} G$. The $+$ sign gives $F=G$ on an interval and therefore everywhere, so we are left with the $-$ sign. Equation (4.1) then implies

$$\text{Re} F = \frac{1}{2}(A + Bs) \quad (4.2)$$

for both physical cuts.

Hence the analytic function $F - \frac{1}{2}(A + Bs)$ is purely real between the cuts, purely imaginary on the cuts and bounded by $|s \log^2 s|$. Hence the function

$$\frac{2F - A - Bs}{\sqrt{(s_0 - s)(u_0 - u)}} \quad (4.3)$$

can only be a constant because it is an entire function bounded by $|\log^2 s|$, since it is purely real on the whole real axis. But the form for F implied by (4.3) is unacceptable, if only because it has no inelastic threshold. So we are left with $R(s,0) = 1$.

For the extension to general t we use the behaviour at threshold. Provided the s wave scattering length $a_0 \neq 0$, it follows from unitarity at threshold that

$$\text{Im } F(s_0, t) = a_0^2 \frac{k}{\sqrt{s_0}} + O(k^3) \quad (4.4)$$

with a similar expression at the u channel threshold. G satisfies a similar equation with the same a_0 , because we can show that $\text{Re } F(s_0, t) = \text{Re } G(s_0, t)$ as follows.

At $s = s_0$ the amplitude is real and $|F| = |G|$ implies $|\text{Re } F(s_0, t)| = |\text{Re } G(s_0, t)|$. However, $\text{Re } F(s_0, t)$ and $\text{Re } G(s_0, t)$ are analytic functions of t which are equal and (by assumption) different from zero at $t = 0$. The only possibility is therefore $\text{Re } F(s_0, t) = \text{Re } G(s_0, t)$. Similarly if the s wave scattering length in the u channel differs from zero we also have at $u = u_0$ $\text{Re } F = \text{Re } G$. Writing (3.4) as

$$R = \pm \frac{\phi(s, t) + \sqrt{(s_0 - s)(u_0 - u)(t_0 - t)}}{\phi(s, t) - \sqrt{(s_0 - s)(u_0 - u)(t_0 - t)}} \quad (4.5)$$

with

$$\phi = \frac{A(t) + s B(t) + s^2 C(t)}{a(t) + s b(t)}$$

we see that if $\phi(s_0, t) \neq 0$ we have to choose the $+$ sign in (4.5); then it is also necessary to have $\phi(s, t) \neq 0$ for $u = u_0$, otherwise the sign in (4.5) would be wrong. If, on the other hand, ϕ vanishes for $s = s_0$ and any t , and for $u = u_0$ and any t , we can multiply the numerator and the denominator of (4.5) by

$$\frac{a(t) + s b(t)}{\sqrt{(s_0 - s)(u_0 - u)}} \sqrt{t_0 - t}$$

and redefine a new ϕ which does not vanish at the thresholds.

Next it follows from the equal moduli and the effective range expansion (4.4) that

$$(\operatorname{Re} F)^2 - (\operatorname{Re} b)^2 = (\operatorname{Im} G)^2 - (\operatorname{Im} F)^2 = O(k^4)$$

so that, always assuming $a_0 \neq 0$

$$\operatorname{Re} F - \operatorname{Re} b = O(k^4) \quad (4.7)$$

Then

$$\frac{F}{G} = \frac{\operatorname{Re} F + i \operatorname{Im} F}{\operatorname{Re} G + i \operatorname{Im} G} = 1 + i O(k^3) + O(k^4); \quad (4.8)$$

on the other hand from (4.5)

$$\frac{F}{G} = 2i \frac{\sqrt{(s-s_0)(u_0-u)(t_0-t)}}{\phi(s, t)} + 1 + O(k^2)$$

so we must have

$$\phi(s, t) \sim \frac{1}{s-s_0} \sim \frac{1}{k^2} \quad (4.9)$$

hence

$$a(t) + s_0 b(t) \equiv 0$$

By a similar argument in the u channel

$$a(t) + (\Sigma M^2 - u_0 - t) b(t) \equiv 0$$

Hence $a(t) \equiv b(t) \equiv 0$ and F is identical to G .

The result is not dependent on the assumption $a_0 \neq 0$, but the argument is a little more complicated if $a_0 = 0$.

5. - IMPLICATIONS FOR PRACTICAL ANALYSES

The ultimate weapon of phase shift analysis is a practical procedure for computing the physical scattering amplitude from the data in a reliable way which avoids unjustifiable ad hoc assumptions or models in removing the continuum phase ambiguity. Our results show that a lot of extra information on the general structure of the amplitude is needed and it is far from straightforward to devise a practical scheme which has these conditions built in. It is clearly of interest to look at current methods of phase shift analysis to see how they relate to these results ¹⁰⁾.

The shortest path method is based on the assumption that, in its energy variation, the true amplitude always chooses the shortest path in through the continuum of possible paths in the space of partial wave amplitudes, which we know is untrue in some circumstances - it seems that it can only work reliably when the ambiguity continua are small compared to the energy variation of the amplitude. Explicit analytic forms for the energy behaviour of the amplitude are simple models, which may or may not come close to the truth.

In the case of analyses based on fixed t dispersion relations ¹¹⁾, however, there seems to be more hope. Even as models they have the advantage of containing some sound dynamics, and the following argument suggests that they may actually contain enough constraints to form the basis of a sound route to uniqueness. One of us ⁸⁾ has shown that if an amplitude satisfies fixed t dispersion relations for all t in a range $-T \leq t \leq 0$, and if its the partial wave expansion in z_t converges and satisfies both positivity of the imaginary part and a bound $|f_\ell| < 1$, then the amplitude is analytic in the strip $-T \leq t \leq 4m_\pi^2$ and further satisfies the Froissart bound. Now this analyticity and positivity are exactly the main inputs for our uniqueness theorems, so there is a suggestion that those who use this method (they can usually also fix the threshold condition) are working on a basis which can give uniqueness.

We do state this conviction more strongly than this because our results refer to perfect data, and the full error analysis of a given procedure will be far from trivial. In practice it is only too easy to get a solution by over-restricting a parametrisation and reverting to what is only a model amplitude. In this method, the practical problems of the high energy cross-sections, of the extrapolation to the low energy unphysical

regions, and of the threshold conditions must be faced ; in each case it must be shown that the amplitude is stable under truncation of the complete expansions and errors in the input data, and the sensitivity to such errors determined. Only then is the method manifestly reliable. Nonetheless, the fixed t dispersion relations seem to hold more hope of such justification than anything else suggested so far.

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