Ref.TH.2035-CERN

SUPERSYMMETRIC GAUGE THEORIES IN TWO DIMENSIONS

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ABSTRACT

Supersymmetric Yang-Mills Lagrangians in two dimensions are constructed. It turns out that the Yang-Mills multiplet contains a pseudo-scalar and a spinor beyond the vector boson, while the matter multiplet contains a scalar and a spinor. Contrary to the four-dimensional case, the interaction is polynomial in any supersymmetric gauge. In the Abelian case, with a charged massless multiplet, the supersymmetric extension of Schwinger QED is obtained. It is found that, in contrast to the Schwinger model, the photon self-energy is logarithmically divergent in the one-loop approximation.

Ref.TH.2035-CERN

12 June 1975

In the present note we construct supersymmetric gauge theories in two dimensions. For the pure supersymmetric Yang-Mills interaction, we find that the Lagrangian contains the usual Yang-Mills interaction of a vector boson, a spinor and a pseudoscalar, and an additional Yukawa interaction among the pseudoscalar and the spinor. All these fields belong to the adjoint representation of the Yang-Mills group. When a supersymmetric matter field is added (an additional scalar and a spinor are present), the Lagrangian contains, beyond the new Yang-Mills couplings, additional Yukawa and quartic interactions. In the Abelian situation we have what should be called the supersymmetric extension of Schwinger QED ¹⁾.

Let us start with the construction of such supersymmetric Lagrangians. We use the technique of superfields in two dimensions by a straightforward adaptation of the known technique in four dimensions ²⁾.

A scalar superfield has the following structure

$$S(x, \theta) = A(x) + i \bar{\theta} \psi(x) + \frac{i}{2} \bar{\theta} \theta F(x) \quad (1)$$

where θ, ψ are Majorana spinors, and A, ψ, F are fields which transform under a supersymmetry transformation according to the definition given in Ref. 3).

The covariant derivative is given by

$$\bar{D}_\alpha = \frac{\partial}{\partial \theta} + i \bar{\theta} \gamma_\alpha \partial \quad (2)$$

From (1) one can construct the multiplet

$$\frac{i}{2} \bar{D} D S = F + i \bar{\theta} \gamma_\alpha \partial \psi + \frac{i}{2} \bar{\theta} \theta \square A \quad (3)$$

The free-massless equation is given by

$$\bar{D} D S = 0 \quad (4)$$

Therefore the massless free field Lagrangian is the last component of the following multiplet

$$\frac{i}{2} S \bar{D} D S = \dots + i \bar{\theta} \theta \left(\frac{1}{2} A \square A + \frac{1}{2} F^2 - \frac{i}{2} \bar{\psi} \gamma_\alpha \partial \psi \right) \quad (5)$$

More general fields are obtained providing a spinor index to a scalar multiplet.

The vector multiplet has to be identified with the spin- $\frac{1}{2}$ superfield. In fact, we have

$$V_\alpha = \xi_\alpha + \frac{1}{2} \gamma_5 \gamma_\mu \theta B^\mu + \frac{1}{2} M \theta + \frac{1}{2} N \gamma_5 \theta + \frac{i}{2} \bar{\theta} \theta S_\alpha \quad (6)$$

where ξ_α, ζ_α are Majorana spinors, B_μ is a vector, M and N , respectively, a pseudoscalar and a scalar.

An Abelian gauge transformation on (6) is defined as

$$V_\alpha \rightarrow V_\alpha + i (\gamma_5 D)_\alpha \Lambda \quad (7)$$

Λ being a scalar superfield.

Therefore M and $\lambda = \zeta - \gamma \cdot \partial \xi$ are gauge-invariant fields, B_μ transforms as usual $B_\mu \rightarrow B_\mu + \partial_\mu \Lambda$ and ξ, N are submitted to arbitrary translations under gauge transformations. The two-dimensional Wess-Zumino gauges⁴⁾ are therefore defined as follows

$$\xi = N = 0 \quad \delta M = \delta \lambda = 0 \quad \delta B_\mu = \partial_\mu \Lambda \quad (8)$$

Λ being a gauge function.

