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DIMENSIONAL REGULARIZATION OF MASSLESS THEORIES
IN SPHERICAL SPACE-TIME

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A B S T R A C T

The use of space-time curvature as an infra-red cut-off has been suggested for massless theories. In this paper we investigate the renormalization of massless theories in a spherical space-time (Euclidean version of de Sitter space) using dimensional regularization. Naive expectations are confirmed, namely that the coupling constant and wave function renormalizations are independent of the curvature. Furthermore the curvature does not induce divergent mass terms or vacuum field values as would be possible on purely dimensional grounds. Although we have investigated only scalar field theories, ϕ^4 theory in four dimensions and ϕ^3 theory in six, these results are encouraging for an application of the method to gauge theories.

Formally massless theories are conformally invariant so the formulation of the theory in a spherical space ought to be equivalent to its formulation in flat space. In fact the renormalization procedure breaks conformal invariance and removes this equivalence. We show that to achieve the flat space limit it is necessary to invoke the aid of the renormalization group. Thus the zero curvature limit can be achieved for infra-red stable theories (ϕ^4) but not for infra-red unstable theories (ϕ^3) as might be expected.

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1. INTRODUCTION

Massless theories are interesting in both the ultraviolet and infrared limits. Their short distance behaviour is expected to be relevant to field theories in general¹⁻³⁾, while the presence of infrared divergences poses a problem of considerable theoretical interest. Yang-Mills theory⁴⁾ is an important case in point.

The problem in studying infrared behaviour is to find a tractable method of introducing a cut-off at large distances. Different methods can be characterized, in part, by the invariance groups they preserve. For example, the simple approach of confining the fields to a box will completely destroy Poincaré invariance. An alternative, which preserves this group, is the introduction of a mass term for the fields. However, this may involve giving up gauge invariance or even renormalizability in the case of Yang-Mills theory.

By contrast, the infrared cut-off introduced by giving space-time a constant non-zero curvature leaves the theory with a large invariance group (de Sitter) and is compatible with gauge invariance. The investigation of field theory in a de Sitter universe has, moreover, a twofold interest, since it is also a step towards understanding field quantization and renormalization in large-scale gravitational fields.

Adler⁵⁾ has already exploited the advantages of spherical space-time in evaluating a certain class of conformally invariant diagrams in massless electrodynamics. In this paper, we investigate the structure of the renormalization constants in scalar field theories using the technique of dimensional regularization^{6,7)}. The simplicity and precision with which this technique exhibits the relationship between bare and renormalized quantities allows us to confirm what we should naively expect, namely that the wave function and coupling constant renormalizations are completely unaffected by the curvature of space-time. What is less obvious, but nevertheless true, is that the curvature does not induce divergent mass terms or vacuum field values in the Lagrangian. These results are encouraging for any future application of the method to gauge theories.

As a matter of technical convenience, we shall investigate Euclidean versions of field theory. This means that the de Sitter invariance is replaced by the rotational invariance of a compact spherical space-time. We do not expect the renormalization constants to be affected by this manoeuvre. The de Sitter version of the theory can be recovered, we presume, by some appropriate analytic continuation of the Green's functions. However, we do not pursue this question in this paper.

2. SPHERICAL FORMULATION

In order to arrive at the correct spherical formulation of our massless theories, we shall exploit their (formal) conformal invariance. We suppose that, in the first instance, the theory is formulated on a flat space-time, which we think of as an n -dimensional plane in an $(n+1)$ -dimensional space. Elements of the conformal group in $(n+1)$ -space will in general, map the plane into a sphere (we shall always discuss Euclidean theories). More generally they map spheres into spheres. The conformal group in n -space is that subgroup which maps the space-time plane into itself. Other isomorphic subgroups map specific spheres into themselves. Because the field theories we are considering are invariant under the n -dimensional conformal group they can be given a formulation which is covariant under the $(n+1)$ -dimensional group.

For these reasons, we may regard the spherical formulation as being the most general and natural one for massless theories. The flat space formulation may be retrieved either by using a suitable element of the $(n+1)$ -dimensional group or by letting the radius of the sphere become infinite. It is important to remember that these two operations are not equivalent for the renormalized theories.

The choice of transformation which takes space-time into a sphere is, for many purposes, arbitrary. However, we shall use the standard transformation⁵⁾ $x \rightarrow \eta$, where x and η are $(n+1)$ -vectors,

$$\eta = Rh + 2 \frac{x - h x^2/R^2}{1 - 2h \cdot x/R + x^2/R^2}, \quad (2.1)$$

and h is a unit vector in the $(n+1)$ -direction. Since Eq. (2.1) implies that

$$\eta^2 - R^2 = 4Rh \cdot x / (1 - 2h \cdot x/R + x^2/R^2) \quad (2.2)$$

it follows that the space-time plane

$$x \cdot h \equiv x_{n+1} = 0,$$

is mapped into the sphere $\eta^2 = R^2$. On the sphere we find

$$\eta_{n+1} = R \frac{1 - x^2/R^2}{1 + x^2/R^2} \quad (2.3)$$

$$\eta_\mu = \frac{2 x_\mu}{1 + x^2/R^2}$$

We easily check the standard results that

$$(\eta - \eta')^2 = KK'(x - x')^2, \quad (2.4)$$

where

$$K = \frac{2}{1+x^2/R^2} = 1 + \eta_{n+1}/R, \quad (2.5)$$

and that the Jacobian of the transformation is

$$\frac{\partial(\eta)}{\partial(x)} = K^{n+1}. \quad (2.6)$$

The volume of the space-time element is, of course, related to the surface of the sphere by

$$d^n x = dA / K^n. \quad (2.7)$$

Space-time derivatives ∂_μ ($\mu = 1, 2, \dots, n$) are mapped into transverse derivatives along the sphere. These are

$$D_a = \left(\delta_{ab} - \frac{\eta_a \eta_b}{R^2} \right) \frac{\partial}{\partial \eta_b} \quad (2.8)$$

that is

$$D_a = \frac{1}{R^2} \eta_b L_{ba}, \quad (2.9)$$

where

$$L_{ab} = \eta_a \frac{\partial}{\partial \eta_b} - \eta_b \frac{\partial}{\partial \eta_a}. \quad (2.10)$$

It is easily checked that

$$D_a^2 = \frac{1}{2R^2} L_{ab}^2 = \frac{1}{2R^2} L^2, \quad (2.11)$$

and that

$$L_{\mu\nu} = x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} \quad (2.12)$$

$$L_{n+1,\mu} = \frac{1}{R} x_\mu x_\nu \frac{\partial}{\partial x_\nu} + \frac{1-x^2/R^2}{2} R \frac{\partial}{\partial x_\mu}.$$

The spherical operator corresponding to the D'Alembertian ∂^2 , is M , where

$$M = D^2 - \frac{n(n-2)}{4R^2} = \frac{L^2 - \frac{1}{2}n(n-2)}{2R^2}. \quad (2.13)$$

Equations (2.12) allow us to show that it has the property

$$M \Phi = \frac{1}{K^{\frac{n}{2}+1}} \partial^2 \phi, \quad (2.14)$$

where

$$\Phi = \frac{1}{K^{\frac{n}{2}-1}} \phi. \quad (2.15)$$

Equation (2.15), therefore, gives the transformation law for a scalar field ϕ into its spherical form Φ . For example, the kinetic part of the action in spherical form is

$$\frac{1}{2} \int dA \Phi M \Phi = \frac{1}{2} \int d^n x \phi \partial^2 \phi. \quad (2.16)$$

The propagator for a free massless field on the sphere is $D_{\eta\eta'}$, which obeys

$$M D_{\eta\eta'} = \delta_{\eta\eta'}, \quad (2.17)$$

and the δ -function is normalized so that

$$\int dA \delta_{\eta\eta'} = 1. \quad (2.18)$$

It follows that

$$D_{\eta\eta'} = - \frac{B_n}{|\eta - \eta'|^{n-2}}, \quad (2.19)$$

and

$$B_n = \Gamma(\frac{n}{2} - 1) / (4\pi^{n/2}). \quad (2.20)$$

The two interactions we shall consider are $(\lambda_0/4!)\phi^4$ and $(g_0/3!)\phi^3$. In terms of Φ these are

$$K^{2n-4} \frac{\lambda_0}{4!} \Phi^4 \quad \text{and} \quad K^{\frac{3n}{2}-2} \frac{g_0}{3!} \Phi^3.$$

The spherical form of the action is then

$$\int dA \left\{ \frac{1}{2} \Phi M \Phi - K^{n-4} \frac{\lambda_0}{4!} \Phi^4 - K^{\frac{n}{2}-3} \frac{g_0}{3!} \Phi^3 \right\} \quad (2.21)$$

If we recall that $\kappa = (1 + \eta_{n+1}/R)$, we see that (of course) the theory is rotationally invariant on the sphere when $n = 4$ if we retain only the Φ^4 interaction and when $n = 6$ if we retain only the Φ^3 interaction.

