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ADLER'S THEOREM IN FINITE MASSLESS Q.E.D. AND
POSSIBLE EXTENSIONS TO NON-ABELIAN GAUGE THEORIES

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A B S T R A C T

A new proof of Adler's theorem in finite massless Q.E.D. is given and generalizations to non-Abelian gauge theories are studied.

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I. INTRODUCTION

The purposes of this paper are twofold :

- 1) we present a novel proof of Adler's theorem ¹⁾ in massless QED; i.e., if at some $\alpha = \alpha'$

$$\beta(\alpha') \equiv m \frac{d}{dm} Z_3 \left(\frac{\Lambda^2}{m^2}, \alpha \right) \Big|_{\alpha=\alpha'} = 0 \quad (1)$$

where

$$\alpha = \alpha \left(\frac{\Lambda^2}{m^2}, e_0 \right) = Z_3 \alpha_0 \quad (2)$$

and m^2 is a Euclidean subtraction point. Then

$$\frac{d^n}{d\alpha^n} \beta(\alpha) \Big|_{\alpha=\alpha'} = 0 \quad (3)$$

- 2) we shall indicate what form its extensions to non-Abelian gauge theories may take. The motivation for this is clear if one believes that, in the real world, electromagnetism is simply a component of a unified gauge theory.

II. ADLER'S THEOREM

We shall present this argument in the Coulomb gauge. This is, of course, not essential to the result since β is gauge invariant. Furthermore, it complicates the work. However, we do so for two reasons :

- 1) In this gauge there are no problems of positivity in a sense that will become clear as we proceed.
- 2) In the non-Abelian case - specifically, the massless Yang-Mills theory - in which Z_3 is gauge dependent, it turns out that Z_3 is positive in the Coulomb gauge. We will use this fact later and hence we shall use the Coulomb gauge throughout.

We begin by reminding the reader of the relevant equations for QED in the Coulomb gauge ²⁾. We call the four-component potential A_μ

$$A_\mu(x) = (\underline{A}(x), A_0(x)) \quad (4)$$

We demand that

$$\underline{\nabla} \cdot \underline{A}(x) = 0 \quad (5)$$

in all Lorentz frames. This means that A_μ is not a Lorentz four-vector. In momentum space we may express the change in $A_\mu(x)$ when we transform under an infinitesimal Lorentz transformation by the equation

$$A'_\mu(k') = ((1 + \epsilon) A(k))_\mu + \frac{k_\mu k_\nu}{|k|^2} \epsilon_{\nu i} A^i(k) \quad (6)$$

where we use the metric

$$K^2 = |k|^2 - k_0^2 \quad (7)$$

We define $\underline{j}^{Tr}(x)$ by the equation

$$\square^2 \underline{A}(x) = e_0 \underline{j}^{Tr}(x) \quad (8)$$

and

$$\underline{\nabla}^2 A_0(x) = e_0 \rho(x). \quad (9)$$

The four-vector current $J_\mu(x)$ satisfies

$$\partial_\mu \underline{J}_\mu(x) = 0 = \underline{\nabla} \cdot \underline{J}(x) + \frac{\partial}{\partial \tau} J_0(x). \quad (10)$$

Hence we can write

$$\underline{j}^{Tr}(x) \equiv \underline{J}(x) + \underline{\nabla} \frac{\partial}{\partial \tau} \int \frac{d^3 x'}{4\pi} \frac{\rho(x', \tau)}{|x - x'|} \quad (11)$$

so that $\underline{j}^{Tr}(x)$ is a non-local, transverse function whose transformation under Lorentz transformations is defined by the corresponding transformation of $\underline{A}(x)$.

At equal times we have

$$[A_i(\underline{x}, \tau), \dot{A}_j(\underline{x}', \tau)] = i \delta_{ij}^{\tau}(\underline{x} - \underline{x}') \quad (12)$$

with

$$\delta_{ij}^{\tau}(\underline{x}) = \int d^3 \underline{k} e^{i \underline{k} \cdot \underline{x}} \left(\delta_{ij} - \frac{k_i k_j}{|\underline{k}|^2} \right) \frac{1}{(2\pi)^3} \quad (13)$$

Thus

$$\text{Trace } \delta_{ij}^{\tau}(\underline{x}) = 2 \delta^j_i(\underline{x}). \quad (14)$$

The state containing one-transverse photon is defined by

$$\langle 0 | \underline{A}(\underline{x}) | P \rangle = \sqrt{Z_3} \langle 0 | \underline{A}_{in}(\underline{x}) | P \rangle. \quad (15)$$

Exploiting the equal-time commutation relations and inserting a complete set of normalized states, we may establish the following sum rule for Z_3

$$1 = Z_3 \left(\frac{\Lambda^2}{m^2}, \alpha \right) + \int_0^{\Lambda^2} \frac{dx}{x} \Pi(x, \alpha) \quad (16)$$

where

$$\begin{aligned} \Pi\left(\frac{q^2}{m^2}, \alpha\right) &= q^2 \sum_n' \langle 0 | A_i(0) | n \rangle \langle n | A_i(0) | 0 \rangle \int^y (p_n - q) \\ &= \frac{e_0^2}{q^2} \sum_n' \langle 0 | \dot{A}_i^{\tau}(0) | n \rangle \langle n | \dot{A}_i^{\tau}(0) | 0 \rangle \int^y (p_n - q). \end{aligned} \quad (17)$$

(The sums exclude the state of one photon.)