The quantity

$$\bar{D} V = \bar{D}_\alpha V_\alpha = M - i \bar{\theta} \lambda + \frac{i}{2} \bar{\theta} \theta G \quad (9)$$

being $G = \epsilon_{\mu\nu} \partial^\mu B^\nu$, is manifestly gauge invariant because $\bar{D}_{V_5} D = 0$. Note in fact that in terms of component fields, $\delta M = \delta \lambda = \delta G = 0$ under gauge transformations.

The free Lagrangian for the vector multiplet is the last component of the following scalar multiplet

$$\begin{aligned} \frac{i}{2} \bar{D} V \bar{D} D \bar{D} V &= \dots + i \bar{\theta} \theta \left[\frac{G^2}{2} + \frac{1}{2} M \square M - \frac{i}{2} \bar{\lambda} \gamma \cdot \partial \lambda \right] = \\ &= \dots + i \bar{\theta} \theta \left[-\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} (\partial_\mu M)^2 - \frac{i}{2} \bar{\lambda} \gamma \cdot \partial \lambda \right] \end{aligned} \quad (10)$$

(up to derivatives).

In order to construct the interaction with a matter multiplet, we consider the gauge transformation of a charged scalar multiplet

$$\delta S_1 = g \wedge S_2 \quad \delta S_2 = -g \wedge S_1 \quad (11)$$

where $S = S_1 + iS_2$. Therefore

$$\delta S = -ig \wedge S \quad \delta S^* = ig \wedge S^* \quad (12)$$

From (7), (11) and (12) one can easily construct the following covariant derivative

$$\nabla_\alpha = D_\alpha + g (\gamma_5 V)_\alpha \quad (13)$$

in fact under a gauge transformation

$$\delta \nabla_\alpha S \rightarrow -ig \wedge \nabla_\alpha S \quad (14)$$

The gauge invariant Lagrangian for the matter field turns out to be given by the last component of the following superfield

$$-\frac{1}{2} \bar{\nabla} S^* \nabla S \quad (15)$$

Therefore the complete Lagrangian of the Abelian gauge theory is given by (last component) the following scalar multiplet

$$\frac{1}{2} \bar{D} V \bar{D} D \bar{D} V - \frac{1}{2} \bar{D} S^* D S - \frac{1}{2} g^2 \bar{V} V S^* S - g \bar{V} \gamma_5 S^* \bar{D} S \quad (16)$$

Working out the Lagrangian in terms of component fields, after having put $N = \xi = 0$, we get

$$\begin{aligned}
 \mathcal{L} = & -\frac{1}{4} F_{\mu\nu}^2 - \frac{i}{2} \bar{\lambda} \gamma \cdot \partial \lambda - \frac{1}{2} (\partial_\mu M)^2 - \frac{i}{2} \bar{\Psi}_1 \gamma \cdot \partial \Psi_1 - \frac{i}{2} \bar{\Psi}_2 \gamma \cdot \partial \Psi_2 \\
 & - \frac{1}{2} (\partial_\mu A_1)^2 - \frac{1}{2} (\partial_\mu A_2)^2 - e B^\mu A_1 \overleftrightarrow{\partial}_\mu A_2 + i e \bar{\Psi}_1 \gamma_\mu \bar{\Psi}_2 B^\mu \\
 & + i e M \bar{\Psi}_1 \gamma_5 \Psi_2 + i e (A_1 \bar{\Psi}_2 - A_2 \bar{\Psi}_1) \gamma_5 \lambda - \\
 & - \frac{e^2}{2} (B_\mu^2 + M^2) (A_1^2 + A_2^2)
 \end{aligned}
 \tag{17}$$

where $e = 2g$.

One can, of course, add a mass term $m \bar{\Psi} \Psi$ to the above Lagrangian which would give, after the elimination of the auxiliary fields F_1, F_2 , the mass term for A_1, A_2 and Ψ_1, Ψ_2 , i.e.,

$$-\frac{1}{2} m^2 (A_1^2 + A_2^2) - \frac{i}{2} m (\bar{\Psi}_1 \Psi_1 + \bar{\Psi}_2 \Psi_2)$$

The above Lagrangian ($m=0$) provides the supersymmetric extension of Schwinger QED.

