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ADLER'S THEOREM IN FINITE MASSLESS QED AND
POSSIBLE EXTENSIONS TO NON-ABELIAN GAUGE
THEORIES -- AN ADDENDUM

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ABSTRACT

The indefinite metric produced by the ghost fields in the Coulomb gauge in Yang-Mills theories is discussed. It is shown that the ghosts greatly complicate the job of proving, or disproving, an Adler theorem in this gauge. An old result of Schwinger for Coulomb gauge Yang-Mills theories is also found to be compromised by ghosts.

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In a recent note ¹⁾ we have attempted to analyze the consequences of assuming an eigenvalue condition that would render Yang-Mills theories finite. In particular, we wished to see if the vacuum polarization function for the Yang-Mills quanta would develop an essential singularity at the putative eigenvalue. That this happens in finite massless QED was first shown by Adler ²⁾. In our note we presented a new proof of this result making use of the Callan-Symanzik equations to simplify the combinatorial analysis. We were able to prove that since, at the eigenvalue,

$$\langle 0 | j_\mu(0) | n \rangle_{\alpha=\alpha'} = 0 \quad (1)$$

then also

$$\frac{\partial}{\partial \alpha^n} \langle 0 | j_\mu(0) | n \rangle_{\alpha=\alpha'} = 0 \quad (2)$$

for all n . Here $j_\mu(x)$ is the electromagnetic current, α' is the eigenvalue, and $|n\rangle$ is any of the complete set of states on the mass shell that must be included in the sum over states in the Lehmann-Källén weight function for Z_3 . Since, in QED, Z_3 is gauge-independent, the indefinite metric terms in the sum will simply cancel themselves out. Hence, we chose to work in the Coulomb gauge where they are not there in the first place. In this gauge no problems of positivity of the metric arise.

In going over to the Yang-Mills case, one is at once confronted by the fact that Z_3 is gauge dependent and can be positive, negative, or even zero, depending on the gauge. However, in 1962, Schwinger ³⁾ made the following observation: he noted

$$\begin{aligned} & \frac{1}{i} \sum_{i,a} \langle 0 | [A_i^a(0), \dot{A}_i^a(0)] | 0 \rangle \\ & = 2 \sum_{i,a} \langle 0 | A_i^a(0) H A_i^a(0) | 0 \rangle > 0 \end{aligned} \quad (3)$$

in the Coulomb gauge for Yang-Mills theories, where H is the Coulomb gauge Hamiltonian. It is just this quantity that enters the sum rule for Z_3 . Implicit in this argument is the use of a definite metric. At the time that Schwinger made his quantization, it was not known that ghost states -- which have an indefinite metric -- had also to be included in this sum. At the

one-loop level in perturbation theory, including ghosts, we verified ⁴⁾ by an explicit calculation that this inequality still obtains. It turns out that this result is also contained in an earlier work by Mohapatra ⁵⁾, who in fact shows that the unphysical poles -- at $p^2 = 0$ -- simply cancel out for one loop in the Yang-Mills propagator.

Recently, Zinn-Justin ⁶⁾ raised the question of whether or not this was some accident of low-order perturbation theory, or whether one is dealing with some general feature of the Coulomb gauge. The remarks that follow were the result of many discussions with Zinn-Justin which we are very pleased to acknowledge. It appears as if these perturbation theory results were accidental, and if the Schwinger inequality is true it can only be the result of complicated cancellations between definite and indefinite metric states -- cancellations that appear to be hopelessly difficult to verify in general. The bearing of this on the generalization of the Adler theorem will be discussed below.

The basic technique to be used is the Ward-Slavnov-Taylor ⁷⁾ identities for the Coulomb gauge. To derive these identities, we consider a Lagrangian composed as follows

$$\begin{aligned} \mathcal{L}(x) = & \mathcal{L}(x)^{Y.M.} + \mathcal{L}(x)^G + \eta_\mu^a(x) A_\mu^a(x) \\ & + \Lambda^a(x) \nabla \cdot \underline{A}^a(x) + \eta^a(x) \Lambda^a(x). \end{aligned} \quad (4)$$

Here $\mathcal{L}^{YM}(x)$ is the Yang-Mills Lagrangian, $\mathcal{L}^G(x)$ is the non-Hermitean ghost Lagrangian, $\eta_\mu^a(x)$ is an external source for the Yang-Mills field, $\Lambda^a(x)$ is a Lagrange multiplier introduced to enforce the gauge condition, and $\eta^a(x)$ is the $\Lambda^a(x)$ source. We assume, for simplicity, that $\Lambda^a(x)$ is invariant under gauge transformations. We may construct the action functional and derive the propagators and Ward identities. We obtain the following mixed propagator matrix; i and j are spatial indices, and 0 is the longitudinal mode. Λ refers to the Λ mode;

$$W_{(\underline{p}, p_0)}^{ab} = \delta_{ab} \quad W(\underline{p}, p_0) = \delta_{ab} \quad i$$

	j	0	1
i	$-\frac{1}{c} \left(\delta_{ij} - \frac{p_i p_j}{\underline{p}^2} \right)$	0	$\frac{p_i}{\underline{p}^2}$
0	0	$\frac{1}{a}$	$-p_0 \frac{b}{a}$
1	$-\frac{p_i}{\underline{p}^2}$	$p_0 \frac{b}{a}$	0

