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DERIVATIVE RELATIONS FOR HELICITY AMPLITUDES

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A B S T R A C T

The derivative relations

$$F_{\Delta\lambda=1} = C_0 (\partial/\partial\sqrt{-t}) F_{\Delta\lambda=0},$$

$$F_{\Delta\lambda=2} = C_1 \sqrt{-t} (\partial/\partial\sqrt{-t}) (F_{\Delta\lambda=1}/\sqrt{-t}), \dots$$

for s channel helicity amplitudes with total helicity transfer $\Delta\lambda$, their physical motivation, their domain of validity and their power for amplitude analyses are discussed. Methods for model-independent tests of these relations are presented and applied to various non-diffractive reactions ($\pi^- p \rightarrow \pi^0 n$, $\gamma p \rightarrow \pi^0 p$, $(-)$ KN CEX) and also to πN diffraction scattering. In all these cases the derivative relations were found to be very well fulfilled.

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1. INTRODUCTION

Polarization measurements in two-body reactions revealed a long time ago¹⁾ that the t -structures and in particular the phases of the various helicity amplitudes are markedly different from each other. Since then it has been a major problem to find a convincing systematics for the *helicity dependence of the amplitudes*.

As is known¹⁾, crossed-channel Regge cut models did not really bring a systematic solution. However, geometrical ideas in the direct channel -- phrased in terms of impact parameter amplitudes -- seem to lead towards simple and global characteristics for the helicity dependence. One basic feature seems to be, for example, that the t -structures of the amplitudes depend only on the total s -channel helicity transfer²⁾ $\Delta\lambda = |\lambda_c - \lambda_d - \lambda_a + \lambda_b|$ rather than on all single s -channel helicities $\{\lambda_i\} = \lambda_c, \lambda_d, \lambda_a, \lambda_b$.

This paper is devoted to a discussion and test of the following, much further reaching hypothesis^{*)}, for the helicity dependence

$$F_{\{\lambda_i\}}(s,t) = C_{\{\lambda_i\}}(s) \sqrt{-t}^{\Delta\lambda} \left(\frac{1}{\sqrt{-t}} \frac{\partial}{\partial \sqrt{-t}} \right)^{\Delta\lambda} F_{\Delta\lambda=0}(s,t), \quad (1)$$

which looks, for example, for $0\frac{1}{2} \rightarrow 0\frac{1}{2}$ reactions simply like

$$F_{\Delta\lambda=1}(s,t) = C(s) \frac{\partial}{\partial \sqrt{-t}} F_{\Delta\lambda=0}(s,t). \quad (2)$$

These derivative relations for the s -channel helicity amplitudes $F_{\{\lambda_i\}}(s,t)$ are simple, powerful, and extremely appealing: all t -dependences are related to one single t -dependence, the one for $\Delta\lambda = 0$, say.

Their formulation in terms of impact parameter amplitudes points to a simple geometrical origin:

$$f_{\{\lambda_i\}}(\theta,s) = \tilde{C}_{\{\lambda_i\}}(s) e^{\Delta\lambda} f_{\Delta\lambda=0}(\theta,s) \quad (3)$$

*) As we shall see later (see Sections 3 and 4), these relations are expected to hold to a very good approximation in specific kinematical regions (and not everywhere!).

Which are the *physical motivations* for these derivative relations (1)?

- i) *Nuclear physics*³⁾: As is the case with many concepts in particle physics, similar derivative rules have been used quite successfully in nuclear physics already long ago, they were inspired by non-relativistic optical model calculations in the presence of a weakly varying (real) spin orbit potential^{*)}.
- ii) *Dual peripheral model*^{5,6)} (an s-channel approach for non-diffractive reactions at high energies): To our knowledge this is the only model in particle physics which necessarily incorporates the derivative relations (1). In fact we shall see in Section 2 that it is the basic hypothesis of the dominance of *one single* Regge pole $\alpha(s)$ in the s-channel which leads to the automatic fulfilment of the derivative relations (1).

The "history" of the derivative relations and the *model-dependent* tests they have been subject to up to now may be summarized as follows: in 1971 Högaasen⁷⁾ first suggested similar derivative relations as an empirical guess based on the qualitative structure of the impact parameter profiles of the $\pi^-p \rightarrow \pi^0n$ amplitudes at 6 GeV/c.

Next they were proposed⁶⁾ and implicitly tested^{5,6)} in the framework of the dual peripheral model (1972-74) versus a large number of non-diffractive two-body reactions. In view of the small number of free parameters, these tests have been quite successful. However, the dual peripheral model has limitations: it does not hold at and near $t = 0$, and it is not applicable to diffraction scattering. It is clearly interesting to ask whether the derivative relations also hold beyond these limitations.

The authors quoted in Refs. 8 were the first to formulate the derivative relations as a property detached from any model; in its impact parameter formulation (3) they call it "b-universality". However, all their tests involve, once again, strong additional model assumptions for the s- and t-dependences of the amplitudes ("Reggeometry") with partly very encouraging and partly failing results.

Independently, Barger and Phillips⁹⁾ recently tested the derivative rules in conjunction with the theoretical bias of a purely Regge-pole-behaved single-flip

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*) These ideas have recently been revived by F. Halzen⁴⁾ and applied to $pp \rightarrow pp$ scattering. However, the derivative relations of the nuclear physics type lead to $\partial/\partial\sqrt{-t} (\log d\sigma/dt) \propto P$. This relation implies $|F_{\text{flip}}|^2 \ll |F_{\text{non-flip}}|^2$ and either $C(s)$ in Eq. (1) purely imaginary or constant relative phases between different helicity amplitudes. In our following analysis we shall, however, find these conditions in general not to be fulfilled. We are therefore unable to establish a clean-cut connection between these relations and the ones discussed in our paper.

helicity amplitude. Their results are to some extent similar to those of the dual peripheral model; however, they describe, in addition, the region around $t = 0$, and also include the new FNAL data for $\pi^- p \rightarrow \pi^0 n$ and $\Delta\sigma(\pi N)$ into their analysis. Their conclusions about the quantitative success of their test is somewhat negative.

Altogether all these *model-dependent tests* are quite encouraging on a qualitative level. Obviously all the aforementioned quantitative failures cannot necessarily be attributed to the failure of the derivative relations; they may well be due to shortcomings of the various additional model assumptions entering the tests. (An illustrative example for how negatively a weak theoretical bias can affect the test of the derivative relations is given in Appendix I.)

What is clearly needed are *model-independent tests* of these interesting and powerful derivative relations. To work out methods for tests, which are as model-independent as possible, and to perform such tests in the case of non-diffractive reactions as well as in diffraction scattering is the main concern of this paper. Sections 5 and 6 are devoted to it. The following Sections 2 to 4 discuss the domain of validity of the derivative relations -- as expected from the dual peripheral model -- and their general limitations.

2. DERIVATIVE RELATIONS AS ABSTRACTED FROM THE DUAL PERIPHERAL MODEL

Let us show that the derivative relations (1) are a basic property of the dual peripheral model and discuss to which extent they are more general.