One could still add a mass term for the vector multiplet (massive vector theory), namely $M^2 \bar{V} V$. In this case, N and ξ cannot be gauged away and ξ becomes a physical degree of freedom. In fact, without interaction, the spinors $\xi + M \bar{\xi}$, $\gamma_5 (\xi - M \bar{\xi})$ both satisfy the Dirac equation with mass M .

Let us now consider the non-Abelian situation, i.e., a supersymmetric Yang-Mills theory. We first consider the pure Yang-Mills interaction. Interaction with matter is straightforward and will be considered later.

The non-Abelian gauge transformation is

$$\delta V_\alpha = i (\partial_5 D)_\alpha \Lambda + g [\Lambda, V_\alpha]
 \tag{18}$$

The quantity

$$\tilde{G} = \bar{D} V - i g \bar{V} \gamma_5 V
 \tag{19}$$

is covariant; in fact from (18) we have

$$\delta \tilde{G} \rightarrow g [\wedge, \tilde{G}] \quad (20)$$

under Yang-Mills transformations.

As usual V_α, Λ are $n \times n$ dimensional matrices, namely $V_\alpha = \lambda^i V_\alpha^i$ $i=1 \dots N^2-1$ for $SU(N)$, and λ^i are the generators in the vector representation.

Consider now the quantity

$$\nabla_\alpha \tilde{G} = D_\alpha \tilde{G} + i g [\gamma_5 V, \tilde{G}] \quad (21)$$

using the transformation property (20) for $\tilde{G}$ we get

$$\delta \nabla_\alpha \tilde{G} \rightarrow g [\wedge, \nabla_\alpha \tilde{G}] \quad (22)$$

under a Yang-Mills transformation. Therefore the last component of the following scalar multiplet

$$-\frac{1}{2} \text{Tr} \bar{\nabla}^\alpha \tilde{G} \nabla_\alpha \tilde{G} \quad (23)$$

provides the supersymmetric two-dimensional pure Yang-Mills theory. In terms of components, it looks like

$$\text{Tr} \left(-\frac{1}{4} \tilde{G}_{\mu\nu} \tilde{G}^{\mu\nu} - \frac{i}{2} \bar{\lambda} \gamma_\mu \nabla^\mu \lambda - \frac{1}{2} (\nabla_\mu M)^2 + i g M \bar{\lambda} \gamma_5 \lambda \right) \quad (24)$$

where ∇_μ are the usual covariant derivatives. So in contrast to the four-dimensional case ⁵⁾ it is not true that the Yang-Mills interaction of B_μ , M and λ is supersymmetric when M and λ are in the adjoint representation. One has to add the extra Yukawa coupling $\text{Tr}(g M \bar{\lambda} \gamma_5 \lambda)$. Such a coupling was not present in four-dimension because of the absence of the pseudoscalar component field in the vector multiplet.

The interaction with matter is straightforward. The covariant derivative for the matter field S_i is simply given by

$$\nabla_{\alpha ij} = D_{\alpha} \delta_{ij} - g T_{ij}^a V_{\alpha}^a \quad (25)$$

where T_{ij}^a are the N^2-1 generator matrices of the representation to which S_i belongs. Therefore the supersymmetric Yang-Mills interaction of the matter multiplet is provided by the following expression

$$-\frac{1}{2}(\bar{\nabla} S^*)_i (\nabla S)_i + i m S_i^* S_i \quad (26)$$

Supersymmetric perturbation theory can be worked out having the propagators of the superfields, using the functional formalism in superspace according to Ref. 6). We give for completeness the superfield propagators in two dimensions.

For the matter field S we have the free equation of motion

$$\frac{i}{2} \bar{D} D S - m S = J \quad (27)$$

and the functional derivative

$$\frac{\delta S(1)}{\delta S(2)} = \frac{1}{2} \delta^2(x_1 - x_2) \bar{\theta}_{12} \theta_{12} \quad \theta_{12} = \theta_1 - \theta_2 \quad (28)$$

multiplying (27) by $i/2(\bar{D}D)$ and using the identity $-1/4(\bar{D}D)^2 = \square$, one has