(5)

In this expression, a, b, c are arbitrary functions of $\underline{p}^2$ and p_0^2 . There is also an extra condition that implies that

$$\lim_{\underline{p}^2 \rightarrow 0} b \sim \sqrt{\frac{ca}{\underline{p}^2}} \frac{1}{p_0} \quad (6)$$

The essential Ward identity is the equation

$$\begin{aligned} & \Gamma_{abc}^{\mu' \nu' j}(\kappa, p, q) \frac{q_j}{q^2} W^{\mu' \mu}(\kappa) W^{\nu' \nu}(p) \\ & - \Gamma_{abc}^{\mu' \nu' 0}(\kappa, p, q) q_0 \frac{b}{a} W^{\mu' \mu}(\kappa) W^{\nu' \nu}(p) \\ & = K_\mu \Gamma_{bac}^{\nu'}(p, \kappa, q) W^{\nu' \nu}(p) G(\kappa) G(q) \\ & + i \Gamma_{abc}^{\nu' \mu}(\kappa, p, q) W^{\nu' \nu}(p) G(q) \\ & + p_\nu \Gamma_{abc}^{\mu'}(\kappa, p, q) W^{\mu' \mu}(\kappa) G(p) G(q) \\ & + i \Gamma_{abc}^{\mu' \nu}(\kappa, p, q) W^{\mu' \mu}(\kappa) G(q) \end{aligned} \quad (7)$$

The notation is the following: call FT a Fourier transform and let "amp" mean "amputated" -- no Green's function legs -- then

$$\Gamma_{abc}^{\mu' \nu' \lambda}(\kappa, p, q) \equiv \text{FT} \langle 0 | T(A_\mu^a(x) A_\nu^b(y) A_\lambda^c(z)) | 0 \rangle_{\text{amp}} \quad (8)$$

$$\int_{ab} G(\kappa) \cdot \equiv \cdot \text{FT} \langle 0 | T M^{ab}(x, y)^{-1} | 0 \rangle. \quad (9)$$

This defines the ghost propagator, and hence

$$\Gamma_{abc}^{\mu}(\kappa, \nu, q) \cdot \equiv \cdot \text{FT} \langle 0 | T (A_{\mu}^a(x) M^{bc}(y, z)^{-1}) | 0 \rangle_{\text{amp}} \quad (10)$$

$$\Gamma_{abc}^{\mu\nu}(\kappa, \nu, q) \cdot \equiv \cdot \text{FT} g \epsilon_{ab'c'} \langle 0 | T (A_{\mu}^a(x) A_{\nu}^{b'}(y) M^{c'c}(y, z)^{-1}) | 0 \rangle_{\text{amp}} \quad (11)$$

There is, in addition, the equation

$$\begin{aligned} \Delta_x^{\lambda} M^{ab}(x, y)^{-1} + g \epsilon_{ars} \partial_{xi} (A_i^r(x) M^{sb}(x, y)^{-1}) \\ = \int_{ab} \delta^4(x-y). \end{aligned} \quad (12)$$

These equations enable us to analyze the sum

$$\sum_{i, a, n} \langle 0 | A_i^a(0) | n \rangle \langle n | A_i^a(0) | 0 \rangle$$

for $|n\rangle$ including all two-particle states; i.e.,

- a) two transverse A's
- b) a transverse and a longitudinal A
- c) two longitudinal A's
- d) two ghosts

in the neighbourhood of the unphysical singularity

$$p^2 = \kappa^2 = q^2 \sim 0.$$

Without giving the details, which are straightforward applications of the Ward identities, we state our results.

- i) If the ghost couplings could be ignored then, in fact, the residues of the poles of the remaining terms precisely cancel. This is, in another language, essentially what Schwinger found.

- ii) If we could put the external four momentum, say k on the mass shell, this cancellation would persist even with the ghost states.
- iii) However, when the external momentum is not on the mass shell, the cancellation does not occur and there is no a priori reason to believe the Schwinger inequality. A new argument, if there is one, must be found.

We may now discuss the bearing of this on the generalization of the Adler theorem ²⁾. All proofs of the Adler theorem, known to us, use positivity in an essential way. This causes no problems in QED because one is dealing with gauge invariant objects. In the Yang-Mills theory, positivity of the metric poses a very non-trivial problem for this program. Standard techniques do not seem to be adequate. Indeed, one is led to wonder if the striking results of Adler ²⁾ are peculiar to Abelian gauge theories alone.

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