In later sections we investigate the renormalization of these theories. However, it is already obvious that just as renormalization breaks conformal invariance in the flat space version of the theory, so it will in the spherical

formulation. Since the equivalence of the two formulations depends on the conformal invariance, this too is removed by renormalization. The invariance group, which is retained, differs in the two versions. In flat space we have Poincaré invariance, in spherical space rotational (de Sitter) invariance. We would like to emphasize that from a mathematical point of view one version is no more to be preferred than another.

3. SPHERICAL HARMONICS

In field theory calculations it is often convenient to work in momentum space. The corresponding transform space in our case uses spherical harmonics⁸⁾ $\{Y_{\alpha}^N(\eta)\}$ in $(n+1)$ -dimensions and is rather more complicated. We collect together in this section some useful information on spherical harmonics.

We shall normalize the $\{Y_{\alpha}^N(\eta)\}$ so that

$$\int dA Y_{\alpha}^N(\eta) Y_{\alpha'}^{N'}(\eta) = \delta_{NN'} \delta_{\alpha\alpha'}, \quad (3.1)$$

and

$$\sum_{N, \alpha} Y_{\alpha}^N(\eta) Y_{\alpha}^N(\eta') = \delta_{\eta\eta'}. \quad (3.2)$$

The harmonics satisfy the eigenvalue equation

$$M Y_{\alpha}^N(\eta) = \lambda_N Y_{\alpha}^N(\eta), \quad (3.3)$$

where

$$\lambda_N = -\left(N + \frac{n}{2} - 1\right)\left(N + \frac{n}{2}\right)/R^2. \quad (3.4)$$

The label " α " may be chosen in a variety of ways (we shall not need to specify which one) and has a range of $h(N, n)$ different values, where

$$h(N, n) = \frac{(2N+n-1)\Gamma(N+n-1)}{\Gamma(n)\Gamma(N+1)}. \quad (3.5)$$

The harmonics are related to the Gegenbauer polynomials,

$$\sum_{\alpha} Y_{\alpha}^N(\eta) Y_{\alpha}^N(\eta') = F(N, n) C_N^{\frac{n-1}{2}}(z), \quad (3.6)$$

where

$$F(N, n) = (2N+n-1)\Gamma\left(\frac{n-1}{2}\right)/(4R^n \pi^{n/2}), \quad (3.7)$$

with $z = \eta \cdot \eta' / R^2$.

We shall need the formula⁸⁾

$$\int_{-1}^1 dz (1+z)^{\frac{n}{2}-1} (1-z)^{\beta} C_N^{\frac{n-1}{2}}(z) = \frac{2^{\beta+\frac{n}{2}} \Gamma(\beta+1) \Gamma(\frac{n}{2}) \Gamma(N+n-1) \Gamma(N+\frac{n}{2}-\beta-1)}{\Gamma(N+1) \Gamma(n-1) \Gamma(N+\frac{n}{2}+\beta+1) \Gamma(\frac{n}{2}-\beta-1)} \quad (3.8)$$

This allows us to deduce that

$$(1-z)^{\nu} = \sum_N a_N C_N^{\frac{n-1}{2}}(z), \quad (3.9)$$

where

$$a_N = \frac{2^{\nu+n-2} (2N+n-1) \Gamma(\frac{n-1}{2}) \Gamma(\nu+\frac{n}{2}) \Gamma(N-\nu)}{\sqrt{\pi} \Gamma(N+n+\nu) \Gamma(-\nu)} \quad (3.10)$$

We can see immediately from Eqs. (3.2) and (3.3) that

$$D_{\eta\eta'} = \sum_{N,\alpha} \frac{1}{\lambda_N} \gamma_{\alpha}^N(\eta) \gamma_{\alpha}^N(\eta') \quad (3.11)$$

However, we could also have obtained this result from Eq. (3.10) with $\nu = 1 - (n/2)$.

More generally, we have the very useful result

$$[(\eta-\eta')^2]^{\nu} = \sum_{N,\alpha} \frac{(2R)^{2\nu+n} \pi^{\frac{n}{2}} \Gamma(\nu+\frac{n}{2}) \Gamma(N-\nu)}{\Gamma(N+n+\nu) \Gamma(-\nu)} \gamma_{\alpha}^N(\eta) \gamma_{\alpha}^N(\eta') \quad (3.12)$$

4. RENORMALIZATION OF Φ^4 THEORY

The Green's functions of Φ^4 theory are generated by the functional

$$Z[J] = \int d\Phi \exp \int dA \left\{ \frac{1}{2} \Phi M \Phi - \frac{\lambda_0}{4!} \Phi^4 - J \Phi \right\} \quad (4.1)$$

The Feynman rules for the perturbation series are the standard ones, namely

- i) a factor of $(-D_{\eta_1 \eta_2})$ for each line joining the points η_1 and η_2 ,
- ii) a factor of $(-\lambda_0)$ for each vertex,
- iii) standard symmetry factors,
- iv) integration over the positions of internal vertices.

Since the divergences of the diagrams occur when the integration points coincide, the degree of divergence is independent of the curvature of space-time. The simplest divergent loop diagrams for Φ^4 theory in four dimensions are shown in Fig. 1.

We shall adopt the renormalization procedure of 't Hooft and Veltman^{6,7)}. In the present case, this means we shall suppose that the spherical Feynman rules hold in n -dimensions and evaluate the diagrams for a range of n for which they are finite. We then substitute for the bare coupling constant the series⁷⁾

$$\lambda_0 = \mu^{4-n} \left(\lambda + \sum_1^{\infty} \frac{a_\nu(\lambda)}{(n-4)^\nu} \right), \quad (4.2)$$

where μ is an arbitrary renormalization mass, and extract a field renormalization factor $Z^{\frac{1}{2}}$ for each external leg, where

$$Z = 1 + \sum_1^{\infty} \frac{c_\nu(\lambda)}{(n-4)^\nu}. \quad (4.3)$$

The coefficients a_ν and c_ν are power series in λ , which are adjusted so that when the renormalized Green's functions are continued in n back to the point $n = 4$, they remain finite. Our aim is to evaluate a_ν and c_ν for the spherical theory, at least to low order in λ . We shall find that they are identical with the flat-space expressions.

Because of the complexity of the transform space, it is actually more convenient for us to discuss renormalization in configuration space. We are aided in this procedure by the compact nature of the underlying space-time. We shall also find it more convenient to discuss Green's functions rather than proper vertices, though this is not strictly necessary.

The four point Green's function to second order in the coupling constant is

$$\begin{aligned} G^{(4)}(\eta_1, \eta_2, \eta_3, \eta_4) = & -\lambda_0 \int dA D_{\eta_1 \eta} D_{\eta_2 \eta} D_{\eta_3 \eta} D_{\eta_4 \eta} \\ & + \frac{\lambda_0^2}{2} \int dA dA' D_{\eta_1 \eta} D_{\eta_2 \eta} [D_{\eta \eta'}]^2 D_{\eta' \eta_3} D_{\eta' \eta_4} \\ & + (\text{two similar terms}). \end{aligned} \quad (4.4)$$

The second contribution to the right side of this equation corresponds to Fig. 1a and has a divergence at $n = 4$ which occurs when $\eta = \eta'$. This divergence is contained in the expression

$$\frac{\lambda_0^2}{2} \int dA dA' D_{\eta_1 \eta} D_{\eta_2 \eta} D_{\eta_3 \eta} D_{\eta_4 \eta} [D_{\eta \eta'}]^2, \quad (4.5)$$

where η' has been set equal to η everywhere except in $D_{\eta\eta'}$. Now

$$\int dA' [D_{\eta\eta'}]^2 = (2R)^{4-n} [\Gamma(\frac{n}{2}-1)]^2 \Gamma(2-\frac{n}{2}) / (16\pi^{n/2}) \quad (4.6)$$

$$\approx -\frac{1}{8\pi^2} \frac{1}{(n-4)} .$$

That is as a distribution equation we can write

$$\approx -\frac{1}{(n-4)} \frac{1}{R^2(4\pi)^4} \left\{ 1 + (n-4) [\chi - \ln 4\pi R^2 - 1] \right\} \quad (4.18)$$

and

$$\frac{1}{R^2} \int dA' [D_{\eta\eta'}]^3 (\eta-\eta')^2 = - \frac{2 B_n^3 \Omega_n \Gamma(4-n) \Gamma(\frac{n}{2})}{(2R)^{2n-6} \Gamma(4-\frac{n}{2})} \quad (4.19)$$

$$\approx \frac{1}{(n-4)} \frac{4}{R^2(4\pi)^4} \left\{ 1 + (n-4) [\chi - \ln 4\pi R^2 + \frac{1}{2}] \right\} .$$