$$\begin{aligned} \square S - \frac{i}{2} m \bar{D} D S &= \frac{i}{2} \bar{D} D J \\ \square S - m^2 S - m J &= \frac{i}{2} \bar{D} D J \\ (\square - m^2) S &= \left(\frac{i}{2} \bar{D} D + m \right) J \\ S &= \frac{1}{\square - m^2} \left(\frac{i}{2} \bar{D} D + m \right) J \end{aligned} \quad (29)$$

$$\Delta(x, x_2, \theta_1, \theta_2) = \delta S(x, \theta_1) / \delta J(x_2, \theta_2) \quad (30)$$

$$\begin{aligned}\Delta(k\theta_1, -k\theta_2) &= \frac{1}{k^2+m^2} \left(\frac{i}{2} \bar{D}D + m \right) \frac{1}{2} \bar{\theta}_{12} \theta_{12} = \\ &= \frac{1}{k^2+m^2} e^{\bar{\theta}_1 \gamma \theta_2 k} \left(i + \frac{m}{2} \bar{\theta}_{12} \theta_{12} \right)\end{aligned}\quad (31)$$

For the vector field one has

$$\frac{\delta V_\alpha(x_1\theta_1)}{\delta V_\beta(x_2\theta_2)} = \frac{1}{2} \int_{\alpha\beta} \delta^2(x_1-x_2) \bar{\theta}_{12} \theta_{12} \quad (32)$$

Adding a gauge breaking term to the Lagrangian of the form

$$\frac{1}{\alpha} \bar{D}\gamma_5 V \bar{D}D \bar{D}\gamma_5 V$$

one gets for the vector propagator, the following expression

$$\langle V_\alpha(k\theta_1) V_\beta(-k\theta_2) \rangle = \frac{e^{\bar{\theta}_1 \gamma \theta_2 k}}{k^4} \left[\gamma \cdot k (1-\alpha) + (1+\alpha) \delta_{\alpha\beta} k^\mu \bar{\theta}_{12} \theta_{12} \right] \quad (33)$$

As a final point, we would like to consider the vacuum polarization graph at the one-loop level for the Abelian theory, and to explore what happens when the mass of the matter fields goes to zero. We know that in spinor QED ¹⁾ one gets the Schwinger anomaly connected to the generation of the mass $m_B^2 = e^2/\pi$ for the boson B_μ . (In Schwinger QED, the one-loop graph gives the exact result because higher-loop graphs vanish identically.)

Using the Pauli-Villars technique, which contrary to the dimensional regularization is a manifestly supersymmetric procedure, we find that the self-energy in massive spinor QED, at the one-loop level, is given by

$$q^2 \Pi(q^2) = \frac{e^2}{\pi} \int_0^1 dx \frac{x(1-x)}{\frac{m^2}{q^2} + x(1-x)} \quad (34)$$

while in the supersymmetric version, is

$$q^2 \Pi_{\text{SUSY}}(q^2) = \frac{e^2}{4\pi} \int_0^1 dx \frac{1}{\frac{m^2}{q^2} + x(1-x)} \quad (35)$$

They are connected by the relation

$$q^2 \pi_{\text{susy}}(q^2) = -\frac{q^2}{4m^2} \left(q^2 \pi(q^2) - \frac{e^2}{\pi} \right) \quad (36)$$

Such a relation can be understood from the fact that the self-energy graph of the supersymmetric theory is automatically transverse, while the self-energy graph of spinor QED is transverse only after the Pauli-Villars regulation has been introduced which just adds the constant term connected to the Schwinger anomaly. In this approximation for both theories, the photon is massless for $m \neq 0$, being

$$\pi(q^2) \underset{q^2 \rightarrow 0}{\sim} \frac{e^2}{6\pi} \quad \pi_{\text{susy}}(q^2) \underset{q^2 \rightarrow 0}{\sim} \frac{e^2}{4\pi}$$

but in the limit of zero mass, one gets

$$q^2 \pi(q^2) \underset{m \rightarrow 0}{=} \frac{e^2}{\pi} \quad q^2 \pi_{\text{susy}}(q^2) \underset{m \rightarrow 0}{\sim} \log m^2$$

i.e., the Schwinger mass generation for the spinor QED, and a divergent result for the supersymmetric QED.

ACKNOWLEDGEMENTS

I would like to thank R. Barbieri, R.J. Crewther, J. Zinn-Justin and B. Zumino for many useful discussions.

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