Several comments are in order :

1) A priori, since we are in the Coulomb gauge where manifest covariance is lost, Π , as defined, could be a general dimensionless function of q_0 , $|\underline{q}|$ and m^2 . In QED this does not happen ²⁾ and Π is only a function of the arguments shown. In the non-Abelian case this is no longer true and we must examine what happens.

2) In general; i.e., for general values of α , and all x ,

$$\Pi(x, \alpha) > 0. \quad (18)$$

3) We know that Z_3 can be expressed as a log series in α ; i.e.,

$$Z_3\left(\frac{\Lambda^2}{m^2}, \alpha\right) = \sum_{i=0}^{\infty} C_i(\alpha) \log\left(\frac{\Lambda^2}{m^2}\right)^i + O\left(\frac{m^2}{\Lambda^2}\right). \quad (19)$$

Hence we can express

$$\Pi(x, \alpha) = \sum_{i=0}^{\infty} C'_i(\alpha) \log(x)^i. \quad (20)$$

The explicit dependence of Z_3 on Λ^2/m^2 comes from the upper limit on the integral

$$\int_0^{\frac{\Lambda^2}{m^2}} \frac{\Pi(x, \alpha)}{x} dx,$$

as indicated.

We may now proceed to the proof of Adler's theorem. We first write down the Callan-Symanzik equation obeyed by Z_3 ¹⁾

$$\left(m \frac{\partial}{\partial m} + \beta(\alpha) \left(\alpha \frac{\partial}{\partial \alpha} - 1\right)\right) Z_3\left(\frac{\Lambda^2}{m^2}, \alpha\left(\frac{\Lambda^2}{m^2}, e_0\right)\right) = 0. \quad (21)$$

In this way of writing the equation e_0 is held fixed, along with any arguments not being differentiated. This equation has the solution

$$Z_3\left(\frac{\Lambda^2}{m^2}, \alpha\right) = Z_3\left(1, \bar{\alpha}(\tau, \alpha)\right) \frac{\bar{\alpha}(0)}{\bar{\alpha}(\tau)} \quad (22)$$

where

$$\tau = \frac{1}{2} \log\left(\frac{\Lambda^2}{m^2}\right) \quad (23)$$

and

$$\frac{d\bar{\alpha}}{d\tau} = \bar{\alpha} \beta(\bar{\alpha}). \quad (24)$$

Z_3 is defined by the equation

$$d_c\left(\frac{g^2}{m^2}, \alpha\right) Z_3\left(\frac{\Lambda^2}{m^2}, \alpha\right) = d\left(\frac{g^2}{\Lambda^2}, e_0\right) \quad (25)$$

and the normalization

$$d_c(1, \alpha) = 1 \quad (26)$$

where

$$\begin{aligned} D_{ij}(\tau^2) &= \left(\delta_{ij} - \frac{q_i q_j}{|q|^2} \right) \frac{d\left(\frac{q^2}{\Lambda^2}, e_0\right)}{q^2} \\ &\equiv -i \int d^4x \langle 0 | T(A_i(x) A_j(0)) | 0 \rangle e^{i(qx)} \end{aligned} \quad (27)$$

From Eq. (22)

$$\overline{\alpha}(0) = \alpha(1, e_0). \quad (28)$$

We give these details, which are well known, for two reasons :

- 1) we will contrast them later with the non-Abelian case;
- 2) if β has an eigenvalue, α' ; i.e., if

$$\beta(\alpha') = 0 \quad (29)$$

then

$$\overline{\alpha}(\tau) \Big|_{\tau \rightarrow \infty} = \alpha' \quad (30)$$

which implies that Z_3 is finite as $\Lambda \rightarrow \infty$.

However, we have not been able to find an argument ³⁾ that shows that this condition is sufficient to guarantee that

$$\left. \frac{\partial^n Z_3}{\partial \alpha^n} \right|_{\alpha = \alpha'} < \infty \quad (31)$$

In convergent perturbation theory we could expand

$$Z_3 = \sum_i d_i \alpha^i \log\left(\frac{\Lambda^2}{m^2}\right)^i \quad (32)$$

implying that the function and its derivatives are regular within the radius of convergence of the expansion. However, Adler's theorem implies that Z_3 cannot be expanded in this way at the eigenvalue. The derivatives of Z_3 with respect to α are finite, but zero. In our method of proof we assume that the derivatives are finite and then show that they are zero. This procedure is not circular, but it is inelegant, and constitutes a loophole in our proof of Adler's theorem. It may be that this loophole can be plugged or that it also enters the other proofs ¹⁾ in various guises. This point deserves further analysis.