To this end we briefly recall the main ideas of the dual peripheral model^{5,6}. It consists of an s-channel Regge approach to non-diffractive reactions; the high-energy amplitudes (t fixed, $s \rightarrow \infty$) are given in terms of *one single* complex (non-linear) s-channel Regge pole $\alpha(s)$ and its analytic continuation $\alpha(u)$ to large negative values of $u \approx -s$. The idea is that this single Regge pole $\alpha(s)$ interpolates the (empirically dominant) peripheral s-channel resonances with spins $J_{\text{res}} + \frac{1}{2} \approx R \cdot k_{\text{cms}}$, such that

$$\begin{aligned} \text{Re } \alpha(s) + \frac{1}{2} &\approx R \cdot \sqrt{s}/2, & R \approx 1 \text{ fermi} = \text{universal} \\ &s \text{ large} > 0 & \text{hadronic interaction radius} \end{aligned} \quad (4)$$

$$\left[\text{hence by analyticity}^{5,6} \right] \alpha(s) + \frac{1}{2} \underset{|s| \text{ large}}{\approx} - \frac{R}{\pi} \frac{\sqrt{-s}}{2} \log \frac{-s}{s_0} \quad (5)$$

The s-u crossing $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ contributions to the s-channel helicity amplitudes have then the following form:

$$F_{\{\lambda_i\}}^{(\pm)}(s,t) = G_{\{\lambda_i\}}^{(\pm)}(s,t) \pm G_{\{\lambda_i\}}^{(\pm)}(u,t) \quad (6)$$

$$= \beta_{\{\lambda_i\}}^{(\pm)}(s) \frac{d_{\lambda, -\mu}^{\alpha(s)}(-\cos \theta_s)}{\sin \pi(\alpha(s) - \lambda)} \pm (s \leftrightarrow u) \quad (7)$$

with $\lambda = \lambda_a - \lambda_b$, $\mu = \lambda_c - \lambda_d$. It is instructive to introduce an approximation of the rotation functions $d_{\lambda, -\mu}^{\alpha}$ by Hankel functions, which is excellent for fixed $t \neq 0$ and large s (in practice for $p_{\text{lab}} \geq 2.5$ GeV/c already):

$$G_{\{\lambda_i\}}^{(\pm)}(s,t) \approx \beta_{\{\lambda_i\}}^{(\pm)}(s) i H_{\Delta\lambda}^{(1)}\left(\frac{\alpha(s) + 1/2}{\sqrt{s/2}} \sqrt{-t}\right), \quad (8)$$

$$\Delta\lambda = |\lambda - \mu|.$$

Obviously, due to the "single Regge pole" hypothesis, the whole t -dependence and in addition the whole helicity dependence of the t -structures is known; in fact -- within this very good Hankel-function approximation -- the t -structures depend only on the total helicity transfer $\Delta\lambda$, rather than on the individual helicities.

Moreover we discover the derivative relations (1) as an immediate consequence of the derivative relations¹⁰⁾ valid for all Bessel functions^{*}). In fact, the Hankel functions in Eq. (8) and therefore also our model amplitudes $G_{\{\lambda_i\}}^{(\pm)}$ are equivalently characterized by the simultaneous validity of the following two "upward" and "downward" derivative relations (plus a boundary condition for $-t \rightarrow \infty$) using $n = \Delta\lambda$:

$$G_n^{(\pm)}(s,t) = C_n^{(\pm)}(s) \sqrt{-t}^n \left(\frac{1}{\sqrt{-t}} \frac{\partial}{\partial \sqrt{-t}} \right)^n G_{n=0}^{(\pm)}(s,t) \quad (9a)$$

$$G_{n=0}^{(\pm)}(s,t) = D_n^{(\pm)}(s) \left(\frac{1}{\sqrt{-t}} \frac{\partial}{\partial \sqrt{-t}} \right)^n \left\{ \sqrt{-t}^n G_n^{(\pm)}(s,t) \right\} \quad (9b)$$

with

$$D_n(s) = \left(\frac{\alpha(s) + 1/2}{\sqrt{s/2}} \right)^{-2n} \frac{(-1)^n}{C_n(s)}.$$

*) Clearly -- without the Hankel function approximation -- quite similar, though much more complicated derivative relations, based on the recursion relations¹¹⁾ for the rotation functions $d_{\lambda, -\mu}^{\alpha}$, may be derived [$\partial/\partial\sqrt{-t}$ being substituted by $(1/k_{\text{cms}})(\partial/\partial\theta)$, etc.]. Actually we consider these relations as the "correct" ones, and only use the much simpler approximate ones for convenience.

Thus the "one *single* Regge pole" hypothesis is equivalent to the simultaneous validity of the two derivative relations (independent of the actual form of the trajectory!). Clearly using only one of the two relations is a more general (i.e. weaker) theoretical constraint and may also have a more general origin. In particular the unwanted logarithmic singularity at $t = 0$ in the dual peripheral model is only enforced by the simultaneous validity of the relations (9a) and (9b) and not necessarily present if only one of them is required to hold. This makes it obviously interesting to study the one of the two relations which has a chance to be valid beyond the scope of the dual peripheral model. This seems to be the first relation (9a), since this "downward" derivative relation is formally equivalent to the "b-universality" relation

$$f_{\lambda_2,3}(b,s) = \tilde{C}_{\lambda_2,3}(s) b^{\Delta\lambda} f_{\Delta\lambda=0}(b,s) \quad (10)$$

which implies the (naive) "threshold" expectation for small b

$$f_{\lambda_2,3}(b,s) = \int_0^\infty \sqrt{-t} d\sqrt{-t} J_{\Delta\lambda}(b\sqrt{-t}) F_{\lambda_2,3}(s,t) \quad (11)$$

$\tilde{C}_{\lambda_2,3}(s) b^{\Delta\lambda}$
 $b \rightarrow 0$

(from the corresponding behaviour of the $J_{\Delta\lambda}$ Bessel functions).

3. DOMAIN OF VALIDITY OF THE DERIVATIVE RELATIONS

Here we ask the questions:

- i) Which is the domain of validity of the derivative relations?
- ii) What can be said about the helicity weights $C_{\{\lambda_i\}}(s)$ in relation (1)?