It is easy to check then that to $O(\lambda^2)$

$$G(\eta_1, \eta_2) \approx -D_{\eta_1\eta_2} - \frac{\lambda^2}{12(4\pi)^4 (n-4)} \int dA D_{\eta_1\eta} \left(\frac{L^2 - \frac{1}{2}n(n-2)}{2R^2} \right) D_{\eta\eta_2} \quad (4.20)$$

+ (finite) ,

that is

$$G(\eta_1, \eta_2) \approx - \left(1 + \frac{1}{(4\pi)^4} + \frac{\lambda^2}{12} \frac{1}{(n-4)} \right) D_{\eta_1\eta_2} + \dots \quad (4.21)$$

Since the renormalized propagator is

$$G^R(\eta_1, \eta_2) = \frac{1}{Z} G(\eta_1, \eta_2) , \quad (4.22)$$

we see that to $O(\lambda^2)$

$$Z = 1 + \frac{\lambda^2}{12(4\pi)^4 (n-4)} + \dots \quad (4.23)$$

Once again this is the standard result obtained in conventional ϕ^4 theory⁹⁾.

In order to extract the pole at $n = 4$ we must proceed with care since the integral diverges already at $n = 3$. First we expand $D_{\eta'\eta_2}$ in a "spherical" Taylor series about $\eta' = \eta$.

$$D_{\eta'\eta_2} = \left(1 - \frac{1}{2} \omega_{ab} L_{ab} + \frac{1}{8} \omega_{ab} \omega_{cd} L_{ab} L_{cd} + \dots\right) D_{\eta\eta_2}, \quad (4.12)$$

where $\{\omega_{ab}\}$ are the parameters of a rotation which carries η into η' . That is

$$\omega_{ab} = \frac{\theta}{R^2} (\xi_a \eta_b - \xi_b \eta_a), \quad (4.13)$$

where θ is the angle between η and η' and ξ is a vector of length R perpendicular to η in the plane of η and η' . We will require the results

$$\begin{aligned} d\Omega'_{n+1} &= d\theta \sin^{n-1} \theta d\Omega_n(\xi) \\ \int d\Omega_n(\xi) \xi_a &= 0 \\ \int d\Omega_n(\xi) \xi_a \xi_b &= (\Omega_n R^2 / n) (\delta_{ab} - \frac{\eta_a \eta_b}{R^2}), \end{aligned} \quad (4.14)$$

which allow us to symmetrize the integrals which occur when Eq. (4.12) is substituted into Eq. (4.11). We easily find

$$\begin{aligned} \int d\Omega'_{n+1} \omega_{ab} &= 0 \\ \int d\Omega'_{n+1} \omega_{ab} \omega_{cd} L_{ab} L_{cd} &= \frac{4}{n} \int d\Omega'_{n+1} \theta^2 \frac{L^2}{2}. \end{aligned} \quad (4.15)$$

If in addition we use the approximation

$$R^2 \theta^2 = (\eta - \eta')^2 \quad (4.16)$$

which does not influence the divergence, we find that the poles in $G(\eta_1, \eta_2)$ arising from the loop integrals, are contained in the expression

$$- \frac{\lambda_0^2}{6} \int dA D_{\eta_1 \eta} \int dA' [D_{\eta \eta'}]^3 \left(1 + \frac{(\eta - \eta')^2}{2nR^2} \frac{L^2}{2}\right) D_{\eta \eta_2} \quad (4.17)$$

Now

$$\int dA' [D_{\eta \eta'}]^3 = - \frac{\Omega_n B_n^3 \Gamma(3-n) \Gamma(n/2)}{2(2R)^{2n-6} \Gamma(3-n/2)} \quad (4.18)$$

$$\approx -\frac{1}{(n-4)} \frac{1}{R^2(4\pi)^4} \left\{ 1 + (n-4) [\gamma - \ln 4\pi R^2 - 1] \right\} \quad (4.18)$$

and

$$\begin{aligned} \frac{1}{R^2} \int dA' [D_{\eta\eta'}]^3 (\eta - \eta')^2 &= - \frac{2 B_n^3 \Omega_n \Gamma(4-n) \Gamma(\frac{n}{2})}{(2R)^{2n-6} \Gamma(4-\frac{n}{2})} \\ &\approx \frac{1}{(n-4)} \frac{4}{R^2(4\pi)^4} \left\{ 1 + (n-4) [\gamma - \ln 4\pi R^2 + \frac{1}{2}] \right\} \end{aligned} \quad (4.19)$$

It is easy to check then that to $O(\lambda^2)$

$$G(\eta_1, \eta_2) \approx -D_{\eta_1\eta_2} - \frac{\lambda^2}{12(4\pi)^4} \frac{1}{(n-4)} \int dA D_{\eta_1\eta} \left(\frac{L^2 - \frac{1}{2}n(n-2)}{2R^2} \right) D_{\eta\eta_2} + (\text{finite}) \quad (4.20)$$

that is

$$G(\eta_1, \eta_2) \approx - \left(1 + \frac{1}{(4\pi)^4} \frac{\lambda^2}{12} \frac{1}{(n-4)} \right) D_{\eta_1\eta_2} + \dots \quad (4.21)$$

Since the renormalized propagator is

$$G^R(\eta_1, \eta_2) = \frac{1}{Z} G(\eta_1, \eta_2) \quad (4.22)$$

we see that to $O(\lambda^2)$

$$Z = 1 + \frac{\lambda^2}{12(4\pi)^4} \frac{1}{(n-4)} + \dots \quad (4.23)$$

Once again this is the standard result obtained in conventional Φ^4 theory⁹⁾.

5. TRANSFORM SPACE

The finite parts of Green's functions are most easily expressed in terms of transform variables. Although the general Green's function depends on the transform variables in a complicated fashion, the propagator is relatively simple, since rotational invariance guarantees that it is a function only of the "angular momentum" variable N . That is

$$G(\eta_1, \eta_2) = \sum_{N, \alpha} G(N) Y_{\alpha}^N(\eta_1) Y_{\alpha}^N(\eta_2) \quad (5.1)$$

From Eq. (3.3) we deduce that

$$[D_{\eta\eta'}]^3 = - \frac{\phi(n)}{(n-4)} \sum_{N, \alpha} R^{6-2n} \frac{\Gamma(N+\frac{3n}{2}-3)}{\Gamma(N-\frac{n}{2}+3)} Y_{\alpha}^N(\eta) Y_{\alpha}^N(\eta') \quad (5.2)$$

where

$$\phi(n) = \frac{2^{6-2n} \pi^{n/2} \Gamma(5-n) B_n^3}{\Gamma(\frac{3n}{2}-3)(n-3)} \quad (5.3)$$

so that $\phi(4) = 1/[2(4\pi)^4]$. In transform space, therefore, Eq. (4.11) becomes

$$G(N) = \frac{R^2}{(N+\frac{n}{2})(N+\frac{n}{2}-1)} + \frac{\lambda^2 \phi(n) (\mu^2 R^2)^{4-n}}{6(n-4)} \frac{R^2 \Gamma(N+\frac{3n}{2}-3)}{(N+\frac{n}{2})^2 (N+\frac{n}{2}-1)^2 \Gamma(N-\frac{n}{2}+3)} \quad (5.4)$$

The residue of the pole in the second term is

$$\frac{\lambda^2}{12(4\pi)^4} \frac{R^2}{(N+1)(N+2)} \quad (5.5)$$

which is the same result as Eq. (4.20) expressed in transform variables. The

't Hooft prescription then tells us to subtract the term

$$\frac{1}{(n-4)} \frac{\lambda^2}{12(4\pi)^4} \frac{R^2}{(N+\frac{n}{2})(N+\frac{n}{2}-1)} \quad (5.6)$$

with the result that the renormalized propagator for $n = 4$ is

$$G(N) = \frac{R^2}{(N+1)(N+2)} + \frac{R^2}{(N+1)(N+2)} \frac{\lambda^2}{12(4\pi)^4} \left\{ \frac{1}{2} \psi(N+1) + \frac{3}{2} \psi(N+3) - \frac{2N+3}{2(N+1)(N+2)} - \ln 4\pi \mu^2 R^2 - \gamma - \frac{13}{4} \right\} \quad (5.7)$$

where $\psi(z) = [\ln \Gamma(z)]'$ and $\gamma = \psi(1)$.