To carry on, note that

$$m \frac{\partial Z_3}{\partial m} = - \Lambda \frac{\partial Z_3}{\partial \Lambda} = 2\pi \left(\frac{\Lambda'}{m^2}, \alpha \right) \quad (33)$$

Thus, at the eigenvalue

$$\pi \left(\frac{\Lambda'}{m^2}, \alpha' \right) = 0 \quad (34)$$

which, by positivity, guaranteed in the Coulomb gauge since $|n\rangle$ never contains longitudinal photons, means that

$$\langle 0 | A_i(0) | n \rangle_{\alpha'} = 0 \quad (35)$$

for all $|n\rangle$, not including the one-photon state. Hence

$$\langle 0 | j_i^{\text{Tr}}(0) | n \rangle_{\alpha'} = 0 \quad (36)$$

for all $|n\rangle$ including the one-photon state. Thus

$$j_i^{\text{Tr}}(0) | 0 \rangle_{\alpha'} = 0 \quad (37)$$

One cannot use the Federbush-Johnson theorem ⁴⁾ to conclude that at the eigenvalue

$$j_i^{\text{Tr}}(0) \equiv 0 \quad (38)$$

since j^{Tr} is not local. Moreover, it follows that

$$\rho(x) | 0 \rangle_{\alpha'} = 0 \quad (39)$$

from Lorentz covariance. The extra piece in the appropriate version of Eq. (6) also annihilates the vacuum at α' . Hence, we can argue, at the eigenvalue, as if $(j^{\text{Tr}}(x), \rho(x))$ were a true four-vector, when applied to the vacuum state.

To prove Adler's theorem, we differentiate Eq. (21) with respect to α repeatedly. To see what is going on, let us record the result of the first two differentiations

$$\frac{\partial \beta(\alpha)}{\partial \alpha} \left(\alpha \frac{\partial}{\partial \alpha} - 1 \right) Z_3 + \left(m \frac{\partial}{\partial m} + \beta(\alpha) \left(\alpha \frac{\partial}{\partial \alpha} - 1 \right) \right) \frac{\partial Z_3}{\partial \alpha} = 0 \quad (40)$$

and

$$\begin{aligned} & \frac{\partial^2 \beta(\alpha)}{\partial \alpha^2} \left(\alpha \frac{\partial}{\partial \alpha} - 1 \right) Z_3 + \frac{\partial \beta(\alpha)}{\partial \alpha} \left(\alpha \frac{\partial}{\partial \alpha} - 1 \right) \frac{\partial Z_3}{\partial \alpha} \\ & + m \frac{\partial}{\partial m} \frac{\partial^2}{\partial \alpha^2} Z_3 + \frac{\partial}{\partial \alpha} \left(\beta(\alpha) \left(\alpha \frac{\partial}{\partial \alpha} - 1 \right) \frac{\partial Z_3}{\partial \alpha} \right) = 0 \end{aligned} \quad (41)$$

and so on. Hence if

$$\left. \frac{\partial^n Z_3}{\partial \alpha^n} \right|_{\alpha=\alpha'} < \infty$$

it is clear that Adler's theorem can be established if we can show that

$$\left. \frac{\partial^n}{\partial \alpha^n} m \frac{\partial}{\partial m} Z_3 \right|_{\alpha=\alpha'} = 0. \quad (42)$$

From Eq. (33), it will suffice to show that

$$\frac{\partial^n}{\partial \alpha^n} \left\langle 0 \left| j^{\text{Tr}}(0) \right| n \right\rangle_{\substack{\alpha=\alpha' \\ p_n^2 = \Lambda^2}} = 0 \quad (43)$$

where $|n\rangle$ is any state not including the one-photon state. To prove this we argue as follows; suppose we create the state $|n\rangle$ with a local creation operator $B(x)$ satisfying

$$\Lambda^2 B(x) = (\Box^2 + \Lambda^2) \phi(x) \quad (44)$$

where Λ has been put in, in this way, for dimensional reasons; i.e.,

$$\langle 0 | \phi(x) | B(p) \rangle = e^{i(p \cdot x)} \quad (45)$$

Then

$$\langle 0 | j_i^{\text{Tr}}(0) | B(p) \rangle = \int d^4x e^{i(p \cdot x)} \langle 0 | T(j_i^{\text{Tr}}(0) B(x)) | 0 \rangle \quad (46)$$

with

$$p^2 = \Lambda^2 \quad (47)$$

We shall now expand Eq. (46) in terms of e_0 .