For this discussion we shall use the dual peripheral model^{5,6} as a guideline. We find the following two restrictions

- i) the derivative relations (1) should be applied only in the region of large s and $-t \ll s$ (or with $t \not\approx u$: for large s and $-u \ll s$). (12)

This is due to the fact that $G^{(\pm)}(s,t)$ fulfils a derivative relation with ∂^s fixed $\partial/\partial\sqrt{-t}$, whereas $G^{(\pm)}(u,t)$ fulfils one with the differential ∂^u fixed $\partial/\partial\sqrt{-t}$. These two differentials become equal only in the limit of t fixed $s \rightarrow \infty$, where $u \approx -s$. In addition, at very low energies the Hankel function approximation becomes bad and the derivative relation for $G^{(\pm)}(s,t)$ has to be modified (see footnote in Section 2). Thus an application of the derivative relations (1), e.g. in the phase-shift region, is not expected to work. This explains consistently the

failure of the derivative relations applied to finite energy sum rules in Ref. 9, which is only weak for the imaginary parts [since it only involves $G^{(\pm)}(s,t)$] and strong for the real parts [involving also $G^{(\pm)}(u,t)$].

ii) As we have seen, in the dual peripheral model the derivative relations (1) hold for $G^{(+)}(s,t)$ and $G^{(-)}(s,t)$ separately, in general with different helicity weights $C_{\{\lambda_i\}}^{(\pm)}(s)$. This leads us to consider the derivative relations for the amplitudes $F^{(\pm)} = G^{(\pm)}(s,t) \pm G^{(\pm)}(u,t)$ of definite s-u crossing symmetry ($\hat{=}$ definite t-channel signature). However, they only hold for these amplitudes, if the ratios of the helicity weights

$$\frac{C_{\{\lambda_i\}}^{(+)}(s)}{C_{\{\lambda_i\}}^{(+)}(u)} \quad \text{and} \quad \frac{C_{\{\lambda_i\}}^{(-)}(s)}{C_{\{\lambda_i\}}^{(-)}(u)} \quad \text{are helicity independent.} \quad (13)$$

Let us again abstract from the dual peripheral model those parts of the amplitudes $F^{(\pm)}$ for which we may expect this helicity independence (13) to hold. In the model the energy and helicity dependence of the s-channel Regge residues $\beta_{\{\lambda_i\}}^{(\pm)}(s)$ at high energies is given⁶⁾ by the asymptotic expansion

$$\beta_{\{\lambda_i\}}^{(\pm)}(s) \propto \sum_{\substack{\text{terms with different} \\ \text{values for } \alpha_0}} A_{\{\lambda_i\}}^{(\pm)} \left(\frac{-s}{s_0}\right)^{\alpha_0} \left(\log \frac{-s}{s_0}\right)^{\Delta\lambda + 2 - 2\alpha_0} e^{i\frac{\pi}{2}\Delta\lambda} \quad (14)$$

with $A_{\{\lambda_i\}}^{(\pm)}$ being real constants.

Obviously it is suggestive to interpret different contributions to this sum as arising from different "t-channel exchanges" [in particular since

$$iH_{\Delta\lambda}^{(+)}\left(\frac{\alpha(s)+1/2}{\sqrt{s/2}}\sqrt{-t}\right) \underset{-t \text{ large}}{\propto} \frac{1}{\sqrt{\frac{R}{2} \log \frac{-s}{s_0} \sqrt{-t}}} \left(\frac{-s}{s_0}\right)^{-\frac{R}{\pi}\sqrt{-t}} e^{-i\frac{\pi}{2}\Delta\lambda} \quad (15)$$

generates a t-channel Regge-type behaviour with a $\sqrt{-t}$ type effective trajectory; see Refs. 6]. Here a "t-channel exchange"^{*)} is characterized *only* by its t-channel signature ($\pm$) and the (effective) trajectory intercept α_0 .

*) In the t-channel Regge language:

"t-channel exchange" = all Regge poles and Regge cuts with a common signature and intercept and *no further* characteristics like definite naturality, etc.

Since for each of these contributions in Eq. (14) the requirement (13) of helicity independence is fulfilled (but not for their sum), one expects

that in general the derivative relations (1) hold for each "t-channel exchange" separately. (16)

(Only in the case of a pair of exchange degenerate t-channel exchanges can they be treated as one contribution to the amplitude.)

Thus the dual peripheral model asks us the price of treating each "t-channel exchange" separately with in general different helicity weights $C_{\{\lambda_i\}}(s)$; however then it makes us the present of suggesting that these functions actually reduce to real constants:

$$C_{\{\lambda_i\}}^{\text{"t ch. exch."}}(s) = \text{real and independent of } s. \quad (17)$$

This becomes obvious with Eq. (14), the specific form (5) for $\alpha(s)$ and

$$\begin{aligned} \sqrt{-t} \Delta\lambda \frac{\partial}{\partial \sqrt{-t}} \left(H_{\Delta\lambda}^{(1)} \left(\frac{\alpha(s)+1/2}{\sqrt{s/2}} \sqrt{-t} \right) / \sqrt{-t} \Delta\lambda \right) \\ = -i \frac{R}{\pi} \log \frac{s}{s_0} H_{\Delta\lambda+1}^{(1)} \left(\frac{\alpha(s)+1/2}{\sqrt{s/2}} \sqrt{-t} \right). \end{aligned} \quad (18)$$

Clearly the result (17) is a very strong prediction.

As emphasized already, we shall subsequently consider the derivative relations (1) as a property detached from the dual peripheral model. Nevertheless in our following tests we shall take the suggestions (12) and (16) as a hint and use only reactions or data involving one single "t-channel exchange" and only at large enough energies and sufficiently small momentum transfers. We shall, however, keep the freedom of complex and energy-dependent functions $C_{\{\lambda_i\}}(s)$, only to find later on from a comparison with data that they actually turn out to be approximately real and constant in the energy range tested.

4. GENERAL LIMITATIONS

We want to stress here that the derivative relations in the form (1) cannot likely be continued to time-like values of t . As discussed already in Ref. 12, the reasons are as follows:

i) A simple pole at $t = m^2$ in the $\Delta\lambda = 0$ amplitude would become a double pole in the $\Delta\lambda = 1$ amplitude, etc., according to the derivative relations. Probably for $t \leq 0$ this limitation becomes visible only in case of π -exchange reactions, where the pion pole is very close to the physical region. An interesting attempt to deal with the derivative relations in the presence of π exchange can be found in Ref. 12.

ii) In principle the large b -behaviour is governed by the lowest t -channel threshold at $t = t_{th}$ [e.g. $t_{th} = (2 m_\pi)^2$] as

$$f_{\{2,3\}}(b,s) \underset{b \rightarrow \infty}{\sim} e^{-\sqrt{t_{th}} b} \quad (19)$$

This behaviour does not reflect a b -universality (3). The question is, however, At how large values of b ($b > 3$ fermi?) does this behaviour set in? Or in other words, How well is its effect confined to the near neighbourhood of $t \approx t_{th} > 0$ only?

The successful tests of the derivative relations give us strong confidence that all these limitations (except for the π exchange!) do not become effective for $t \leq 0$.

5. METHODS FOR MODEL-INDEPENDENT TESTS OF THE DERIVATIVE RELATIONS

We consider a reaction as suitable for a "model-independent" test

- i) if a completely model-independent amplitude analysis is available (as, for example, in $\pi N \rightarrow \pi N$ at 6 GeV/c); or
- ii) if only an additional weak theoretical bias (as, for example, fixed- t dispersion relations) has entered the amplitude analysis from the data; or finally
- iii) where a model with a sufficiently large number of free parameters gives a perfect fit to an almost complete set of experimental information, thus providing quite reliable amplitudes (as, for example, the Barger and Phillips amplitudes for $\pi N \rightarrow \pi N$).