If we take the limit $R, N \rightarrow \infty$ with $p = N/R$ held fixed, then we recover the result for the propagator in flat space, namely

$$\Delta(p^2) \cong G(Rp) \cong \frac{1}{p^2} + \frac{1}{p^2} \frac{\lambda^2}{12(4\pi)^4} \left\{ \ln \frac{p^2}{\mu^2} - \gamma - \ln 4\pi - \frac{5}{4} \right\} \quad (5.8)$$

We can also use transform space to elucidate the vanishing of the contribution from Fig. 1b. From Eq. (4.10) we see that this term is proportional to $D_{\eta\eta}$. Now

$$D_{\eta\eta} = -R^2 \sum_{N,\alpha} \frac{\gamma_\alpha^N(\eta) \gamma_\alpha^N(\eta)}{(N+\frac{n}{2})(N+\frac{n}{2}-1)} \quad (5.9)$$

Using the properties of Gegenbauer polynomials we find

$$D_{\eta\eta} \propto \sum_{N=0}^{\infty} \frac{(2N+n-1) \Gamma(N+n-1)}{(N+\frac{n}{2})(N+\frac{n}{2}-1) \Gamma(N+1)} \quad (5.10)$$

When $n < 2$ the first term in the above sum has a sign opposite to the rest. Hence the vanishing of $D_{\eta\eta}$ for $n < 2$ comes about "because"

$$\sum_{N=1}^{\infty} \frac{(2N+n-1) \Gamma(N+n-1)}{(N+\frac{n}{2})(N+\frac{n}{2}-1) \Gamma(N+1)} = \frac{4 \Gamma(n)}{n(2-n)} \quad (5.11)$$

Now, for large values of N , the terms in the sum behave like $2N^{n-3}$, so the sum has an ultraviolet divergence of the form

$$2 \sum_N N^{n-3} \approx \frac{2}{2-n} \quad (5.12)$$

The ultraviolet pole at $n = 2$ then is cancelled as it ought to be by the pole in the longest wavelength component in the sum.

As a matter of fact, we can observe a similar phenomenon in the flat space calculation. In that case, the diagonal elements of the propagator are proportional to the integral

$$I = \int \frac{d^n k}{k^2} = \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty (k^2)^{\frac{n}{2}-2} dk^2 \quad (5.13)$$

We interpret this integral as

$$I = \frac{\pi^{n/2}}{\Gamma(n/2)} \left(\int_0^A + \int_A^\infty \right) (k^2)^{\frac{n}{2}-2} dk^2 \quad (5.14)$$

We evaluate the two contributions for the (different) ranges of n for which they are defined. We obtain

$$I = \frac{\pi^{n/2}}{\Gamma(n/2)} \left[\frac{A^{n/2-1}}{n/2-1} - \frac{A^{n/2-1}}{n/2-1} \right] \quad (5.15)$$

Once again we see the infrared pole at $n = 2$ cancelling the ultraviolet divergence. The curved space version of this argument is the more straightforward, however, since ill-defined summations like Eq. (5.13) are entirely avoided.

6. TWO LOOP CONTRIBUTION TO λ_0

In Section 4 we showed that λ_0 and Z were independent of R to $O(\lambda^2)$. We shall confirm in this section that the same is true of the higher order contributions which arise from the diagrams in Figs. 2 and 3. The way in which this comes about involves a cancellation between these diagrams and their counter-terms and suggests a general argument as to why λ_0 and Z are independent of R to all orders.

All contributions to $G^{(4)}$ from diagrams like Fig. 2a are zero for reasons we have already discussed. One of the contributions to $G^{(4)}$ from diagrams like Fig. 2b is

$$G^{(4)}(\eta_1, \eta_2, \eta_3, \eta_4) \sim -\frac{\lambda_0^3}{4} \int dA dA' dA'' D_{\eta_1 \eta_2} D_{\eta_2 \eta_3} [D_{\eta_3 \eta_4}]^2 [D_{\eta_1 \eta_2}]^2 D_{\eta_1 \eta_3} D_{\eta_2 \eta_4} \quad (6.1)$$

Making use of Eq. (4.7) we can write this as

$$G^{(4)} \sim -\frac{\lambda_0^3}{4} \left\{ \frac{(\mu^2)^{n-4}}{(n-4)^2} \frac{1}{64\pi^4} \int dA D_{\eta_1 \eta_2} D_{\eta_2 \eta_3} D_{\eta_3 \eta_4} D_{\eta_1 \eta_4} \right. \\ \left. - \frac{2\mu^{n-4}}{(n-4)} \frac{1}{8\pi^2} \int dA dA' D_{\eta_1 \eta_2} D_{\eta_2 \eta_3} K_{\eta_1 \eta_2} D_{\eta_3 \eta_4} D_{\eta_1 \eta_4} \right. \\ \left. + \int dA dA' dA'' D_{\eta_1 \eta_2} D_{\eta_2 \eta_3} K_{\eta_1 \eta_2} K_{\eta_3 \eta_4} D_{\eta_1 \eta_3} D_{\eta_2 \eta_4} \right\} \quad (6.2)$$

Now the appropriate counter-term arises from diagrams like Fig. 2d and yields a total contribution

$$-2 \frac{(\mu^2)^{4-n}}{(n-4)} \frac{\lambda^2}{16\pi^2} \frac{\lambda}{2} \int dA dA' D_{\eta_1 \eta} D_{\eta_2 \eta} [D_{\eta \eta'}]^2 D_{\eta' \eta_3} D_{\eta' \eta_4} \quad (6.3)$$

That is a term

$$- \frac{(\mu^2)^{4-n}}{(n-4)} \frac{\lambda^3}{16\pi^2} \left\{ - \frac{\mu^{n-4}}{(n-4)} \frac{1}{8\pi^2} \int dA D_{\eta_1 \eta} D_{\eta_2 \eta} D_{\eta_3 \eta} D_{\eta_4 \eta} \right. \\ \left. + \int dA dA' D_{\eta_1 \eta} D_{\eta_2 \eta} K_{\eta \eta'} D_{\eta' \eta_3} D_{\eta' \eta_4} \right\}. \quad (6.4)$$

Combining this term with the right side of Eq. (6.3) we see that the complicated single pole terms vanish and we obtain the result to $O(\lambda^3)$,

$$G^{(4)} \sim \frac{\mu^{4-n}}{(n-4)^2} \frac{\lambda^3}{(4\pi)^4} \int dA D_{\eta_1 \eta} D_{\eta_2 \eta} D_{\eta_3 \eta} D_{\eta_4 \eta} \\ - \frac{(\mu^3)^{4-n}}{4} \frac{\lambda^3}{(4\pi)^4} \int dA dA' dA'' D_{\eta_1 \eta} D_{\eta_2 \eta} K_{\eta \eta''} K_{\eta'' \eta'} D_{\eta' \eta_3} D_{\eta' \eta_4}. \quad (6.5)$$

This shows that the remaining divergence is proportional to

$$H^{(4)} = \int dA D_{\eta_1 \eta} D_{\eta_2 \eta} D_{\eta_3 \eta} D_{\eta_4 \eta}, \quad (6.6)$$

and can therefore be removed by a counter-term proportional to Φ^4 . If we note that there are three different terms of this type, then we deduce that λ_0 receives a contribution

$$\mu^{4-n} \frac{3\lambda^3}{(4\pi)^4} \frac{1}{(n-4)^2}. \quad (6.7)$$

With the remaining graphs we shall not check the renormalization procedure in detail, but simply extract the renormalization constants.