$$\begin{aligned} \langle 0 | T(j_i^{\text{Tr}}(0) B(x)) | 0 \rangle &= \int \int d^4x_1 \dots d^4x_n \frac{1}{(2n)!} e_0^{2n} (-i)^{2n} \\ &\times \langle 0 | T(j_i^{\text{Tr}}(0) \hat{B}(x) \hat{j}_1^{\text{Tr}}(x_1) \hat{A}(x_1) \dots \hat{j}_n^{\text{Tr}}(x_n) \hat{A}(x_n)) | 0 \rangle \\ &+ \text{Coulomb Terms} \end{aligned} \quad (48)$$

The symbols $\hat{B}(x)$, and so on, mean interaction representation operators with no e_0 dependence. The subscript c means the connected part and we have not written down the Coulomb terms explicitly since they complicate the notation horribly. Using Eq. (39), one can verify that the Coulomb terms simply follow along with their own powers of e_0 and factorials and do not alter the result that we are about to write down; namely, with $|n\rangle$ any state except the one-photon state,

$$\begin{aligned} e_0 \frac{\partial}{\partial e_0} \langle 0 | j_i^{\text{Tr}}(0) | n \rangle &= e_0^2 \int d^4x' d^4x'' \hat{D}_{im}^0(x' - x'') \\ &\times \langle 0 | T(j_i^{\text{Tr}}(0) B(x) \hat{j}_l^{\text{Tr}}(x') \hat{j}_m^{\text{Tr}}(x'')) | 0 \rangle \end{aligned} \quad (49)$$

where

$$\hat{D}_{lm}^0(x' - x'') = -i \langle 0 | T(\hat{A}_l(x') \hat{A}_m(x'')) | 0 \rangle \quad (50)$$

This result is established by first contracting all of the A_i with each other in Eq. (48). Such contractions of $A_1 \dots A_{2n}$ produce a combinatorial factor of $2n!/(n!2^n)$. One then differentiates with respect to e_0 and re-sums the series, producing Eq. (49). To obtain the higher derivatives we differentiate Eq. (49) again and go through the same summing and re-summing. Thus, for example

$$e_0^2 \frac{\partial^2}{\partial e_0^2} \langle 0 | \hat{J}_i^{\tau_1}(0) | n \rangle = e_0^2 \int d^4x' d^4x'' D_{im}^0(x'-x'') \langle 0 | T(\hat{J}_i^{\tau_1}(0) B(x) \hat{J}_e^{\tau_2}(x') \hat{J}_m^{\tau_3}(x'')) | 0 \rangle$$

$$+ e_0^4 \int d^4x' d^4x'' d^4x''' d^4x'''' D_{im}^0(x'-x'') D_{ms}^0(x''-x''') D_{se}^0(x'''-x''') \langle 0 | T(\hat{J}_i^{\tau_1}(0) B(x) \hat{J}_e^{\tau_2}(x') \hat{J}_m^{\tau_3}(x'') \hat{J}_s^{\tau_4}(x''') \hat{J}_e^{\tau_5}(x''')) | 0 \rangle \quad (51)$$

In general, the n th derivative will be a linear combination of T products of the form

$$\langle 0 | T(\hat{J}_i^{\tau_1} B \hat{J}_e^{\tau_2} \dots \hat{J}_{jn}^{\tau_n}) | 0 \rangle$$

However, by Eq. (37), these all vanish at $\alpha = \alpha'$. Before we can use this result, we must relate derivatives with respect to e_0 to those with respect to e ;

$$e = \sqrt{z_3} e_0 \quad (52)$$

since Eq. (50) and Eq. (51) and their generalizations involve e_0 derivatives while Eq. (42) involves α derivatives. This connection uses chain rule manipulations and, when the dust settles, we derive the equation

$$\frac{\partial}{\partial e_0} = \frac{\sqrt{z_3}}{(1 - \frac{1}{2} \frac{\partial z_3}{\partial e} \frac{e}{z_3})} \frac{\partial}{\partial e} \quad (53)$$

For purposes of the argument all we want is that

$$[\quad] \equiv \frac{\sqrt{z_3}}{1 - \frac{1}{2} \frac{\partial z_3}{\partial e} \frac{e}{z_3}} \quad (54)$$

is neither finite nor zero at the eigenvalue. The former is guaranteed by the eigenvalue condition, while the latter is discussed in the remarks surrounding Eq. (31). We may iterate Eq. (53). For example, symbolically

$$\frac{\partial^2}{\partial e^2} = [\]^2 \frac{\partial^2}{\partial e^2} + [\] \frac{\partial}{\partial e} ([\]) \frac{\partial}{\partial e} \quad (55)$$

Hence, e_0 derivatives can be reduced to e derivatives and the argument is concluded. Whether this argument is any simpler than Adler's or Johnson's ¹⁾ is clearly a matter of taste. However, it does suggest to us, as we shall now indicate, how we might try to generalize to the non-Abelian case.