In any of these cases we shall consider the amplitudes as being given from the data, and we shall check whether they obey the derivative relations without adding any theoretical bias.

For the tests the following two methods seem appropriate.

Method I: differentiating or integrating amplitudes

Let us start from the derivative relations (1) for the helicity amplitudes with definite " t -channel exchange", written in their recursive form ($\Delta\lambda = n = 0, 1, 2, \dots$):

$$F_{n+1} = \hat{C}_n \sqrt{-t}^n \frac{\partial}{\partial \sqrt{-t}} \left(\frac{F_n}{\sqrt{-t}^n} \right), \quad \hat{C}_n = |C_n| e^{i\varphi_n} \quad (20)$$

$$[\hat{C}_n = C_{\{2,3\} \text{ with } \Delta\lambda = n+1} / C_{\{2,3\} \text{ with } \Delta\lambda = n}]$$

In practice we shall perform tests up to $n = 1$, i.e. test the relations $F_{\Delta\lambda=2} \neq F_{\Delta\lambda=1}$ and $F_{\Delta\lambda=1} \neq F_{\Delta\lambda=0}$. Each Eq. (20) consists of two equations, one for the imaginary part and one for the real part.

Since the absolute phases of the amplitudes are not measurable, it is more adequate to rewrite Eqs. (20) equivalently in the following form (with $F_i = |F_i| e^{i\delta_i}$):

$$\frac{\partial}{\partial \sqrt{-t}} \log \frac{|F_n|}{\sqrt{-t}^n} = \frac{1}{|C_n|} \frac{|F_{n+1}|}{|F_n|} \cos(\delta_{n+1} - \delta_n - \varphi_n) \quad (21a)$$

and

$$\frac{\partial}{\partial \sqrt{-t}} \delta_n = \frac{1}{|C_n|} \frac{|F_{n+1}|}{|F_n|} \sin(\delta_{n+1} - \delta_n - \varphi_n). \quad (21b)$$

The Eqs. (21a) involve only measurable quantities, such as moduli of the amplitudes and phase differences; the Eqs. (21b) can be written (by subtractions) such that all but one involve measurables exclusively, and only one is left with an absolute phase on the left-hand side! Thus for a given reaction all but one of the derivative relations (21a) and (21b) allow for a model-independent test.

In practice a differentiation of a quantity with error bars is a delicate procedure! We shall apply it only when the amplitude F_n is very well determined experimentally, as, for example, $F_{\Delta\lambda=0}$ in πN diffraction scattering. Notice that Eqs. (21a and 21b) do not involve a numerical differentiation of a rapid exponential variation in t as Eqs. (20), but at most one of a low-order polynomial in t . This is very valuable for practical applications.

In general we prefer the numerically more stable procedure of integration:

$$\frac{|F_n(t)|}{\sqrt{-t}^n} = \frac{|F_n(t_0)|}{\sqrt{-t_0}^n} \cdot \exp \left\{ \frac{1}{|C_n|} \int_{\sqrt{-t_0}}^{\sqrt{-t}} d\tau \frac{|F_{n+1}|}{|F_n|} \cos(\delta_{n+1} - \delta_n - \varphi_n) \right\} \quad (22a)$$

$$\delta_n(t) = \delta_n(t_0) + \frac{1}{|C_n|} \int_{\sqrt{-t_0}}^{\sqrt{-t}} d\tau \frac{|F_{n+1}|}{|F_n|} \sin(\delta_{n+1} - \delta_n - \varphi_n) \quad (22b)$$

Once again only the numerical integration of a rather weakly varying function is involved.

In either case -- differentiation or integration -- the experimental information for moduli, phase differences and, if available, for absolute phase, is used as input on both sides, t_0 is suitably fixed, and then a simultaneous fit of the right-hand sides to the left-hand sides is performed by varying the common free parameters $|C_n|$ and ϕ_n .

Method II: Laguerre expansion of the amplitudes

The idea is to expand the helicity amplitudes with respect to a suitable complete function system $O_k^{\Delta\lambda}$, $k = 0, 1, \dots$ (at a fixed energy):

$$F_{\lambda_2 i_2} (t) = C_{\lambda_2 i_2} \sum_k f_k O_k^{\Delta\lambda} (t), \quad (23)$$

- such that
- i) $O_k^{\Delta\lambda} (t)$ satisfy the derivative relations for all $k = 0, 1, \dots$
  - ii) all coefficients f_k are *helicity-independent*
  - iii) the series converges rapidly for practical reasons.

The test procedure is then a simultaneous fit of these amplitudes to the available observables such as $d\sigma/dt$, P , and the amplitude analyses with complex parameters f_k , $C_{\lambda_2 i_2}$.

This method is particularly advantageous for reactions where several partly incompatible amplitude analyses exist; it allows for an interpolation between the various analyses and guarantees that $(d\sigma/dt)$ and P are correctly reproduced.

In particular we chose the function system*)

$$O_k^{\Delta\lambda} (t) = \left(-\frac{\partial}{\partial \sqrt{-t}} A(t) \right)^{\Delta\lambda} e^{-A(t)} L_k^{\Delta\lambda} (A(t)) \quad (24)$$

involving the Laguerre polynomials $L_k^{\Delta\lambda}$; it satisfies the derivative relations (1) with $\Delta\lambda = n$ for all $k = 0, 1, 2, \dots$. The condition

$$\frac{\partial}{\partial \sqrt{-t}} A(t) \underset{t \rightarrow 0}{\propto} \sqrt{-t} \quad (25)$$

guarantees angular momentum conservation, and the exponential $e^{-A(t)}$ accounts for the main decrease of the amplitudes, thus providing a good convergence of the series (23).

Remember, however, that parameter saving is only of secondary importance. The parametrization has to be sufficiently flexible in order not to introduce a theoretical bias, in addition to the derivative relations.

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*) First suggested by G. Cohen-Tannoudji (private communication).

6. RESULTS OF THE TESTS

6.1 Test of the " $\Delta\lambda = 0 \leftrightarrow \Delta\lambda = 1$ " relation in the odd-signature (" ρ ") reaction $\pi^-p \rightarrow \pi^0n$

6.1.1 Fixed energy: $p_{\text{lab}} = 6 \text{ GeV}/c$

Since there exist two partly incompatible polarization measurements, and correspondingly several slightly different sets of amplitude analyses, we applied the test method II of a Laguerre expansion of the amplitudes in the specific form (A2.1) given in Appendix 2. Figures 1a and 1b show the best fit solution which is in very good agreement with all data¹³⁻¹⁸⁾ $[(d\sigma/dt), P, \text{ and amplitudes}]$. Notice that the existence of a solution is highly non-trivial. The parameters in Eq. (26) corresponding to this solution are given in Appendix 2. We find furthermore:

$$1/|C| = 2.4 \text{ GeV}^{-1}, \quad \pi - \varphi_C = -0.15 \approx -8.6^\circ \quad (26)$$

i.e. $C(s)$ turns out to be approximately real -- as expected.