There are six terms associated with Fig. 2c. A typical one is

$$G^{(4)} \sim - \frac{\lambda_0^3}{2} \int dA dA' dA'' D_{\eta\eta} D_{\eta\eta} D_{\eta\eta'} D_{\eta\eta''} D_{\eta\eta'} [D_{\eta'\eta''}]^2 D_{\eta''\eta} . \quad (6.8)$$

The divergences which influence λ_0 are contained in a term, obtained from Eq. (6.8), by putting $\eta' = \eta'' = \eta$ everywhere except in the singular factors, namely

$$G^{(4)} \sim - \frac{\lambda_0^3}{2} I H^{(4)} \quad (6.9)$$

where

$$I = \int dA' dA'' D_{\eta\eta'} [D_{\eta'\eta''}]^2 D_{\eta''\eta} \quad (6.10)$$

We show how to evaluate I in the Appendix. The result, so far as the pole terms are concerned, is

$$I \approx \frac{2(R^2)^{4-n}}{(4\pi)^4} \frac{1}{(n-4)^2} \left\{ 1 + (n-4) \left[\gamma - \ln 4\pi - \frac{1}{2} \right] \right\} \quad (6.11)$$

From Eq. (6.9) we see that the relevant poles in $G^{(4)}$ are to $O(\lambda^3)$

$$G^{(4)} \sim - \mu^{4-n} \frac{\lambda^3}{(4\pi)^4} \frac{H^{(4)}}{(n-4)^2} \left\{ 1 + (n-4) \left[\gamma - \ln 4\pi \mu^2 R^2 - \frac{1}{2} \right] \right\} \quad (6.12)$$

The appropriate counter-term again arises from Fig. 2d and is (so far as poles are concerned)

$$\mu^{4-n} \frac{\lambda^3}{(4\pi)^4} \frac{H^{(4)}}{(n-4)^2} \left\{ 2 + (n-4) \left[\gamma - \ln 4\pi \mu^2 R^2 \right] \right\} \quad (6.13)$$

Combining the two terms we see that the terms in $\ln \mu R$ cancel between them. The result implies that λ_0 has a contribution

$$\mu^{4-n} \frac{\lambda^3}{(4\pi)^4} \left\{ \frac{6}{(n-4)^2} + \frac{3}{(n-4)} \right\} , \quad (6.14)$$

where a factor of 6 has been included to take account of all the different terms of this form.

Finally, we must include a contribution to λ_0 associated with the field re-normalization constant, which originates in Figs. 2e and 2f. It has the form

$$-\frac{\lambda^3}{(4\pi)^4} \frac{1}{6} \frac{1}{(n-4)} \quad (6.15)$$

Adding the three contributions of Eqs. (6.7), (6.14) and (6.15), and including the result of Section 4 for completeness, we have

$$\lambda_0 = \mu^{4-n} \left\{ \lambda + \frac{1}{(n-4)} \left[-\frac{3\lambda^2}{(4\pi)^2} + \frac{17\lambda^3}{6(4\pi)^4} \right] + \frac{1}{(n-4)^2} \frac{9\lambda^3}{(4\pi)^4} + \dots \right\} \quad (6.16)$$

This is precisely the flat space result¹¹⁾. We can now see in general why λ_0 is independent of R . Each term of any order in λ , which contributes to λ_0 , will be a polynomial in $\ln \mu R$. Therefore, if we expect a finite limit to exist for each order in λ , as $R \rightarrow \infty$, we must expect that the R dependence will cancel out entirely from the sum of all contributions. This argument holds good also for Z as we will verify to $O(\lambda^3)$ in the next section.

7. THREE LOOP CORRECTION TO THE PROPAGATOR

The relevant diagrams which do not vanish in a trivial way are shown in Fig. 3. Associated with Fig. 3a we have a contribution to the propagator of the form

$$G(\eta_1, \eta_2) = \frac{\lambda_0^3}{12} \int dA dA' dA'' D_{\eta_1 \eta_2} D_{\eta_1 \eta_2} D_{\eta_1 \eta_2} [D_{\eta_1 \eta_2}]^3 D_{\eta_1 \eta_2} \quad (7.1)$$

This is proportional to the integral

$$J = \int dA' dA'' D_{\eta_1 \eta_2} [D_{\eta_1 \eta_2}]^3 D_{\eta_1 \eta_2} \quad (7.2)$$

and represents a mass term. We evaluate this integral in the Appendix and show that although it is formally divergent when $n = 4$ it does not have any poles there and so necessitates no subtractions and therefore no divergent mass counter-term in the Lagrangian. This is satisfactory since, in the flat space theory, such a graph yields a contribution which is identically zero^{9,10)}. However, it would have been quite compatible with this if the spherical version of the theory had demanded a bare mass $m_0 = O(1/R)$. Whether or not the graph yields a non-vanishing finite contribution we have not determined.

Finally, we come to the graph in Fig. 3b and its counter-terms associated with Fig. 3c. From Fig. 3b we obtain a contribution

$$G(\eta_1, \eta_2) = \frac{\lambda_0^3}{4} \int dA dA' dA'' D_{\eta_1 \eta} [D_{\eta \eta''}]^2 [D_{\eta'' \eta'}]^2 D_{\eta' \eta} D_{\eta' \eta_2} \quad (7.3)$$

By an argument of the same kind as that used for the propagator divergence in Section 4, we find that the poles which influence the field renormalization are contained in the expression

$$G(\eta_1, \eta_2) \sim \frac{\lambda_0^3}{4} \int dA dA' dA'' D_{\eta_1 \eta} [D_{\eta \eta''}]^2 [D_{\eta'' \eta'}]^2 D_{\eta' \eta} \times \left(1 + \frac{(\eta - \eta')^2}{2\pi R^2} \frac{L^2}{2} \right) D_{\eta \eta_2} \quad (7.4)$$

We must compute the integrals

$$Q_1 = \int dA' dA'' [D_{\eta \eta''}]^2 [D_{\eta'' \eta'}]^2 D_{\eta' \eta} \quad , \quad (7.5)$$

$$Q_2 = \int dA' dA'' [D_{\eta \eta''}]^2 [D_{\eta'' \eta'}]^2 D_{\eta' \eta} (\eta - \eta')^2$$

The evaluation of these integrals is again relegated to the Appendix. The pole contributions are

$$Q_1 \approx \frac{1}{R^2} \frac{8}{3(4\pi)^6} \frac{1}{(n-4)^2} \left\{ 1 + (n-4) \left[\frac{3}{2} (\gamma - \ln 4\pi R^2) - 2 \right] \right\} \quad , \quad (7.6)$$

$$Q_2 = - \frac{32}{3(4\pi)^6} \frac{1}{(n-4)^2} \left\{ 1 + (n-4) \left[\frac{3}{2} (\gamma - \ln 4\pi R^2) + \frac{1}{2} \right] \right\}$$

When these results are inserted into Eq. (7.4) we find for the poles

$$G(\eta_1, \eta_2) \approx - \frac{\lambda^3}{3(4\pi)^6} \frac{1}{(n-4)^2} \left\{ 1 + (n-4) \left[\frac{3}{2} (\gamma - \ln 4\pi \mu^2 R^2) + \frac{1}{4} \right] \right\} \times \int dA D_{\eta \eta} \left(\frac{L^2 - \frac{1}{2} n(n-2)}{2R^2} \right) D_{\eta \eta_2} \quad (7.7)$$

$$- \frac{\lambda^3}{(4\pi)^6} \frac{4}{(n-4)} \int dA D_{\eta\eta} \frac{1}{2R^2} D_{\eta\eta_2} .$$

In this expression we see that although the leading, double pole term, has the correct balance between the two terms contributing to M, the single pole term does not. However, this defect is remedied when the right side of Eq. (7.7) is combined with the counter-terms. There are two of these, both associated with diagrams like Fig. 3c. Their total contribution is

$$G(\eta_1, \eta_2) \approx -2 \frac{\lambda_0 \delta \lambda_0}{6} \int dA dA' D_{\eta\eta} [D_{\eta\eta'}]^3 D_{\eta'\eta_2} \quad (7.8)$$

Using Eqs. (4.9), (4.18) and (4.19), we see that this yields pole terms of the form

$$G(\eta_1, \eta_2) \approx \frac{\lambda^3}{2(4\pi)^6} \frac{1}{(n-4)^2} \left\{ 1 + (n-4) \left[\gamma - \ln 4\pi \mu^2 R^2 + \frac{1}{4} \right] \right\} \\ \times \int dA D_{\eta\eta} \left(\frac{L^2 - \frac{1}{2} n(n-2)}{2R^2} \right) D_{\eta\eta_2} \quad (7.9) \\ + \frac{\lambda^3}{(4\pi)^6} \frac{4}{(n-4)} \int dA D_{\eta\eta} \frac{1}{2R^2} D_{\eta\eta_2} .$$

The result of adding Eqs. (7.7) and (7.9) is

$$G(\eta_1, \eta_2) \approx \frac{\lambda^3}{(4\pi)^6} \left\{ \frac{1}{6} \frac{1}{(n-4)^2} + \frac{1}{24} \frac{1}{(n-4)} \right\} \int dA D_{\eta\eta} \left(\frac{L^2 - \frac{1}{2} n(n-2)}{2R^2} \right) D_{\eta\eta_2} \quad (7.10)$$

This implies that to $O(\lambda^3)$, Z has the form [recall Eq. (4.23)]

$$Z = 1 + \frac{1}{(n-4)} \left[\frac{1}{12} \frac{\lambda^2}{(4\pi)^2} + \frac{1}{24} \frac{\lambda^3}{(4\pi)^6} \right] + \frac{1}{(n-4)^2} \frac{1}{6} \frac{\lambda^3}{(4\pi)^6} + \dots \quad (7.11)$$

Not only have we seen in this calculation that, as expected, Z is independent of R, but also that no divergent mass term is generated of $O(1/R^2)$. We presume that this result remains true to all orders in λ , though we do not have a general argument for it.