III. THE NON-ABELIAN CASE

For purposes of this note, we shall consider simply the massless, self-coupled Yang-Mills theory as the prototype of a non-Abelian gauge theory. It might turn out that the program of Baker, Johnson, Adler ¹⁾ and many others, to determine the coupling constant from an eigenvalue condition such as Eq. (1) has a solution, if it has a solution, for only some specific gauge group with some selections of representations - the gauge group of the world. In any event it seems reasonable, since we are charting new terrain, to start with the simplest case involving no fermions. We have not, as yet, worked out all of the details corresponding to the results of the last section for this case. However, we have gone far enough so that the outlines of what is involved here become clear. In this note we shall report on this, the rest being under active investigation.

We shall attempt to imitate the methods of the previous section as closely as possible and to note in what way the non-Abelian character of the problem changes it. We work in the Coulomb gauge. We shall not give all of the details of the quantization in this gauge ⁵⁾ since we will need relatively few of them for present purposes. We begin with the definition of the bare and renormalized propagation functions in this gauge for the transverse; i.e., the physical Yang-Mills quanta, the analogue of Eq. (27). Thus

$$D_{ij}^{ab}(|\underline{q}|, q_0, |\underline{\Lambda}|) \equiv -i \int d^4x e^{i(\underline{q}x)} \langle 0 | T(B_i^a(x) B_j^b(0)) | 0 \rangle$$

$$= \int_{ab} \left(\delta_{ij} - \frac{q_i q_j}{|\underline{q}|^2} \right) \frac{1}{q^2} d\left(\frac{q_0^2}{|\underline{q}|^2}, \frac{|\underline{q}|^2}{|\underline{\Lambda}|^2}, g_0\right) \quad (56)$$

with

$$\underline{\nabla} \cdot \underline{B}^a(\underline{x}, \tau) = 0. \quad (57)$$

In this case d cannot be written in a manifestly covariant way and we use a three-vector cut-off. There could, in principle, be a dependence on $q_0/|\underline{\Lambda}|$ in d . But d is a series in $\log(|\underline{\Lambda}|)$ so this variable can be re-expressed. To illustrate, a specific computation⁶⁾ of d to order g_0^2 in the Coulomb gauge reveals that

$$d\left(\frac{q_0^2}{|\underline{q}|^2}, \frac{|\underline{q}|^2}{|\underline{\Lambda}|^2}, g_0\right) = 1 + \frac{g_0^2}{4\pi^2} \log\left(\frac{\underline{\Lambda}}{|\underline{q}|}\right)$$

$$+ \frac{g_0^2}{|\underline{q}|^2} \int_1^\infty \frac{f(x)}{\left[x - \frac{q_0^2}{|\underline{q}|^2}\right] x} dx + \text{const.} \quad (58)$$

The precise definitions of $f(x)$ and the value of the constant are not relevant here. We are only interested in illustrating the dependence of d . The divergent part of d has been isolated as shown. Henceforth we shall call

$$|\underline{\Lambda}| \equiv \Lambda \quad (59)$$

and the subtraction point

$$|\underline{\mu}| \equiv \mu \quad (60)$$

Thus

$$d_c\left(\frac{q_0^2}{|\underline{q}|^2}, \frac{|\underline{q}|^2}{\mu^2}, g\right) Z_3\left(\frac{\Lambda^2}{\mu^2}, g\right) = d\left(\frac{q_0^2}{|\underline{q}|^2}, \frac{|\underline{q}|^2}{\Lambda^2}, g_0\right) \quad (61)$$

where we fix Z_3 by the normalization condition

$$d_c(0,1,g) = 1. \quad (62)$$

The second complication in the non-Abelian case arises from the relation between g and g_0 ; i.e.,

$$g = \frac{Z_3(\frac{\Lambda^2}{\mu^2}, g)^{3/2}}{Z_1(\frac{\Lambda^2}{\mu^2}, g)} g_0. \quad (63)$$

This combination of Z 's is gauge invariant ⁷⁾ - Z_1 is the three-point function renormalization - but we have not found any way of using the combination directly via a sum rule to make a useful statement about β :

$$\beta \equiv \mu \frac{dg}{d\mu}. \quad (64)$$

In fact, in the Coulomb gauge, we will show

$$Z_3 > 0 \quad (65)$$

while to order g_0^2 ⁶⁾

$$Z_1 = 1 - \frac{g_0^2}{12\pi^2} \log\left(\frac{\Lambda}{\mu}\right) \quad (66)$$

To the same order

$$Z_3 = 1 + \frac{g_0^2}{4\pi^2} \log\left(\frac{\Lambda}{\mu}\right) \quad (67)$$

as follows directly from Eq. (57). It might be that some other gauge, such as the axial gauge, where

$$Z_1 = Z_3 \quad (68)$$

will turn out to be a better gauge in which to study our problem. But, in these gauges there are positivity questions and we have not seen how to resolve them.

