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NULL PLANE FIELD THEORY AND COMPOSITE MODELS

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ABSTRACT

A sketch is given of an approach to null plane field theory which (it is hoped) illuminates the relation between the relativistic parton model, the non-relativistic quark model, and various $SU(6)$ and $SU(6)_W$ broken symmetry schemes, including those of Melosh.

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1. INTRODUCTION

This is a sketch of an approach to null plane field theory ¹⁾ designed to meet the following requirements :

- 1) to exhibit, quite explicitly, the null plane approach as a method in ordinary quantum field theory, as distinct from a fresh departure from the classical Lagrangian;
- 2) to make it easy to pass to a non-relativistic approximation, and so hold closely together in one formalism the relativistic parton model and the non-relativistic quark model;
- 3) to extend to interacting field theory ²⁾ considerations on the Melosh transformations previously made for free field theory ³⁾, for a potential model ⁴⁾, and for a quasi-potential model ⁵⁾.

None of these requirements has been definitely and convincingly met. This is a preliminary account, to an informal meeting, of undigested ideas.

2. COMPOSITE SYSTEMS ON THE NULL PLANE

There is a naïve way of regarding complex systems in field theory as made up of non-interacting particles. Imagine the interaction switched off at some instant. The state then indeed becomes one of free particles, and this free particle state can be used to characterize the complex system. In this sense the physical vacuum at zero time is represented by

$$T e^{-i \int d^4x \Theta(-t) \mathcal{H}_I(x)} | \text{bare vacuum} \rangle \quad (1)$$

where T denotes time-ordering, Θ is the step function, $\mathcal{H}_I$ is the interaction Hamiltonian in the interaction representation, and an adiabatic switch-on is implicit. To deal with states other than vacuum suitable source terms can be added to $\mathcal{H}_I$.

In this picture even the vacuum is complicated. The instantaneous switch-off does not conserve energy, so that even the vacuum gives rise to a sea of bare particles and antiparticles.

However, why switch off everywhere at the same time ? There is nothing sacred about simultaneity. Let us switch off instead on a null plane, taking instead of (1)

$$T e^{-i \int d^4x \Theta(-t-z) \mathcal{L}_I(x)} | \text{bare vacuum} \rangle \quad (2)$$

Again energy is not conserved, and now also the third component of momentum is not conserved. But because the switch-off is invariant under simultaneous displacement in time and along the z direction in space, the sum of energy and momentum (in the z direction) is conserved. So a state containing particles with momenta p can arise from the vacuum only if

$$\sum (p_0 + p_3) = 0$$

where

$$p_0 = \sqrt{m^2 + \vec{p}^2} > |p_3|$$

(excluding the case of zero mass). Since all the terms in the sum are positive only the zero particle state can arise from the vacuum.

In this picture real and bare vacua are identical. This is very convenient for composite models, since the constituents of the composite do not have to be disentangled from those of the vacuum.

The null plane switch off is not rotation invariant (except for rotation about the 3-axis). The description of rotation is correspondingly complicated, becoming interaction dependent. On the other hand, the null plane switch-off is invariant for Lorentz boosts in the 3-direction. The description of this part of Lorentz invariance is correspondingly simplified.

3. NULL PLANE DYNAMICS

We can make (2) look more like (1) by introducing

$$\hat{\mathcal{L}}_I(t, x, y, z) = \mathcal{L}_I(t-z, x, y, z) \quad (3)$$

Then (2) becomes

$$T e^{-i \int d^4x \Theta(-t) \hat{\mathcal{L}}_I(x)} | \text{bare vacuum} \rangle \quad (4)$$

at least if

$$[\mathcal{L}_I(x), \mathcal{L}_I(y)] = 0 \quad (5)$$

for $|x_0 - y_0| \leq |\vec{x} - \vec{y}|$

These commutators arise because of the reordering in going from (2) to (4); ordering operators $\hat{\mathcal{L}}_I(t, x, y, z)$ with respect to t is not the same as ordering operators $\mathcal{L}(t, x, y, z)$. We will tentatively suppose (4) to be equivalent to (2), at least in some suitably regulated version of the theory.