It is perhaps worth remarking that so far as we know the contribution $O(\lambda^3)$ to Z has not been calculated in the flat space theory, so this numerical result is actually new.

8. Φ^3 THEORY

In this section, we consider a theory with an interaction term

$$\mathcal{L}_I = - \frac{g_0}{3!} \Phi^3, \quad (8.1)$$

because in contrast to the previous case it is infrared unstable. The Feynman rules are the same as for Φ^4 theory, except that we have $(-g_0)$ at each three-vertex.

The one loop contribution to the vertex function is shown in Fig. 4a. To $O(g_0^3)$ we have

$$\begin{aligned} G(\eta_1, \eta_2, \eta_3) = & -g_0 \int dA D_{\eta_1 \eta} D_{\eta_2 \eta} D_{\eta_3 \eta} \\ & + g_0^3 \int dA dA' dA'' D_{\eta_1 \eta'} D_{\eta_2 \eta'} D_{\eta_3 \eta''} \\ & D_{\eta \eta'} D_{\eta \eta''} D_{\eta' \eta''}. \end{aligned} \quad (8.2)$$

The divergence of the loop contribution is contained in the term

$$G(\eta_1, \eta_2, \eta_3) \sim g_0^3 \int dA D_{\eta_1 \eta} D_{\eta_2 \eta} D_{\eta_3 \eta} J, \quad (8.3)$$

where

$$J = \int dA' dA'' D_{\eta \eta'} D_{\eta' \eta''} D_{\eta \eta''}. \quad (8.4)$$

It is convenient to express J , which is independent of η , as a trace; thus

$$J = \frac{1}{R^n \Omega_{n+1}} \int dA dA' dA'' D_{\eta \eta'} D_{\eta' \eta''} D_{\eta \eta''}. \quad (8.5)$$

If we evaluate this expression in terms of transform variables we find

$$J = \frac{1}{R^n \Omega_{n+1}} \sum_N h(N, n) \left[\frac{-R^2}{(N + \frac{n}{2})(N + \frac{n}{2} - 1)} \right]^3. \quad (8.6)$$

The pole at $n = 6$ comes from the large N contribution to the sum; we easily find

$$J \sim - \frac{1}{R^n \Omega_{n+1}} \frac{2}{\Gamma(n)} \sum_N N^{n-7} \sim \frac{1}{(4\pi)^3} \frac{1}{(n-6)} . \quad (8.7)$$

In the 't Hooft method¹²⁾ we write

$$g_0 = \mu^{\frac{3}{2}-n} \left(g + \frac{A_1(g)}{n-6} + \dots \right) , \quad (8.8)$$

and adjust the pole residues to cancel the ultraviolet divergences. The above calculation then implies that to $O(g^3)$

$$A_1(g) = \frac{g^3}{(4\pi)^3} , \quad (8.9)$$

which is of course the standard result for ϕ^3 theory in six dimensions¹²⁾.

The contribution to the propagator from Fig. 4b is

$$G(\eta_1, \eta_2) = \frac{g_0^2}{2} \int dA dA' D_{\eta\eta} [D_{\eta\eta'}]^2 D_{\eta\eta_2} . \quad (8.10)$$

In transform space this is

$$G(N) = \frac{g_0^2}{2} \left(\frac{-R^2}{(N+\frac{n}{2})(N+\frac{n}{2}-1)} \right)^2 \left(\frac{B_n^2(2R) \pi^{\frac{n}{2}} \Gamma(\frac{n}{2}) \Gamma(N+n-2)}{\Gamma(N+2) \Gamma(n-2)} \right) \quad (8.11)$$

The pole at $n = 6$ is

$$G(N) \approx \frac{1}{(n-6)} \frac{g_0^2}{6(4\pi)^3} \frac{R^2}{(N+3)(N+2)} . \quad (8.12)$$

This shows that the standard renormalization factor¹²⁾

$$Z = 1 + \frac{1}{n-6} \frac{g^2}{6(4\pi)^3} + \dots , \quad (8.13)$$

will remove the divergence from the propagator.

In order to calculate the renormalized propagator we must subtract from Eq. (8.9), the term

$$\frac{1}{(n-6)} \frac{g^2}{(4\pi)^3} \frac{R^2}{(N+\frac{1}{2})(N+\frac{1}{2}-1)} \quad (8.14)$$

The complete result to $O(g^2)$ for the renormalized $G(N)$ is then

$$G(N) = \frac{R^2}{(N+3)(N+2)} + \frac{R^2}{(N+3)(N+2)} \frac{g^2}{6(4\pi)^3} \left\{ \psi(N+4) - \frac{2N+5}{2(N+3)(N+5)} - \frac{1}{2} \ln 4\pi - \frac{1}{2} \ln R^2 - \frac{\gamma}{2} - \frac{4}{3} \right\} \quad (8.15)$$

Finally, we consider the vacuum expectation value graphs in Fig. 5. The single loop graph vanishes identically for reasons already discussed, since it is proportional to $D_{\eta\eta}$. The two loop graph in Fig. 5b is proportional to

$$I = \int dA dA' D_{\eta\eta'} [D_{\eta'\eta''}]^2 D_{\eta''\eta} \quad (8.16)$$

This integral has already appeared in Eq. (6.10) and is evaluated in the Appendix. When we examine it near $n = 6$, we find that it is finite. This means that no divergent counter-term of the form $K\phi$ need be added to the Lagrangian. *A priori* such a term is not excluded, since $K = O(1/R^2)$, and so would certainly vanish as $R \rightarrow \infty$. We assume that the absence of such a term, as well as that of an anomalous mass term, remains true to all orders in λ . It would be interesting to construct a general argument to show this.

9. THE LIMIT $R \rightarrow \infty$ AND THE RENORMALIZATION GROUP

In Section 5 we set the "angular momentum" quantum number $N = Rp$ and took the limit $R \rightarrow \infty$ with p fixed. The result for the propagator coincides with flat space perturbation theory when p is interpreted as the momentum flowing through the propagator. To take the limit in this fashion, however, is to throw away any advantage the spherical space approach might have in its treatment of long wavelength components.

A limit suited to the examination of long wavelengths is $R \rightarrow \infty$ with N fixed. However, we cannot take this limit term by term in the perturbation series since, as Eq. (5.7) suggests, the effective expansion parameter is $\lambda \ln \mu R$. The series, therefore, only retains its significance when λ is small and $R \sim 1/\mu$. In order to investigate this limit it is necessary to invoke the aid of the renormalization group¹³⁾.

Since the relationship between bare and finite parameters remains the same as in conventional theories, we can use 't Hooft's analysis of scaling⁶⁾ to tell us how λ or g changes as we alter μ . We then use dimensional analysis to determine the R dependence of the Green's functions.

Accordingly, we have that when $\mu \rightarrow \mu e^s$, $\lambda \rightarrow \lambda(s)$, $g \rightarrow g(s)$, where

$$\begin{aligned} \frac{d\lambda(s)}{ds} &= \beta(\lambda(s)) \\ \beta(\lambda) &= (1 - \lambda \frac{\partial}{\partial \lambda}) A_1(\lambda) = \frac{3\lambda^2}{(4\pi)^2} + \dots, \end{aligned} \quad (9.1)$$

and

$$\begin{aligned} \frac{dg(s)}{ds} &= \beta(g(s)) \\ \beta(g) &= \frac{1}{2} (1 - g \frac{\partial}{\partial g}) A_1(g) = -\frac{g^3}{(4\pi)^3} + \dots. \end{aligned} \quad (9.2)$$

In a range of s for which $\lambda(s)$ or $g(s)$ remains small, we have the approximate results

$$\lambda(s) = \lambda / \left(1 - \frac{3\lambda s}{(4\pi)^2} \right), \quad (9.3)$$

and

$$g(s) = g^2 / \left(1 + \frac{2g^2 s}{(4\pi)^3} \right). \quad (9.4)$$

Φ^4 theory is infrared stable and Eq. (9.3) remains a good approximation as $s \rightarrow -\infty$. The opposite is true of Eq. (9.4), since Φ^3 theory is infrared unstable.