We will now write down the sum rule for Z_3 in the Coulomb gauge - the analogue of Eq. (17). The first thing to notice is that

$$\langle 0 | [B_i^a(x, \tau), \dot{B}_j^b(x', \tau)] | 0 \rangle \neq i \int_{\text{ab}}^{\tau} (x - x') \quad (69)$$

This is because $\dot{B}_i^a(x, t)$ is not canonical to $B_i^a(x, t)$. Following Schwinger ⁵⁾ we can decompose the canonical momentum Π_i^a into its transverse and longitudinal parts

$$\Pi_i^a(x, \tau) = \Pi_i^{a\tau} (x, \tau) + \Pi_i^{a\text{long}} (x, \tau) \quad (70)$$

The precise form of these operators will not concern us here. However, the quantization condition is

$$[B_i^a(x, \tau), \Pi_j^{b\tau}(x', \tau)] = i \int_{\text{ab}}^{\tau} (x - x') \quad (71)$$

We must use the equations of motion to find $\dot{B}_i^a$ and then compute the correct form of Eq. (69). Thus

$$\begin{aligned} & \int d^3x e^{i\mathbf{q} \cdot \mathbf{x}} \langle 0 | [B_i^a(x, \tau), \dot{B}_j^b(0, \tau)] | 0 \rangle \\ &= i \left(\int_{\text{ab}} - \frac{\mathbf{q}_i \mathbf{q}_j}{|\mathbf{q}|^2} \right) \int_{\text{ab}} \lim_{\substack{|\mathbf{q}|^2 \text{ fixed} \\ q_0 \rightarrow +\infty}} d\left(\frac{\mathbf{q}_i}{|\mathbf{q}|^2}, \frac{|\mathbf{q}|^2}{\Lambda^2}, q_0\right) \end{aligned} \quad (72)$$

this being the Bjorken-Johnson-Low ⁸⁾ definition of the equal time commutator. Note that ⁹⁾

$$\begin{aligned} & \sum_i \int d^3x e^{i\mathbf{q} \cdot \mathbf{x}} \frac{1}{i} \langle 0 | [B_i^a(x, \tau), \dot{B}_i^a(0, \tau)] | 0 \rangle \\ &= \sum_{n,i} P_n^a \int^3 (\mathbf{q} - \mathbf{P}_n) | \langle 0 | B_i^a(0) | n \rangle |^2 \end{aligned} \quad (73)$$

It follows from Eq. (73) that in the Coulomb gauge

$$\lim_{\substack{|\mathbf{q}|^2 \text{ fixed} \\ q_0 \rightarrow +\infty}} d\left(\frac{\mathbf{q}_i}{|\mathbf{q}|^2}, \frac{|\mathbf{q}|^2}{\Lambda^2}, q_0\right) \gg Z_3\left(\frac{\Lambda}{|\mathbf{q}|}, g\right) > 0 \quad (74)$$

There are three relevant Callan-Symanzik equations in this gauge :

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \gamma(g) \right) Z_c = 0 \quad (75)$$

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma(g) \right) Z_3 = 0 \quad (76)$$

and

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \frac{3}{2} \gamma(g) + \frac{\beta(g)}{g} \right) Z_1 = 0 \quad (77)$$

where

$$\gamma(g) = \frac{1}{Z_3} \mu \frac{\partial}{\partial \mu} Z_3 \quad (78)$$

In Abelian QED there is the - exceptional - simplification that β and the "anomalous dimension" γ are proportional to each other. In general these functions are independent. Indeed, β does not depend on the gauge while γ does. Equations (77) and (78) can be solved to give

$$Z_3(\lambda', g) = Z_3(1, \bar{g}) \exp \left(- \int_0^\tau \gamma(\bar{g}(g, x)) dx \right) \quad (79)$$

and

$$Z_1(\lambda', g) = Z_1(1, \bar{g}) \exp \left(- \int_0^\tau \left(\frac{3}{2} \gamma(\bar{g}) - \frac{\beta(\bar{g})}{\bar{g}} \right) dx \right) \quad (80)$$

with

$$\lambda = \frac{\Lambda}{\mu} \quad (81)$$

$$\tau = \frac{1}{2} \log(\lambda') \quad (82)$$

$$\frac{d\bar{g}}{d\tau} = \beta(\bar{g}(g, \tau)) \quad (83)$$

and

$$\bar{g}(0) = g(1, g_0). \quad (84)$$

The correctness of the factors occurring in Eqs. (79) and (80) is confirmed by solving for the gauge invariant combination $Z_2^{3/2}/Z_1$. Thus,

$$\begin{aligned} \frac{Z_2^{3/2}(\tau, g)}{Z_1(\tau, g)} &= \frac{Z_2(1, \bar{g})^{3/2}}{Z_1(1, \bar{g})} \exp \left(\int_0^\tau \frac{\beta}{\bar{g}} dx \right) \\ &= \frac{Z_2(1, \bar{g})^{3/2}}{Z_1(1, \bar{g})} \frac{\bar{g}(\tau)}{\bar{g}(0)}. \end{aligned} \quad (85)$$

In Eq. (85) all reference to the gauge dependent quantity γ has cancelled out.

