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EFFECTIVE COUPLING CONSTANT EXPANSION
OF THE MUON ANOMALOUS MAGNETIC MOMENT

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A B S T R A C T

The relationship between the renormalization group and the muon anomalous magnetic moment calculation in QED is reconsidered. A very simple analysis shows that the contribution to the muon anomaly from muon vertex graphs with electron loop insertions in the photon propagators is function of only one dimensionless effective coupling constant $\alpha_{\mu} \simeq 1/136.0785$. The perturbative expansion coefficients up to the α_{μ}^3 term are given, and comparison is made with the previous calculations.

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In pure quantum electrodynamics the anomalous gyromagnetic ratio of the muon a_μ is a function $a_\mu(\alpha, m_e/m_\mu)$ of the fine structure constant α and of the electron/muon mass ratio m_e/m_μ , so that, when expanding a_μ in powers of α , the coefficients of the expansion are functions of m_e/m_μ . In view of the smallness of the actual value of $m_e/m_\mu \approx 1/200$, it is customary to expand in turn the coefficients, dropping terms vanishing for $m_e/m_\mu \rightarrow 0$. Explicitly, the following structure is found:

$$a_\mu(\alpha, m_e/m_\mu) \approx \sum_{n=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^n \left[\sum_{m=0}^{n-1} B_{nm} \log^m \frac{m_\mu}{m_e} \right] \equiv a_\mu^\infty(\alpha, m_e/m_\mu) \quad (1)$$

In the following we will restrict ourselves to the contributions to Eq. (1):

$$a_\mu^\infty(\text{el.vac.pol.}) = \sum_{n=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^n \left[\sum_{m=0}^{n-1} C_{nm} \log^m \frac{m_\mu}{m_e} \right] \quad (2)$$

coming only from the "electron vacuum polarization" vertex diagrams, i.e. the muon vertex graphs obtained from the graphs with muon lines only, by replacing in them everywhere the bare photon propagator $-i\delta_{\mu\nu}/k^2$ by the dressed renormalized photon propagator containing only electron loops:

$$-i\delta_{\mu\nu}/k^2 \rightarrow -i \frac{\delta_{\mu\nu}}{k^2} d_R(k^2/m_e^2, \alpha) + \text{gauge terms} \quad (3)$$

We will not be concerned with other contributions to Eq. (1) coming from the important electron-loop "light-light" diagrams, starting from order α^3 , and from "mixed" vacuum polarization graphs, starting from order α^4 , where a muon loop is contained in an electron vacuum polarization loop. (Examples are shown in Figs. 1a and 1b, respectively.)

Fig. 1

In particular we want to reconsider here the relationship between the renormalization group and the calculation of $a_\mu^\infty(\text{el.vac.pol.})$. As is well known^{1,2)}, this relationship is based on the observation that in $a_\mu^\infty(\text{el.vac.pol.})$ only the asymptotic part $d_R^\infty(x, \alpha)$ of the function $d_R(x, \alpha)$ -- controlled by the renormalization group -- plays a significant role, since the mean momentum squared q^2 carried by the photon lines is of order m_μ^2 and therefore $x \equiv q^2/m_e^2$ is large. Along those lines Lautrup and de Rafael²⁾, from the knowledge of the Callan-Symanzik function $\beta(\alpha)$, have been able to give the coefficients C_{nm} for $m \neq 0$ in terms of the coefficients C_{n0} *) appearing in Eq. (2). Numerically, using the recently computed³⁾ value for C_{30} we find**)

$$a_\mu^\infty(\text{el.vac.pol.}) = \left(\frac{\alpha}{\pi}\right) 0.5 + \left(\frac{\alpha}{\pi}\right)^2 0.7543 + \left(\frac{\alpha}{\pi}\right)^3 3.139(26) + \left(\frac{\alpha}{\pi}\right)^4 (C_{40} + 16.5) \approx$$

$$\approx 1\,165\,519(23) \times 10^{-9} + \left(\frac{\alpha}{\pi}\right)^4 C_{40} . \quad (4)$$

One notes that strong cancellations occur between the different powers of $\log(m_\mu/m_e)$ in each order, as is clear from the fact that the various coefficients in Eq. (4) are relatively small numbers, whereas the coefficient of $(\alpha/\pi)^3$, for instance, involves up to the second power of $\log(m_\mu/m_e) = 5.3$. This is indeed a rather important observation, which makes one reasonably confident about the good convergence of the expansion in Eq. (3), despite the presence of higher and higher powers of $\log(m_\mu/m_e)$. On the other hand, one would possibly like to have better arguments than this *a posteriori* observation.