Figure 2 shows the amplitudes as resulting from the fit. They fulfil the derivative relations (1) exactly. Two features are remarkable. Firstly the zero structure of the *imaginary parts* seem to be in quite good agreement with the predictions from the *zero positions of the Bessel functions* $J_{\Delta\lambda}(R\sqrt{-t})$ with $R = 1 \text{ fermi}$: for $\Delta\lambda = 0$ the zeroes in Fig. 2 are at $-t = 0.25$ and 1.6 GeV^2 as compared to the predictions of $-t = 0.23$ and 1.2 GeV^2 ; for $\Delta\lambda = 1$ they match exactly the predicted positions of $-t = 0.6$ and 1.9 GeV^2 . Secondly the *flip amplitude*, $F_{\Delta\lambda=1}$, has the welcome feature of *resembling very closely a t-channel (ρ) Regge-pole amplitude*: $\text{Re } F_{\Delta\lambda=1}$ shows a double zero, which is only slightly displaced as compared to the single zero position of $\text{Im } F_{\Delta\lambda=1}$ at $-t \approx 0.6 \text{ GeV}^2$; only at larger values of $-t$ $\text{Re } F_{\Delta\lambda=1}$ does not quite reach the height required by a pure Regge pole. As we know from Ref. 9, a pure Regge-pole behaviour in $F_{\Delta\lambda=1}$ brings trouble for the validity of the derivative relations. Here we learn that already *slight changes* are sufficient to reconcile the derivative relations with the data.

From this analysis we may conclude that there exists an excellent and theoretically very acceptable solution fulfilling the derivative relations exactly.

6.1.2 Energy dependence of $C(s) = 5 \leq p_{\text{lab}} \leq 16 \text{ GeV}/c$

There are available the amplitudes of a) the Hecht-Jakob-Kroll analysis¹⁸⁾ of the πN data involving in addition fixed- t dispersion relations and b) the Barger and Phillips analysis¹⁷⁾ involving in addition experimental low-energy information entering via finite energy sum rules.

We applied test method I using the integrated form (22) with $t_0 = 0$. A good agreement for each energy is found (also in the case of the absolute phase). Figures 3a and 3b show representative examples. The values for $1/|C|$ and $\pi - \phi_C(s)$, which were obtained at each energy, are plotted in Figs. 4a and 4b versus p_{lab} . It appears that in the given energy region, $C(s)$ is to a good approximation real and energy-independent -- in agreement with the predictions of the dual peripheral model.

6.2 Test of the " $\Delta\lambda = 1 \nleftrightarrow \Delta\lambda = 2$ " relation in the odd signature (" $\omega + \rho$ ") reaction $\gamma p \rightarrow \pi^0 p$ at $p_{lab} = 4$ and 6 GeV/c

We chose the weakly model-dependent amplitude analysis by Argyres et al.¹⁹⁾ at $p_{lab} = 4$ GeV/c and the amplitudes at $p_{lab} = 6$ GeV/c from an extensive and very flexible Regge analysis by Worden²⁰⁾ fitting very well the complete experimental information on all $\gamma N \rightarrow \pi N$ processes.

In both cases we applied, once again, test method I in the integrated form (22). The results are shown in Figs. 5a,b. There is good agreement out to $-t \approx 1.5$ GeV². The parameters were determined to be

$$\begin{aligned} 1/|C| &= 6.52 \text{ GeV}^{-1}, \quad \phi_C - \pi = 2.63^\circ \text{ in the Argyres et al. analysis} \\ &\quad \text{(with } t_0 = -0.2 \text{ GeV}^2 \text{)}, \\ \text{and } 1/|C| &= 14.6 \text{ GeV}^{-1}, \quad \phi_C - \pi = 26^\circ \text{ in the Worden analysis} \\ &\quad \text{(with } t_0 = -0.3 \text{ GeV}^2 \text{)}. \end{aligned} \tag{27}$$

The latter result is the only case where we found some deviation from a zero phase.

6.3 Test of the " $\Delta\lambda = 0 \nleftrightarrow \Delta\lambda = 1$ " relation for the even-signature (" A_2 ") contribution to the KN charge exchange reactions at $p_{lab} = 6$ GeV/c

The even-signature amplitudes have been isolated in a weakly model-dependent way by Elvekjaer and Johnson²¹⁾. Test method I in the integrated form (22) with $t_0 = 0$ is applied to these amplitudes. Within the substantial errors a good qualitative agreement has been found, as is shown in Fig. 6. The helicity weight once more turns out to be approximately real:

$$1/|C| = 1.60 \text{ GeV}^{-1}, \quad -\varphi_C = -10.8^\circ \tag{28}$$

6.4 Test of the " $\Delta\lambda = 0 \nrightarrow \Delta\lambda = 1$ " relation for the diffractive $I_t = 0$ ("Pomeron + f") component of πN elastic scattering at $p_{lab} = 6 \text{ GeV/c}$

This is our only example for diffraction scattering. The amplitude analyses^{15-18,22)} at 6 GeV/c are used. All of them agree very well for the extremely well determined modulus of the non-flip amplitude $|F_{\Delta\lambda=0}^{I_t=0}|$. Therefore we used a test method, which essentially amounts to method I in the differentiation version (21a) for $|F_{\Delta\lambda=0}|$ and the integration version (22b) for $\delta_{\Delta\lambda=0}$.

Unfortunately for $-t \gtrsim 0.2 \text{ GeV}^2$ the available amplitude analyses disagree substantially for the very small quantity $|F_{\Delta\lambda=1}| \cos(\delta_{\Delta\lambda=1} - \delta_{\Delta\lambda=0})$. Therefore we used the data for the observables $d\sigma/dt$, P , R , and A directly, and rewrote Eqs. (21a) and (22b) as

$$-2 \operatorname{Re} \{ F_0^0 F_1^{0*} \} = -2 \sqrt{-t} \frac{C}{\cos \varphi_d} |F_0^0|^2 \frac{d}{dt} \log |F_0^0|^2 + \operatorname{tg} \varphi_d (P d\sigma/dt)^{(0)} \quad (29a)$$

$$\delta_0^0(t) = \delta_0^0(0) + \frac{1}{|C| \cos \varphi_d} \int_0^{\sqrt{-t}} d\tau \left(\frac{P}{2} \frac{d\sigma}{d\tau} \right)^{(0)} / |F_0^0|^2 \quad (29b)$$

with²³⁾

$$(P d\sigma/dt)^{(0)} = -\operatorname{tg} \varphi_d \log \frac{|F_0^0(t)|}{|F_0^0(0)|} = \frac{1}{2} \{ (P d\sigma/dt)^{\pi^+P} + (P d\sigma/dt)^{\pi^-P} - (P d\sigma/dt)^{\pi^-P \rightarrow \pi^0 n} \}. \quad (30)$$

The quantity $-2 \operatorname{Re} \{ F_0^0 F_1^{0*} \}$ was calculated from the data²⁴⁾ for the R and A parameters according to the following relation:

$$\begin{aligned} -2 \operatorname{Re} \{ F_0^0 F_1^{0*} \} &= \frac{1}{2} \left[(R \frac{d\sigma}{dt})^{\pi^+P} + (R \frac{d\sigma}{dt})^{\pi^-P} \right] \sin \theta_N \\ &+ \frac{1}{2} \left[(A \frac{d\sigma}{dt})^{\pi^+P} + (A \frac{d\sigma}{dt})^{\pi^-P} \right] \cos \theta_N \\ &+ 2 \operatorname{Re} (F_0^1 F_1^{1*}), \end{aligned} \quad (31)$$

with $\sin \theta_N = \cos(\theta_{cms}/2)/\sqrt{1 - t/4m_N^2}$, θ_N = nucleon recoil angle in the lab. system.