We may now ask, in light of these equations, what would one mean by the analogue of the finite QED programme generalized to the Yang-Mills situation? Presumably, the analogous statement would be to locate values of g such that

$$Z_1(g') < \infty \quad (86)$$

and

$$Z_2(g') < \infty. \quad (87)$$

The first thing to note is that if the Yang-Mills theory is asymptotically free in the sense that

$$\beta(g) < 0 \quad (88)$$

for all g greater than zero, then there is no finite Yang-Mills theory and no Baker-Johnson-Adler¹⁾ et al. program.

This can be seen from Eqs. (79) and (80) along with the perturbation theory results⁶⁾ in the Coulomb gauge to lowest non-trivial order in $\bar{g}$:

$$\beta(\bar{g}) = -\frac{11}{24} \frac{\bar{g}^3}{\pi^2} \quad (89)$$

$$\gamma(\bar{g}) = - \frac{1}{8\pi^2} \bar{g}^2 \quad (90)$$

while

$$\bar{g}'(\tau) = \frac{12\pi^2}{11} \frac{1}{\tau} \quad (91)$$

It is the $1/t$ behaviour of $\bar{g}^2$ in an asymptotically free theory that causes Z_1 and Z_3 to diverge.

It is simply not known how β and γ behave away from small values of their argument. But in the spirit of the finite QED programme we may ask what behaviour would be needed to render the Yang-Mills theory ultra-violet finite. To see what is involved, suppose that β has a simple zero somewhere. (This hypothesis may be contradicted by an Adler theorem; but that remains to be seen.) In the neighbourhood of this zero we may write

$$\beta \approx c(\bar{g} - \bar{g}') \quad (92)$$

Thus, from Eq. (83)

$$\bar{g} \approx \bar{g}' + e^{c\tau} \quad (93)$$

It is clear - for example from Eq. (85) - that c must be negative if the theory is going to be finite. Hence it is not enough that β have one zero. It must have at least two zeros since, at the first zero, $c > 0$. (The slope at the origin is negative. There is, of course, the zero of β at the origin, but the zeros we are discussing are away from the origin.) This condition on β is necessary but not sufficient for the finiteness of the Z 's. This can be seen by re-writing, say Eq. (79), in the form

$$Z_3(\lambda, \bar{g}) = Z_3(1, \bar{g}) \exp\left(- \int_{\bar{g}(0)}^{\bar{g}(\tau)} \frac{\gamma(x)}{\beta(x)} dx\right) \quad (94)$$

As

$$\begin{aligned} \tau &\rightarrow \infty \\ \bar{g} &\rightarrow \bar{g}' \end{aligned} \quad (95)$$

and

$$\beta \rightarrow 0$$

Therefore the integral will diverge at its upper limit unless

$$\gamma(g') = \beta(g') = 0. \quad (96)$$

This is a very stringent requirement. It says that β and γ have a common zero. It pins down what is needed if the theory is to be finite in the Coulomb gauge. In other gauges, Eqs. (75)-(77) must be generalized to allow variations in the gauge fixing parameters and there is a corresponding set of conditions. We do not know if the theory meets these conditions. It would also be very interesting to be able to prove that the theory cannot meet these conditions. That would, of course, show that the Adler-Baker-Johnson et al. programme for finite QED is possible, in principle, only for the Abelian case. These are extremely difficult problems to tackle and will require the invention of new non-perturbative techniques.

In any case, let us assume for the sake of argument, that there exists a suitable g' so that Eq. (96) is satisfied, and explore the question of the existence of an Adler-like theorem. In this note we report on the first step in this direction and then finish the exposition with a list of unresolved problems. We begin by considering the right-hand side of Eq. (73) with the state containing one-transverse Yang-Mills quantum removed. Thus,

$$0 \leq \sum_{n,i} P_{ni} \int (\delta - P_{ni}) |\langle 0 | B_i^a(0) | n \rangle|^2$$

$$\equiv \int_0^{\frac{\Lambda}{|g|}} \frac{dx}{x} \int_0^\infty \frac{dy}{y} \Pi(y, x, g) \quad (97)$$

which serves to define the dimensionless, positive definite - in the Coulomb gauge - weight function Π . In consequence of our assumptions - Eq. (96) - it follows that

$$\bigwedge_{j,\Lambda} d \left(\frac{g_0^2}{|g|}, \frac{|g|^2}{\Lambda^2}, g_0 \right) = 0 \quad (98)$$

$g = g'$
 $|g|$ fixed
 g_0 arbitrary

Hence if we can exchange the limit $q_0 \rightarrow \infty$ and differentiation with respect to Λ we can conclude, using Eq. (72), from positivity, that

$$\overline{\Pi}(y, \frac{\Lambda}{\mu}, g)_{g'} = 0 \quad (99)$$

or

$$\langle 0 | B_i^a(0) | n \rangle_{g'} = 0 \quad (100)$$

where $|n\rangle$ is any state except the one-vector meson state.