The field renormalization constants also change $Z \rightarrow Z\zeta(s)$, where

$$\frac{d}{ds} \zeta(s, \lambda) = \lambda \left. \frac{\partial C_1(\lambda)}{\partial \lambda} \right|_{\lambda=\lambda(s)} = \frac{\lambda^2(s)}{6(4\pi)^4} + \dots, \quad (9.5)$$

so that while $\lambda(s)$ remains small

$$\zeta(s, \lambda) = \exp \left\{ \frac{\lambda^2 s}{6(4\pi)^4} / \left(1 - \frac{3\lambda s}{(4\pi)^2} \right) \right\} \quad (9.6)$$

The corresponding result in Φ^3 theory is

$$\zeta(s, g) = \left(1 + \frac{2g^2 s}{(4\pi)^3} \right)^{\frac{1}{12}} \quad (9.7)$$

In Φ^4 theory the scaling analysis applied to the propagator tells us that

$$G(N, \lambda, \mu, R) = \zeta(s, \lambda) G(N, \lambda(s), \mu e^s, R) \quad (9.8)$$

while from dimensional analysis we have

$$G(N, \lambda, \mu, e^{-s} R) = e^{-2s} G(N, \lambda, e^{-s} \mu, R) \quad (9.9)$$

Now suppose the radius of space-time acquires its general value R by being scaled up from its "natural" value $1/\mu$. The parameter s then has the value $(-\ln \mu R)$.

We can use then Eqs. (9.8) and (9.9) to improve Eq. (5.7) for $G(N)$ with the result

$$G(N) = \frac{R^2}{(N+1)(N+2)} \exp \left\{ - \frac{\lambda^2 \ln \mu^2 R^2}{12(4\pi)^4} / \left(1 + \frac{3\lambda}{2} \frac{\ln \mu^2 R^2}{(4\pi)^2} \right) \right\} \\ \times \left[1 + \frac{\lambda^2}{12(4\pi)^4} \left(1 + \frac{3\lambda}{2} \frac{\ln \mu^2 R^2}{(4\pi)^2} \right)^{-2} \left(\frac{3}{2} \psi(N+2) + \frac{1}{2} \psi(N+1) - \frac{2N+3}{2(N+2)(N+1)} \right. \right. \\ \left. \left. - \gamma - \ln 4\pi - \frac{13}{4} \right) \right] \quad (9.10)$$

The correspondingly improved version of Eq. (8.15) is

$$G(N) = \frac{R^2}{(N+2)(N+3)} \left\{ 1 - \frac{g^2 \ln \mu^2 R^2}{(4\pi)^3} \right\}^{\frac{1}{12}}$$

$$\times \left\{ 1 + \frac{g^2}{6(4\pi)^3} \left(1 - \frac{g^2 \ln \mu^2 R^2}{(4\pi)^3} \right)^{-1} \left(\psi(N+4) - \frac{2N+5}{2(N+2)(N+3)} - \frac{1}{2} \ln 4\pi - \frac{\gamma'}{2} - \frac{4}{3} \right) \right\} \quad (9.11)$$

These two equations show clearly the contrast between the infrared stable and unstable theories. Equation (9.10) holds for R arbitrarily large, while Eq. (9.11) breaks down in this limit. For the unstable theory then we cannot (as of course we expect) remove the infrared cut-off without knowing more about the behaviour of $\beta(g^2)$ for example. It would be interesting to pursue Wilson's hypothesis of a possible zero in $\beta(g^2)$ ¹⁴⁾ and, in general, the connection between our formulation of field theory and conformally invariant limiting theories ^{2,3)}. However, in order to get a complete picture of the propagator as $R \rightarrow \infty$ it is probably necessary to investigate limits intermediate between the two considered in this section. That is we might expect to have to consider $N \sim R^\alpha$ for $0 \leq \alpha \leq 1$. However, this presumably requires more information than is contained in the renormalization group as used here and we do not pursue this question further. Nevertheless, it is interesting and satisfying that there is a sensible geometrical context in which perturbation theory will give reasonable information about infrared unstable theories.

Finally, we would like to indicate what field theories look like if we return to flat space by simply reversing the transformation in Eq. (2.1) and leaving R unchanged. In that case, our renormalization prescription amounts to using a Lagrangian, for ϕ^4 theory with an x -dependent coupling constant $\lambda_0(x) = \lambda_0 \kappa(x)^{4-n}$; thus

$$\mathcal{L} = -\frac{1}{2} (\partial \phi)^2 - \frac{\lambda_0}{4!} [\kappa(x)]^{4-n} \phi^4, \quad (9.12)$$

where $\kappa(x) = 2/(1 + x^2/R^2)$. Thus

$$\lambda_0(x) = [\mu \kappa(x)]^{4-n} \left(\lambda + \sum_{\nu=1}^{\infty} \frac{a_\nu(\lambda)}{(n-4)^\nu} \right). \quad (9.13)$$

The space-time dependence can be assigned to either the renormalization mass as above, or to the coupling parameter $\lambda(x)$; thus

$$\lambda_0(x) = \mu^{4-n} \left(\lambda(x) + \sum_{\nu=1}^{\infty} \frac{a_\nu(\lambda(x))}{(n-4)^\nu} \right). \quad (9.14)$$

Equation (9.3) tells us that in the lowest approximation

$$\lambda(x) = \frac{\lambda}{1 + \frac{3\lambda}{(4\pi)^2} \ln K(x)} \quad (9.15)$$

Since $\kappa(x) \rightarrow 0$ as $x \rightarrow \infty$, this equation is only a good approximation in some finite neighbourhood of the origin. This is in contrast to the corresponding result in Φ^3 theory, where

$$g^2(x) = \frac{g^2}{1 - \frac{2g^2}{(4\pi)^3} \ln K(x)} \quad (9.16)$$

which can remain a good approximation uniformly over the whole of space-time. These comments would have to be reconsidered in a true de Sitter universe.

It is perhaps rather unexpected that the renormalization procedure can be carried out with a bare coupling constant given by Eq. (9.13) and one can ask to what extent $\kappa(x)$ is arbitrary. Clearly $\kappa(x)$ can be any local scaling associated with the conformal mapping of flat space onto a sphere. The interesting question is whether $\kappa(x)$ can be chosen outside this class. We suspect that this is not the case.

10. CONCLUSIONS

We have examined in this paper the renormalization of massless Φ^4 theory in spherical space-time to $O(\lambda^3)$. We were able to show that the renormalization constants were identical with their flat space values. A general argument suggests why this must be true to all orders of perturbation theory. Furthermore, we were able to confirm that the curvature of space-time does not necessitate the introduction of a divergent mass counter-term $O(1/R^2)$. We do not have a general argument for excluding such a term to all orders, though we presume it to be absent. Similar results hold true in Φ^3 theory. These conclusions are, of course, in accordance with naive intuition, which suggests that the divergences of field theory are effects localized at a point in space-time. Consequently, they are not influenced by the large-scale structure of the universe. They are also in accord with the results of previous investigations of the effect of external gravitational fields on quantum field theory^{15,16}.

The method of dimensional regularization which we have used lends a new precision, however, to statements about renormalization constants. It should be noted that the numerical value of the contribution $O(\lambda^3)$ to the field renormalization constant $Z(\lambda)$ is, so far as we know, a new result.

We showed how 't Hooft's analysis of scaling could be used to elucidate the dependence of the Green's functions on the radius of space-time and found that we could confidently use perturbation theory in the large R limit for an infrared stable theory though not, of course, for an unstable one. This not unexpected result has an obvious application to Yang-Mills theory. Nevertheless, it is interesting and possibly useful to know that even infrared unstable theories can be computed in perturbation theory, provided the curvature of space-time is large enough to make the effective coupling parameter weak.

Finally, we pointed out the amusing result that our procedure of working in curved space-time is equivalent to working in flat space, but with a space-time dependent bare coupling constant. This may be interpreted as being due to a space-time dependence of the finite coupling parameter.

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APPENDIX

All the integrals in the text can be evaluated in essentially the same way. We shall discuss the first one in detail and simply indicate the details for the others.