The alternative approach that we propose in order to have a better understanding of the relationship between the renormalization group and the evaluation of $a_\mu^\infty(\text{el.vac.pol.})$, starts from the observation that $a_\mu(\text{el.vac.pol.})$ does not depend on the fine structure constant and the electron mass separately, but only through its functional dependence on $\alpha d_R(k^2/m_e^2, \alpha)$:

$$a_\mu(\text{el.vac.pol.}) = \phi\left[\alpha d_R(k^2/m_e^2, \alpha), m_\mu\right] . \quad (5)$$

Let us now consider, instead of the usual dressed photon propagator of Eq. (3), renormalized at $k^2 = 0$, the one renormalized at $k^2 = -m_\mu^2$, which we write

*) More precisely, the $C_{10}, C_{20}, \dots, C_{N0}$ determine $C_{n(n-1)}, C_{n(n-2)}, \dots, C_{n(n-N)}$ for any n .

**) The uncertainty in the over-all result comes primarily from the uncertainty in the value of α .

$$-i \frac{\delta_{\mu\nu}}{k^2} d\left(\frac{k^2}{m_\mu^2}, \frac{m_e^2}{m_\mu^2}, \alpha_\mu\right) .$$

The renormalization group functional relation is

$$\alpha d_R\left(\frac{k^2}{m_e^2}, \alpha\right) = \alpha_\mu d\left(\frac{k^2}{m_\mu^2}, \frac{m_e^2}{m_\mu^2}, \alpha_\mu\right) \quad (6)$$

giving, on account of the renormalization condition $d[-1, (m_e^2/m_\mu^2), \alpha_\mu] = 1$:

$$\alpha_\mu = \alpha d_R\left(-\frac{m_\mu^2}{m_e^2}, \alpha\right) . \quad (7)$$

By combining Eqs. (5) and (6) we get

$$a_\mu(\text{el.vac.pol.}) = \phi\left[\alpha_\mu d\left(\frac{k^2}{m_\mu^2}, \frac{m_e^2}{m_\mu^2}, \alpha_\mu\right), m_\mu\right] . \quad (8)$$

Having performed the renormalization subtraction at the Euclidean point $k^2 = -m_\mu^2$ (and not at $k^2 = 0$), the $m_e \rightarrow 0$ limit at fixed α_μ of Eq. (8) exists and is finite:

$$a_\mu(\text{el.vac.pol.}) \xrightarrow{m_e \rightarrow 0} \phi\left[\alpha_\mu d\left(k^2/m_\mu^2, 0, \alpha_\mu\right), m_\mu\right] . \quad (9)$$

This is essentially ensured, order by order of the perturbative expansion in α_μ , by the Kinoshita theorem⁴⁾ on mass singularities in QED.

Equation (9) is our main result. As a_μ is dimensionless, the r.h.s. of Eq. (9) can depend only on α_μ , since, in performing the renormalization subtraction, we have not introduced any new scale. We can therefore write

$$a_\mu(\text{el.vac.pol.}) \simeq a_\mu^{(2)}\left(\frac{\alpha_\mu}{\pi}\right) + a_\mu^{(4)}\left(\frac{\alpha_\mu}{\pi}\right)^2 + a_\mu^{(6)}\left(\frac{\alpha_\mu}{\pi}\right)^3 + \dots , \quad (10)$$

where the coefficients $a_\mu^{(2n)}$ are constants, in close analogy with the familiar expression for the electron anomaly a_e ,

$$a_e = a_e^{(2)}\left(\frac{\alpha}{\pi}\right) + a_e^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + a_e^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + \dots , \quad (11)$$

where contributions due to muon loops, known to be of order m_e^2/m_μ^2 , are ignored.

All the dependence of α_μ (el.vac.pol.) on $\log(m_\mu/m_e)$ in Eq. (10) is contained in α_μ through Eq. (7). Neglecting again terms vanishing for $m_e/m_\mu \rightarrow 0$, the present knowledge of the asymptotic proper vacuum polarization amplitude $\pi(x, \alpha)$ gives us

$$\begin{aligned} \alpha_\mu^{-1} &= \alpha^{-1} \left[1 + \pi\left(-m_\mu^2/m_e^2, \alpha\right) \right] \approx \\ &\alpha^{-1} \left\{ 1 + \left(\frac{\alpha}{\pi}\right) \left[\frac{5}{9} - \frac{1}{3} \log \frac{m_\mu}{m_e} \right] + \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{5}{24} - \zeta(3) - \frac{1}{4} \log^2 \frac{m_\mu}{m_e} \right] \right. \\ &\quad \left. + \left(\frac{\alpha}{\pi}\right)^3 \left[\text{unknown const.} + \left(\frac{47}{96} - \frac{1}{3} \zeta(3)\right) \log \frac{m_\mu}{m_e} - \frac{1}{24} \log^2 \frac{m_\mu}{m_e} \right] + \dots \right\}. \end{aligned} \quad (12)$$

Inserting the value $\alpha^{-1} = 137.03608(26)$ in Eq. (12), and ignoring the unknown constant, we get

$$\alpha_\mu^{-1} = 136.07875(26) \quad . \quad (13)$$

Note that $\alpha_\mu > \alpha$ as expected for the ultraviolet unstable QED. Still α_μ is sufficiently small to expect a convergence of the series in Eq. (10) that is as good as the usual fine structure constant expansion of the electron anomaly, and to explain therefore the cancellations of the various powers of $\log(m_\mu/m_e)$ in each order of α in Eq. (2) *).