The term $2 \operatorname{Re} \{F_0^1 F_1^{1*}\}$ in Eq. (31) constitutes only a small correction; it has been taken from the Saclay amplitude analysis¹⁶⁾. For $-t < 0.2 \text{ GeV}^2$ we used the amplitude analyses of Refs. 18 and 22 to calculate $-2 \operatorname{Re} \{F_0^0 F_1^{0*}\}$.

Figures 7a and 7b show the result of a simultaneous fit of the right-hand sides of Eqs. (29a,b) to their left-hand sides with $t_0 = 0$ and $|C|$ and ϕ_c as free parameters, which were found to be

$$1/|C| = 42.6 \text{ GeV}^{-1}, \quad \phi_c = -6.5^\circ. \quad (32)$$

Once more we find an almost zero phase ϕ_c . The derivative relations seem to work very well also in diffraction scattering.

Figure 7c is to illustrate how the derivative rules reproduce well also the fine structures: With $\phi_c \approx 0$ we obtain from Eq. (29b) the relation

$$\frac{\partial}{\partial \sqrt{-t}} \delta_0^0 \propto \left(\frac{P}{2} \frac{d\sigma}{dt} \right)^{(0)} / |F_0^0|^2 \quad (33)$$

which predicts an extremum for δ_0^0 if the observable $\frac{1}{2}[P(d\sigma/dt)]^{(0)} / |F_0^0|^2$ vanishes. In fact this quantity shows a zero at $-t \approx 0.8 \text{ GeV}^2$ (see Fig. 7c), and consistently δ_0^0 indicates a maximum in the same t -region (see Fig. 7b).

7. HOW POWERFUL ARE THE DERIVATIVE RELATIONS?

The preceding tests give us some confidence in the derivative relations. Let us now go through the exercise to assume that they hold well in general and to investigate how powerful they could turn out to be for amplitude analyses. This implies in particular the question, Which minimal (incomplete) set of experimental observables is necessary to yield -- in conjunction with the derivative relations -- a unique solution?

For simplicity let us study the case of a $0\frac{1}{2} \rightarrow 0\frac{1}{2}$ reaction with a definite "t-channel exchange", and let us assume the helicity weight $C(s) = C$ to be real. It is easily seen that the derivative relations (1) imply a *non-linear* differential equation for the quantity

$$Y = |F_{\Delta\lambda=0}(s,t)|^2 \quad (34)$$

involving the two observables $d\sigma/dt$ and P as input:

$$\frac{C^2}{4} Y'^2 + Y^2 - \frac{d\sigma}{dt} Y + \left(\frac{P}{2} \frac{d\sigma}{dt} \right)^2 = 0 \quad (35)$$

with

$$y' = \frac{\partial}{\partial \sqrt{-t}} y \quad \text{and} \quad y(0) = \frac{d\sigma}{dt} \Big|_{t=0}$$

and the boundary condition

$$0 \leq \frac{1}{2} \frac{d\sigma}{dt} (1 - \sqrt{1-P^2}) \leq y \leq \frac{1}{2} \frac{d\sigma}{dt} (1 + \sqrt{1-P^2}) \quad (36)$$

(from $-1 \leq \cos(\delta_1 - \delta_0) \leq 1$) ;

$\delta_0(t)$ is obtained from $d\sigma/dt$, P , and the solution y of the differential equation (35) by simple integration

$$\delta_0(t) = \delta_0(0) + \frac{1}{|C|} \int_0^{\sqrt{-t}} d\tau \frac{P}{2} \frac{d\sigma}{dt} \Big|_{y(\tau)}. \quad (37)$$

The main question now is, For which values of C does differential equation (35) have acceptable, i.e. physical, solutions? To find a general answer seems impossible, since $d\sigma/dt$ and P are given in terms of data and not as analytic functions. It is, however, quite probable that the manifold of solutions is substantially reduced owing to the *non-linearity* of the differential equation in conjunction with the *boundary condition* (36). In other words, one is facing an *eigenvalue problem* here.

For us the most interesting cases are

- i) there is no physical solution at all for any C [as, for example, if Eq. (35) has only solutions with $y < 0$]; the fact that this possibility exists shows how strong a constraint the derivative relations may be;
- ii) there is a unique solution given in terms of one eigenvalue C .

We used $\pi^- p \rightarrow \pi^0 n$ in order to get a feeling for the situation in practice: A Laguerre expansion [see Eq. (A2.1)] type fit to $d\sigma/dt$ and P only (without the additional information contained in the amplitude analyses) yields firstly only one solution for $1/C \approx 2.4 \text{ GeV}^{-1}$, and secondly this solution is identical (within the errors) to the one obtained from a fit to the complete experimental information in Section 6.1. This is a *hint* that the case (ii) with a *unique solution* to the *eigenvalue problem* is the one which we might expect more generally.

8. SUMMARY AND OUTLOOK

We discussed the derivative relations for s-channel helicity amplitudes:

$$F_{\lambda_i j}(s, t) = C_{\lambda_i j}(s) \sqrt{-t}^{\Delta\lambda} \left(\frac{1}{\sqrt{-t}} \frac{\partial}{\partial \sqrt{-t}} \right)^{\Delta\lambda} F_{\Delta\lambda=0}(s, t), \quad (38)$$

$$(\Delta\lambda = |\lambda_c - \lambda_d - \lambda_a + \lambda_b|)$$

which are motivated from the dual peripheral model^{5,6)} (an s-channel Regge model), but which seem to hold beyond its limitations.

From the dual peripheral model we abstracted two constraints on the derivative relations:

- i) kinematical restriction: s large, $-t \ll s$ (and for $t \neq u$: s large, $-u \ll s$);
  - ii) validity only for amplitudes with definite "t-channel exchange", characterized by t-channel signature and effective trajectory intercept (and nothing more!);
- and the expectation that the functions

$$C_{\{\lambda_i\}}(s) \text{ are energy-independent and real.} \quad (39)$$

One main aim of this paper was to propose methods for model-independent tests of the derivative relations, and to perform such tests in case of the reactions $\pi^- p \rightarrow \pi^0 n$, $(\pi^\pm p \rightarrow \pi^\pm p)^{I_{t=0}}$, $\pi^- p \rightarrow \pi^0 n$, KN CEX, and $\gamma p \rightarrow \pi^0 p$, where sufficient experimental information is available. In all these cases we found the derivative relations to be very well and, in part, excellently fulfilled. This agreement appears to hold even on the level of the fine structures in the data, which seems highly non-trivial to us. Moreover, we find the functions $C_{\{\lambda_i\}}(s)$ to be approximately constant and real -- as far as tested.