The first integral we wish to consider is

$$I = \int dA' dA'' D_{\eta\eta'} [D_{\eta'\eta''}]^2 D_{\eta''\eta} \quad (A.1)$$

Since it is independent of η we can express it as a trace,

$$I = \frac{1}{R^n \Omega_{n+1}} \int dA dA' dA'' D_{\eta\eta'} [D_{\eta'\eta''}]^2 D_{\eta''\eta} \quad (A.2)$$

Now we use

$$D_{\eta\eta'} = \sum_{N, \alpha} \left(\frac{-R^2}{(N + \frac{n}{2})(N + \frac{n}{2} - 1)} \right) Y_{\alpha}^N(\eta) Y_{\alpha}^N(\eta'), \quad (A.3)$$

and [from Eq. (3.12) with $v = 2 - n$]

$$[D_{\eta\eta'}]^2 = B_n^2 \sum_{N, \alpha} \frac{(2R)^{4-n} \pi^{\frac{n}{2}} \Gamma(2 - \frac{n}{2}) \Gamma(N + \frac{n}{2})}{\Gamma(n-2) \Gamma(N+2)} Y_{\alpha}^N(\eta') Y_{\alpha}^N(\eta'') \quad (A.4)$$

In transform space then the trace in Eq. (A.2) may be expressed as

$$I = \frac{(2R^2)^{4-n} B_n^2 \pi^{\frac{n}{2}} \Gamma(2 - \frac{n}{2})}{\Omega_{n+1} \Gamma(n-2) \Gamma(4-n)} \sum_{N=0}^{\infty} h(N, n) [B(N + \frac{n}{2} - 1, 2)]^2 B(N + n - 2, 4 - n) \quad (A.5)$$

Using the standard representation for B-functions, the above summation over N , which we denote by K , can be written

$$K = \int_0^1 dx dy dz \sum_{N=0}^{\infty} h(N, n) x^{N + \frac{n}{2} - 2} (1-x) y^{N + \frac{n}{2} - 2} (1-y) z^{N + n - 3} (1-z)^{3-n} \quad (A.6)$$

Now we use the fact that

$$h(N, n) = \frac{(2N+n-1)}{n-1} (-1)^N \binom{1-n}{N}, \quad (\text{A.7})$$

to express K as

$$K = K_1 + K_2 \quad (\text{A.8})$$

where

$$K_1 = \int_0^1 dx dy dz x^{\frac{n}{2}-2} (1-x) y^{\frac{n}{2}-2} (1-y) z^{n-3} (1-z)^{3-n} (1-xyz)^{1-n},$$

$$K_2 = 2 \int_0^1 dx dy dz x^{\frac{n}{2}-1} (1-x) y^{\frac{n}{2}-1} (1-y) z^{n-2} (1-z)^{3-n} (1-xyz)^{-n} \quad (\text{A.9})$$

We use the transformation (inspired by the theory of dual models)

$$x = \frac{1-u}{1-uv},$$

$$y = \frac{1-uv}{1-uvw},$$

$$z = 1-uvw, \quad (\text{A.10})$$

with

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{u^2 v}{(1-uv)(1-uvw)},$$

to re-express K_1 and K_2 as

$$K_1 = \int_0^1 du dv dw u^{\frac{n}{2}-2} (1-u)^{\frac{n}{2}-2} v^{5-n} (1-v) w^{3-n} (1-w) \\ \times (1-uv)^{-2} (1-uvw)^{\frac{n}{2}-3},$$

$$K_2 = 2 \int_0^1 du dv dw u^{7-2n} (1-u)^{\frac{n}{2}-2} v^{5-n} (1-v) w^{3-n} (1-w) \times (1-uvw)^{-2} (1-uvw)^{\frac{n}{2}-3} \quad (A.11)$$

Now we expand the factors $(1-uv)^{-2}$ and $(1-uvw)^{(n/2)-3}$ using the binomial theorem to obtain

$$K_1 = \sum_{p,q} (-1)^{p+q} \binom{-2}{p} \binom{\frac{n}{2}-3}{q} B(9+p+q-2n, \frac{n}{2}-1) \times B(6+p+q-n, 2) \times B(4-n+q, 2) \quad (A.12)$$

$$K_2 = 2 \sum_{p,q} (-1)^{p+q} \binom{-2}{p} \binom{\frac{n}{2}-3}{q} B(8+p+q-2n, \frac{n}{2}) \times B(6+p+q-n, 2) \times B(4-n+q, 2)$$

The combined result is

$$K = \Gamma(\frac{n}{2}-1) \sum_{p,q} (-1)^{p+q} \binom{-2}{p} \binom{\frac{n}{2}-3}{q} \frac{\Gamma(8+p+q-2n)}{\Gamma(8+p+q-\frac{3n}{2})} \times \left(\frac{1}{(7+p+q-n)(5-n+q)(4-n+q)} \right) \quad (A.13)$$

It is easily checked that, for the range of n -values of interest the above interchange of integration and summation is justified. In the text the poles in K at $n = 4$ and 6 are explicit in the various terms. None are generated by the infinite summation. The statements in the text are easily verified from Eq. (A.13).

The next integral is

$$J = \frac{1}{R^n \Omega_{n+1}} \int dA dA' dA'' D_{\gamma\gamma'} [D_{\gamma'\gamma''}]^3 D_{\gamma''\gamma} \quad (A.14)$$

We use [from Eq. (3.12) with $v = 3 - (3n/2)$]

$$[D_{\gamma'\gamma''}]^3 = - \frac{B_n^3 (2R)^{6-2n} \pi^{n/2} \Gamma(3-n)}{\Gamma(\frac{3n}{2} - 3)} \sum_N \frac{\Gamma(N + \frac{3n}{2} - 3)}{\Gamma(N + 3 - \frac{n}{2})} Y_{\alpha}^N(\gamma') Y_{\alpha}^N(\gamma'') \quad (A.15)$$

This leads to

$$J \propto H = \sum_N h(N, n) B(N + \frac{3n}{2} - 3, 6-2n) [B(N + \frac{n}{2} - 1, 2)]^2 \quad (A.16)$$

where the missing factor of proportionality is smooth in n at $n = 4$. We can write then $H = H_1 + H_2$, where

$$H_1 = \int_0^1 dx dy dz x^{\frac{n}{2}-2} (1-x) y^{\frac{n}{2}-2} (1-y) z^{\frac{3n}{2}-4} (1-z)^{5-2n} (1-xyz)^{1-n} \quad (A.17)$$

$$H_2 = 2 \int_0^1 dx dy dz x^{\frac{n}{2}-1} (1-x) y^{\frac{n}{2}-1} (1-y) z^{\frac{3n}{2}-3} (1-z)^{5-2n} (1-xyz)^{-n}$$

Following the same pattern as the previous calculation, we find eventually

$$H = \Gamma(\frac{n}{2}-1) \sum_{p,q} (-1)^{p+q} \binom{-2}{p} \binom{n-4}{q} \frac{\Gamma(10+p+q-3n)}{\Gamma(10+p+q-\frac{5n}{2})} \times \left(\frac{1}{(9+p+q-2n)(6+q-2n)(7+q-2n)} \right) \quad (A.18)$$

Again all the poles at $n = 4$ are explicit in the terms and it can be checked that their residues add up to zero.

Finally we come to Q_1 and Q_2 . We shall deal explicitly only with Q_1 , since the calculation for Q_2 is with minor complications identical.

$$Q_1 = \frac{1}{R^n \Omega_{n+1}} \int dA dA' dA'' [D_{\gamma\gamma'}] [D_{\gamma''\gamma'}]^2 D_{\gamma'\gamma} \quad (A.19)$$

Using Eqs. (A.3) and (A.4) we can easily show that

$$Q_1 = -\frac{1}{R^2} \frac{\beta_n^4 \pi^n}{\Omega_{n+1}} \left[\frac{\Gamma'(2-n/2)}{\Gamma(4-n)} \right]^2 (4R^3)^{4-n} T \quad (A.20)$$

where

$$T = \sum_N h(N, n) [B(N+n-2, 4-n)]^2 B(N+\frac{n}{2}-1, 2) \quad (A.21)$$

Now $T = T_1 + T_2$, where

$$T_1 = \int_0^1 dx dy dz x^{\frac{n}{2}-2} (1-x)^{n-3} (1-y)^{3-n} z^{\frac{n}{2}-2} (1-z)^{3-n} (1-xyz)^{1-n} \quad (A.22)$$

$$T_2 = 2 \int_0^1 dx dy dz x^{\frac{n}{2}-1} (1-x)^{n-2} (1-y)^{3-n} z^{\frac{n}{2}-1} (1-z)^{3-n} (1-xyz)^{-n}$$

Again we follow the same pattern as before changing $(x, y, z) \rightarrow (u, v, w)$ and obtain, finally

$$T = \Gamma(\frac{n}{2}-1) \sum_{p, q} (-1)^{p+q} \binom{\frac{n}{2}-3}{p} \binom{n-4}{q} \frac{\Gamma(10+p+q-3n)}{\Gamma(10+p+q-\frac{5n}{2})} \times \frac{B(4+q-n, 4-n)}{(9+p+q-2n)} \quad (A.23)$$

The poles in T at $n = 4$ can all be extracted from this expression.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5