Needless to say, the calculation of the coefficients $a_\mu^{(2n)}$ in the expansion of Eq. (10) is a task which is as difficult as the evaluation of the analogous coefficients $a_e^{(2n)}$ in the expansion of the electron anomaly. Re-expanding Eq. (10) in α , we easily relate them to the coefficients C_{n0} appearing in Eq. (2):

$$\begin{aligned} a_\mu^{(2)} &= C_{10} = 1/2 ; \quad a_\mu^{(4)} = C_{20} + \frac{5}{9} C_{10} = -0.74512 \\ a_\mu^{(6)} &= C_{30} + \frac{10}{9} C_{20} + \left(\frac{335}{648} - \zeta(3) \right) C_{10} = 1.282 \pm 0.026 \quad . \end{aligned}$$

For comparison we recall that the values of the coefficients $a_e^{(2n)}$ are

$$\begin{aligned} a_e^{(2)} &= \frac{1}{2} ; \quad a_e^{(4)} = \frac{197}{144} + \frac{1}{2} \zeta(2) + \frac{3}{4} \zeta(3) - 3\zeta(2) \log 2 = -0.32848 \\ a_e^{(6)} &= 1.195 \pm 0.026 \quad . \end{aligned}$$

*) Of course we cannot prove that α_μ will not blow up when higher-order contributions in Eq. (12) are introduced.

The size of the $a_{\mu}^{(2n)}$'s, compared with that of the $a_e^{(2n)}$'s, is another indication that α_{μ} is the natural expansion parameter for $a_{\mu}(\text{el.vac.pol.})$ in the limit $m_e/m_{\mu} \rightarrow 0$.

As expected, if we insert the computed values for the $a_{\mu}^{(2n)}$'s in Eq. (10), together with the value of α_{μ} given in Eq. (13), we get to the same accuracy the same value as in Eq. (4) for $a_{\mu}^{\infty}(\text{el.vac.pol.})$.

In conclusion -- apart from the actual calculations of the various coefficients -- we can say that the only real information we used to get Eq. (10) and the corresponding numerical value for $a_{\mu}^{\infty}(\text{el.vac.pol.})$ is the Kinoshita theorem on mass singularities. From the renormalization group we have taken the idea of subtracting the photon propagator at a mean momentum transfer value, as well as the "trivial" functional relation given in Eq. (6). By exploiting the "non-trivial" information of the renormalization group, in principle a better value for the effective coupling constant α_{μ} could have been obtained. Standard renormalization group techniques give in fact

$$\alpha_{\mu} = \alpha d_R^{\infty} \left(-\frac{m_{\mu}^2}{m_e^2}, \alpha \right) = \bar{\alpha} d_R^{\infty}(1, \bar{\alpha}) =$$

$$\bar{\alpha} \left[1 - \frac{5}{9} \left(\frac{\bar{\alpha}}{\pi} \right) + \left(\frac{65}{648} + \zeta(3) \right) \left(\frac{\bar{\alpha}}{\pi} \right)^2 \right] .$$

Here $\bar{\alpha}$ is the effective coupling constant given implicitly by the equation:

$$\log \frac{m_{\mu}}{m_e} = \int_{\alpha}^{\bar{\alpha}} \frac{dx}{x\beta(x)} .$$

Solving for $\bar{\alpha}$ by a set of successive approximations, one gets an expression for $\bar{\alpha}^*$, and so for α_{μ} , which contains the three leading powers in $\log(m_{\mu}/m_e)$ to any order in α . However, the numerical evaluation of α_{μ} obtained in this way gives $\alpha_{\mu}^{-1} = 136.07874(26)$, namely a value not significantly different from that of Eq. (13).

For completeness we recall that, to the accuracy at present needed for comparison with the experimental results, the muon anomaly in pure QED consists, besides the contribution

$$a_{\mu}^{\infty}(\text{el.vac.pol.}) = 1\,165\,519(23) \times 10^{-9} ,$$

*) This is done in the Appendix of Ref. 2.

of additional contributions from an α^2 -term vanishing in the limit $m_e/m_\mu \rightarrow 0$ ⁵⁾:

$$\Delta a_\mu = \left(\frac{\alpha}{\pi}\right)^2 \frac{3}{2} \zeta(2) \frac{m_e}{m_\mu} = 64.8 \times 10^{-9} \quad ,$$

and from the sixth-order electron loop "light-by-light" vertex diagrams ⁶⁾

$$a_\mu^\infty(\text{light-by-light}) = \left(\frac{\alpha}{\pi}\right)^3 (19.79 \pm 0.16) = 247(2) \times 10^{-9} \quad .$$

In total

$$a_\mu(\alpha, m_e/m_\mu) = 1\,165\,831(25) \times 10^{-9} \quad .$$

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Figure caption

Fig. 1 : Examples of diagrams whose contribution to the anomaly is not contained in Eq. (2).