We now argue that if

$$\frac{\partial}{\partial g} \langle 0 | B_i^a(0) | n \rangle_{g'} < \infty \quad (101)$$

then

$$\frac{\partial}{\partial g} \sum_{n_i} P_{n_i} \int^3 (q - p_n) |\langle 0 | B_i^a(0) | n \rangle_{g'}|^2 = 0, \quad (102)$$

and thus

$$\frac{\partial}{\partial g} \left(\sum_{n_i} P_{n_i} \int^3 (q - p_n) \frac{|\langle 0 | B_i^a(0) | n \rangle_{g'}|^2}{z_i} \right) = 0. \quad (103)$$

The reason for writing this second expression is that we want to exploit the Callan-Symanzik equation for d_c ; Eq. (75). We assume that

$$d_c^\infty \equiv \lim_{\substack{|q|^2 \text{ Fixed} \\ q_0 \rightarrow \infty}} d_c \left(\frac{q_0^2}{|q|^2}, \frac{|q|^2}{\mu^2}, g \right) \quad (104)$$

obeys Eq. (75) since for every finite value of q_0 the equation is obeyed. From Eq. (103), we conclude that

$$\mu \frac{\partial^2}{\partial \mu \partial g} d_c^\infty \Big|_{g'} = 0. \quad (105)$$

as well as

$$\frac{\partial}{\partial g} d_c^\infty \Big|_{g'} = 0. \quad (106)$$

Thus if we differentiate Eq. (75) with respect to g and exploit these eigenvalue conditions, we learn that

$$\frac{\partial \gamma}{\partial g} d_c^\infty \Big|_{g'} = 0 \quad (107)$$

i.e.,

$$\frac{\partial \gamma}{\partial g} \Big|_{g'} = 0. \quad (108)$$

We are not able to draw any conclusions about β using this method, but we arrive at the beginnings of a form of the Adler theorem involving the anomalous dimension γ . In QED this distinction is irrelevant since, as we have noted, β and γ are proportional to each other. In the concluding section we list the problems suggested by these results in the form of a set of questions.

IV. CONCLUSIONS

Here is a list of questions suggested by the work of the preceding sections :

- 1) Given the assumptions, does it follow that

$$\frac{\partial^n}{\partial g^n} \gamma \Big|_{g'} = 0 \quad (109)$$

for all n ; i.e., is there an essential zero in γ ?

- 2) Can one make a statement about β ? 10).

- 3) Can these methods be utilized in other renormalizable field theories to investigate the analytical properties of the Callan-Symanzik functions in the neighbourhood of assumed eigenvalues ?

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R E F E R E N C E S

- 1) The basic reference is
S.L. Adler, Phys.Rev. D5, 3621 (1972).
In footnote 28) of this reference, Adler notes that the theorem was also known to K. Johnson (unpublished). So, perhaps, it should be called the Adler-Johnson theorem.
K. Johnson and M. Baker, PHYS.Rev. D8, 1110 (1973) have suggested an alternative proof. Our method differs from both of these. In these references the reader will find a full list of additional references to relevant papers on finite QED.
- 2) J.D. Bjorken and S.D. Drell, Relativistic Quantum Fields (McGraw-Hill, New York 1965)
is an excellent source on QED in the Coulomb gauge.
- 3) We are grateful to B. Lautrup for a discussion of this point.
- 4) P.G. Federbush and K. Johnson, Phys.Rev. 120, 1296 (1960).
- 5) See, for example, J. Schwinger, Phys.Rev. 125, 1043 (1962).
- 6) A. Ali, J. Bernstein and A. Zepeda, Phys.Rev. D - in press.
- 7) An argument for this is given in
R.E. Kellosh and I.V. Tyutin, Lebedev preprint 42 (1974).
- 8) J.D. Bjorken, Phys.Rev. 148, 1467 (1966);
K. Johnson and F.E. Low, Progr.Theor.Phys.(Kyoto) Suppl. 37, 74 (1966).
- 9) This equal time commutator has been evaluated in perturbation theory by
A. Ali and J. Bernstein, Phys.Rev. D11, 336 (1975).
- 10) We may be able to exploit the Ward-Taylor identities in the Coulomb gauge.