This whole testing effort has been undertaken in order to get as much confidence as is at present possible into these interesting equations, which relate all helicity amplitudes to a single one.

As future steps, one might consider the following:

- i) To perform *amplitude analyses* with incomplete experimental information in combination with the derivative relations. Their power for amplitude analyses of this type has been pointed out in Section 7.
- ii) Since the derivative relations seem to hold so well and also quite generally for non-diffraction as well as diffraction, one might speculate on a more general origin. They might, for example, be due to an *asymptotic symmetry*. Just as the

symmetry under the rotation group manifests itself in the *independence* of the partial wave amplitudes $f_J(s)$ on M (the third component of J) in the Legendre expansion

$$F(s, t) \propto \sum_J (J + \frac{1}{2}) f_J(s) P_J(\cos \theta) \quad (40)$$

the *independence* of the coefficients f_K on $\Delta\lambda$ in the Laguerre expansion

$$F_{12;3}(s, t) = G_{12;3}(s) \sum_k f_k \left(-\frac{\partial}{\partial \sqrt{-t}} A(t) \right)^{\Delta\lambda} e^{-A(t)} L_k^{\Delta\lambda}(A(t)) \quad (41)$$

may be due to an (asymptotic) symmetry group ($s \rightarrow \infty$, t fixed).

APPENDIX 1

IMPORTANCE OF MODEL INDEPENDENT TESTS

Let us add a short discussion on why we consider the addition of model assumptions as being so dangerous -- or, in other words -- why we are convinced that only a model-independent test can do full justice to the derivative relations.

The reason is the following: a slight modification in the analytic form of the non-flip amplitude may blow up, via the derivation, and give a considerable effect in the flip amplitude. This is best illustrated by means of an over-simplified example.

Let us consider a flip-dominated $0\frac{1}{2} \rightarrow 0\frac{1}{2}$ reaction of the "rotating type" such as, for example, $K^-p \rightarrow \bar{K}^0 n$. Let us furthermore assume a pure Regge behaviour for the flip amplitude:

$$F_{\Delta\lambda=1} = A \left(\frac{s}{s_0}\right)^{\alpha(t)} e^{-i\pi\alpha(t)} \frac{\partial \alpha(t)}{\partial \sqrt{-t}}. \quad (A1.1)$$

According to the derivative rule (1) we find

$$F_{\Delta\lambda=0} = \frac{A}{C} \left(\frac{s}{s_0}\right)^{\alpha(t)} e^{-i\pi\alpha(t)} \frac{1}{\log \frac{s}{s_0}}. \quad (A1.2)$$

Near $t = 0$ we have to require that $\alpha(t)$ is linear in t in order that $\partial\alpha(t)/\partial\sqrt{-t}$ guarantees angular momentum conservation. This is very welcome. Let us now discuss the consequences at larger values of $-t$ ($-t \gtrsim 0.25 \text{ GeV}^2$) of i) a linear form for $\alpha(t)$, and ii) a $\sqrt{-t}$ type behaviour as suggested by the dual peripheral model. We find the following behaviours for the polarization (for, say, $-t \gtrsim 0.25 \text{ GeV}^2$):

$$\text{i) } P \equiv \frac{U \cdot \sqrt{-t}}{V + W \cdot \sqrt{-t^2}} ; \quad \begin{array}{l} U, V, W \text{ being independent of} \\ t \text{ and } W \gtrsim V \text{ because of flip} \\ \text{dominance} \end{array} \quad (A1.3)$$

$$\propto \frac{1}{\sqrt{-t}}, \text{ for } \sqrt{-t} \text{ large}$$

$$\text{ii) } P \text{ independent of } t \quad (A1.4)$$

The characteristic difference between the two polarizations (44) and (45) are exclusively due to the different forms assumed for the trajectory $\alpha(t)$. Thus already the weak bias for the analytic form of $\alpha(t)$ fixes to a large extent the

predicted structure of the observables, unless a large number of parameters in the t -dependence of the amplitudes allows for additional freedom!

Obviously if a certain economical parametrization for the amplitudes, which incorporates the derivative relations, fails to describe the data, one cannot conclude that the derivative relations are not fulfilled (cf. Ref. 9). It may well be due to a too restricted parametrization.

APPENDIX 2

LAGUERRE EXPANSION OF $\pi^- p \rightarrow \pi^0 n$ AMPLITUDES AT 6 GeV/c

We started from the Laguerre expansion given by Eqs. (23)-(25). Taking the s-u crossing antisymmetry of $\pi^- p \rightarrow \pi^0 n$ into account, we finally parametrized the amplitudes in the following suggestive way [with $\alpha(t) = \alpha_0 + \alpha' t$]:

$$F_{\Delta\lambda=0,1}^{\pi N, \text{CEX}} = -\sqrt{2} \left[|C| e^{i\phi_c} \frac{2}{\partial\sqrt{-t}} \alpha(t) \right]^{\Delta\lambda} \left(\frac{s}{s_0} \right)^{\alpha(t)-1} \times \left\{ e^{-i\pi\alpha(t)} \left(\log \frac{-s}{s_0} \right)^{\Delta\lambda} \sum_{k=0}^N (a_k + i b_k) L_k^{\Delta\lambda} \left([\alpha_0 - \alpha(t)] \log \frac{-s}{s_0} \right) - \left(\log \frac{s}{s_0} \right)^{\Delta\lambda} \sum_{k=0}^N c_k L_k^{\Delta\lambda} \left([\alpha_0 - \alpha(t)] \log \frac{s}{s_0} \right) \right\} \quad (\text{A2.1})$$

with real parameters a_k , b_k , and c_k . Notice, however, that the energy dependence in this ansatz is by no means expected to be more than a first approximation, providing the main contributions to the phase.

$$\frac{d\sigma}{dt}^{\text{CEX}} [\text{mb/GeV}^2] = |F_0|^2 + |F_1|^2, \quad (\text{A2.2})$$

$$P = \frac{-2 \text{Im}(F_0 F_1^*)}{|F_0|^2 + |F_1|^2},$$

$$N = 8, \quad \alpha_0 = 0.769, \quad \alpha' = 1.002 \text{ GeV}^{-2}, \quad s_0 = 0.189 \text{ GeV}^2,$$

$$1/|C| = 2.4 \text{ GeV}^{-1}, \quad \phi_c \pm \pi = 0.15 \approx 8.6^\circ,$$

and

k	a_k	b_k	c_k
0	-0.508	1.432	1.581
1	0.430	-0.776	0.569
2	-0.0843	0.295	-1.235
3	0.0054	0.153	0.700
4	-0.0251	-0.0359	-0.0963
5	-0.0311	-0.0309	0.561
6	-0.0069	0.0169	-0.240
7	-0.0061	0.0139	-0.597
8	-0.0059	-0.0212	0.421

a_k, b_k, c_m in $[\sqrt{\text{mb}}/\text{GeV}]$

This is the solution belonging to Figs. 1 and 2.

