Ref.TH.2034-CERN

DIRAC EQUATIONS WITH AN EXACT HIGHER SYMMETRY

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ABSTRACT

A family of external potentials for the Dirac equation is exhibited for which there is an exact $SU(2)$ symmetry leading to exact spin-orbit doublet degeneracy. The family in question, includes an equal superposition of Coulomb-like and Lorentz-scalar potentials.

^{*)} Partially supported by the Swiss National Fund.

Ref.TH.2034-CERN

11 June 1975

We have reported elsewhere ¹⁾ a study of approximate SU(2) symmetries for the Dirac equation with an external potential. The object of that investigation was to help in elucidating the meaning of relativistic SU(6) [or SU(6)_w] symmetry and the Melosh transformation. In the course of the work, we came upon a family of potentials with an exact SU(2) symmetry and exactly degenerate spin-orbit doublets. We did not see any simple relation between this exact symmetry for special potentials and the more general approximate symmetries. So we left it for a separate report, which we now make.

Consider the Dirac equation, for energy E, with a superposition of Coulomb-like and Lorentz scalar potentials

$$\left. \begin{aligned} E \psi &= h \psi \\ h &= \vec{\alpha} \cdot \vec{p} + \beta m + A_0 + \beta u \end{aligned} \right\} \quad (1)$$

where

$$\vec{p} = \frac{\hbar}{i} \frac{\partial}{\partial \vec{x}}$$

Consider the particular case

$$A_0 = u = \frac{1}{2} \omega \quad (2)$$

For the projections

$$\psi_{\pm} = P_{\pm} \psi \quad P_{\pm} = \frac{1}{2}(1 \pm \beta) \quad (3)$$

we have

$$\begin{aligned} (E - m - \omega) \psi_+ &= \vec{\alpha} \cdot \vec{p} \psi_- \\ (E + m) \psi_- &= \vec{\alpha} \cdot \vec{p} \psi_+ \end{aligned} \quad (4)$$

whence

$$\psi_- = (E + m)^{-1} \vec{\alpha} \cdot \vec{p} \psi_+ \quad (5)$$

It is notable that this last relation does not involve the potential. Using it to eliminate ψ_- from the first of equations (4)

$$(E - m - w) \psi_+ = (E + m)^{-1} \vec{p}^2 \psi_+ \quad (6)$$

The Dirac matrices no longer appear, so that we have the infinitesimal symmetry

$$\delta \psi_+ = -\frac{i}{2} \vec{\sigma} \cdot \vec{\epsilon} \psi_+ \quad (7)$$

where $\vec{\epsilon}$ is any real 3-vector, and

$$i \vec{\sigma} = \vec{\alpha} \times \vec{\alpha} \quad (8)$$

From (5) then

$$\left. \begin{aligned} \delta \psi_- &= (E + m)^{-1} \vec{\alpha} \cdot \vec{p} \delta \psi_+ \\ &= (E + m)^{-1} \vec{\alpha} \cdot \vec{p} \frac{\vec{\sigma} \cdot \vec{\epsilon}}{2i} \psi_+ \\ &= \vec{\alpha} \cdot \vec{p} \frac{\vec{\sigma} \cdot \vec{\epsilon}}{2i} \vec{\alpha} \cdot \vec{p} (\vec{p}^2)^{-1} \psi_- \end{aligned} \right\} \quad (9)$$

Combining (7) and (9)

$$\left. \begin{aligned} \delta \psi &= \frac{\vec{\epsilon} \cdot \vec{S}}{2i} \psi \\ \vec{S} &= \vec{\sigma} \left(\frac{1+\beta}{2} \right) + \frac{\vec{\alpha} \cdot \vec{p}}{\vec{p}^2} \vec{\sigma} \frac{\vec{\alpha} \cdot \vec{p}}{2} \left(\frac{1-\beta}{2} \right) \end{aligned} \right\} \quad (10)$$

One can check that

$$[\vec{S}, h] = 0, \quad [S_x, S_y] = 2i S_z, \text{ etc.}$$

We do indeed have an $SU(2)$ symmetry group of the Hamiltonian.

The exact symmetry (10) is rather different from the approximate symmetries of Ref. 1), in having no $\vec{p}$ independent limit for small $(\vec{p})$. However, if attention is restricted to positive energy states, then in the non-relativistic limit the awkward second term of (10) is unimportant.

We would, of course, have obtained similar results for potentials proportional to $(1-\beta)$, interchanging the roles of ψ_+ and ψ_- . More generally, we have such a symmetry for Dirac Hamiltonians of the form

$$h = \vec{\alpha} \cdot \vec{p} + \beta m + \frac{1}{2} w (1 + \vec{\alpha} \cdot \vec{e} + \beta e) \quad (11)$$

where $\vec{e}$ and e are constants such that

$$e^2 + \vec{e}^2 = 1 \quad (12)$$

Let us introduce 4-vectors

$$\underline{\alpha} = (\beta, \vec{\alpha}) \quad \underline{p} = (m, \vec{p}) \quad \underline{e} = (e, \vec{e}) \quad (13)$$

in a Euclidean vector space with inner products

$$\underline{\alpha} \cdot \underline{p} = \beta m + \vec{\alpha} \cdot \vec{p} \quad (14)$$

etc. Then

$$h = \underline{\alpha} \cdot \underline{p} + \frac{1}{2} w (1 + \underline{\alpha} \cdot \underline{e}) \quad (15)$$

$$= \beta' m' + \vec{\alpha}' \cdot \vec{p}' + \frac{1}{2} w (1 + \beta') \quad (16)$$

where the primed quantities are defined by projecting the 4-vectors on to $\underline{e}$ and three 4-vectors, e^x, e^y, e^z , orthogonal to $\underline{e}$ and to one another. The primed Dirac matrices have the same anticommutation rules as the originals:

$$\alpha'_x \beta' + \beta' \alpha'_x = 0, \text{ etc.} \quad (17)$$

So we have symmetry generators

$$\vec{S}' = \frac{\vec{\alpha}' \times \vec{\alpha}'}{i} \left(\frac{1+\beta'}{2} \right) + \frac{\vec{\alpha}' \cdot \vec{p}'}{i |\vec{p}'|^2} \frac{\vec{\alpha}' \times \vec{\alpha}'}{i} \frac{\vec{\alpha}' \cdot \vec{p}'}{i |\vec{p}'|^2} \left(\frac{1-\beta'}{2} \right) \quad (18)$$

We are encouraged to report these observations by the remark of J. Kuti that potentials something like (2) occur in certain "bag models" of hadrons ²⁾.

ACKNOWLEDGEMENTS

We thank Prof. H. Bacry and Dr. Z. Kunszt for useful discussions.

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