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Figure captions

- Fig. 1 : A model-independent test of the derivative relation $F_1 = C(\partial/\partial\sqrt{-t})F_0$ for the s-channel helicity amplitudes $F_{\Delta\lambda=0,1}$ in $\pi^-p \rightarrow \pi^0n$ at 6 GeV/c. The curves are the result of a fit of the amplitudes -- being parametrized as a sum (A2.1) of properly weighted Laguerre polynomials such as to fulfil automatically the derivative relation -- to the complete experimental information available; a) contains $d\sigma/dt$ and P (data from Refs. 13 and 14), b) contains the amplitude analyses of Halzen and Michael¹⁵⁾ [■], Saclay¹⁶⁾ [●], Hecht, Jakob and Kroll¹⁸⁾ (O), and the Barger and Phillips amplitudes¹⁷⁾ (Δ); the latter two analyses also provide the absolute phase $\delta_{\Delta\lambda=0}^{I_t=1}$; for the first two it was extracted by using the Barger and Phillips $I_t = 0$ phase $\delta_{\Delta\lambda=0}^{I_t=0}$ as an input.
- Fig. 2 : The $\pi N, I_t = 1$ helicity amplitudes $F_{\Delta\lambda=0,1}^1$ as they result from the fit shown in Fig. 1. They fulfil the derivative relation $F_1 = C(\partial/\partial\sqrt{-t})F_0$ exactly.
- Fig. 3 : A model-independent test of the derivative relation $F_1 = C(s)(\partial/\partial\sqrt{-t})F_0$ in $\pi^-p \rightarrow \pi^0n$ at various energies. The full curves represent the left-hand sides (l.h.s.) $|F_{\Delta\lambda=0}|$ and $\delta_{\Delta\lambda=0}^1$ of the integrated form (22) of the derivative relations; the broken lines are the result of a fit of the r.h.s. to the l.h.s. with the two free parameters $|C|$ and ϕ_c ; a) shows the results with the Hecht-Jakob-Kroll amplitude analysis¹⁸⁾ at representative energies 5, 10, and 14 GeV/c; b) those with the Barger and Phillips amplitudes¹⁷⁾ at 5, 11, and 16 GeV/c.
- Fig. 4 : The energy dependence of the function $C(s)$ in $\pi^-p \rightarrow \pi^0n$ as obtained model independently, point by point, by means of the procedure described in the caption of Fig. 3; a) from the Hecht-Jakob-Kroll amplitude analysis¹⁸⁾ with $4 \leq p_{lab} \leq 14$ GeV/c; b) from the Barger and Phillips amplitudes¹⁷⁾ with $4 \leq p_{lab} \leq 16$ GeV/c. $1/|C(s)|$ seems to soon reach a constant value and $\pi - \phi_c(s)$ remains small.
- Fig. 5 : A model-independent test of the derivative relation $F_2 = C \cdot \sqrt{-t}(\partial/\partial\sqrt{-t}) \times (F_1/\sqrt{-t})$ for the s-channel helicity amplitudes $F_{\Delta\lambda=2,1}$ in $\gamma p \rightarrow \pi^0p$: a) at 4 GeV/c using the almost model independent amplitude analysis of Argyres et al.¹⁹⁾ for $0 \leq -t \leq 0.7$ GeV²; and b) using the amplitudes from a flexible Regge analysis of Worden²⁰⁾ for $0 \leq -t \leq 1.5$ GeV². Once again the full curves represent the l.h.s., $|F_{\Delta\lambda=1}|$ and $\delta_{\Delta\lambda=1}$, of the integrated form (22) of the derivative relations; the broken lines are the result of a fit of the r.h.s. with the two free parameters $|C|$ and ϕ_c .

Fig. 6 : A model independent test of the derivative relation $F_1 = C(\partial/\partial\sqrt{-t})F_0$ for the A_2 -exchange helicity amplitudes $F_{\Delta\lambda=0,1}$ as obtained from the KN charge exchange reactions at 6 GeV/c in Ref. 21. The full and broken lines are obtained as described in the caption of Fig. 3.

Fig. 7 : A model-independent test of the derivative relation $F_1 = C(\partial/\partial\sqrt{-t})F_0$ in the case of diffraction scattering $(\pi^\pm p \rightarrow \pi^\pm p)^{I_{t=0}}$. The derivative relations have been rewritten in the form (29). The left-hand sides a) $-2 \operatorname{Re} \{F_{\Delta\lambda=0}^{I_{t=0}} F_{\Delta\lambda=1}^{I_{t=0}*}\}$ given for $0 \leq -t \leq 0.2 \text{ GeV}^2$ from the amplitude analyses of Pietarinen²²⁾ [■] and Hecht-Jakob-Kroll¹⁸⁾ [▲], and for $0.2 \leq -t \leq 0.5 \text{ GeV}^2$ given in terms of measurables^{23,24)} $(d\sigma/dt, P, R, A)$ [◆], and b) $\delta_{\Delta\lambda=0}^{I_{t=0}}$ given from the amplitude analyses of Pietarinen²²⁾ [■], Hecht-Jakob-Kroll¹⁸⁾ [▲], Barger and Phillips¹⁷⁾ [△], and Mc Clure and Pitts²⁵⁾ [○], are fitted to the right-hand sides (with common free parameters $|C|$ and ϕ_c), a) solid line, b) [◆], c) illustrates that the quantity $\frac{1}{2}[P(d\sigma/dt)]^{(0)}/|F_{\Delta\lambda=0}^{I_{t=0}}|^2$ shows a zero position at the same place ($-t \approx 0.8 \text{ GeV}^2$), where $\delta_{\Delta\lambda=0}^{I_{t=0}}$ has a maximum, see (b), as required by the derivative relation (29b).

FIG.1a

$\pi N, I_t=1$, amplitude analyses, 6 GeV/c

FIG. 1b

$\pi N, I_t = 1$ helicity amplitudes satisfying derivative relations

FIG. 2

$\pi N, I_t = 1$, Hecht - Jakob - Kroll amplitude analysis

—— original $\Delta\lambda = 0$, — — — generated from $\Delta\lambda = 1$

FIG. 3a

πN , $I_t = 1$, Barger - Phillips amplitudes

— original $\Delta\lambda = 0$; - - - generated from $\Delta\lambda = 1$

FIG.3 b

Energy dependence of C

a) Hecht-Jakob-Kroll amplitude analysis

b) Barger-Phillips amplitudes

FIG. 4

FIG. 5a

FIG. 5 b

A_2 exchange in KN CEX at 6 GeV/c

FIG. 6

FIG. 7c