Comparing (4) and (2), we see that null plane dynamics is the same as equal time dynamics but with a different Hamiltonian. All the usual quantum mechanical formalism is therefore available. When $\mathcal{L}_I$ is a combination of field variables ψ , $\hat{\mathcal{L}}_I$ is the same combination of new fields

$$\hat{\psi}(t, x, y, z) = \psi(t - z, x, y, z) \quad (6)$$

Since there is only a change of time argument here, the $\hat{\psi}$ satisfy the same equations of motion

$$i \frac{\partial \hat{\psi}}{\partial t} = [\hat{\psi}, H_0] \quad (7)$$

where H_0 is the same, original, unmodified, constant, free Hamiltonian. From the interaction picture we can pass to a Schroedinger picture in the usual way, taking $t=0$ in the free fields $\hat{\psi}$. The complete Hamiltonian is

$$\hat{H} = H_0 + \int d^4x \, \delta(t) \, \hat{\mathcal{L}}_I(x) \quad (8)$$

From this Schroedinger picture we can pass in the usual way to a Heisenberg picture. As usual the bound states of composite systems will appear as eigenstates of the total Hamiltonian (8).

Note, however, an important difference of detail between null plane and equal time dynamics. The Hamiltonian (8) does not conserve z momentum, but rather the combination of z momentum and energy :

$$\Delta \sum (p_0 + p_3) = 0 \quad (9)$$

It is usual to write

$$h = p_0 + p_3 \quad (10)$$

and to denote the other components of momentum by $\vec{p}_\perp$. Then the Hamiltonian connects states such that

$$\Delta \sum \vec{p}_\perp = \Delta \sum h = 0 \quad (11)$$

4. NON-RELATIVISTIC QUARK MODEL AND RELATIVISTIC PARTON MODEL

In view of the success of non-relativistic potential models it may be useful to write the null plane Hamiltonian in the form

$$\hat{H} = H_0 + V + \left(\int d^4x \delta(t) \hat{\mathcal{H}}_t(x) - V \right) \quad (12)$$

where V is a potential, i.e., it scatters particles (or antiparticles) on one another but does not produce or absorb them. In null plane dynamics it is natural to choose V such that

$$\Delta \sum h = 0 \quad (13)$$

rather than

$$\Delta \sum p_3 = 0 \quad (14)$$

But when all particles move slowly, i.e.,

$$h = p_3 + m + o(c^{-2})$$

and when particles are not absorbed or produced, (13) implies (14) apart from relativistic corrections. Neglecting in H_0 relativistic corrections to particle energy

$$p_0 \approx m + \vec{p}^2/2m$$

the first two terms of (12) give a non-relativistic potential model.

When the remaining terms of (12) are taken into account, either by formal perturbation theory or more realistically, eigenstates Ψ of the full Hamiltonian are mapped onto eigenstates Ψ' of the model :

$$\Psi = F \Psi' \quad (15)$$

The linear operator F must create the gluons and particle-antiparticle parts contained in the real state Ψ and not in the model state Ψ' - in the relativistic parton model but not in the non-relativistic quark model.

The notion of model state and of the associated transformation to the real state have long been familiar in, for example, nuclear physics, as has been recalled in the quark model context by G. Morpurgo ⁶⁾.

Matrix elements of an operator, for example a current J , can of course be calculated from either state vector :

$$(\Psi, J \Psi) = (\Psi', F^+ J F \Psi') \quad (16)$$

There is a choice between complicated states Ψ and simple operators J , on the one hand, or simple states Ψ' and complicated operators $F^+ J F$ on the other. For this reason we have suggested ⁴⁾ that the two pictures Ψ and Ψ' of the same state be called "current" and "constituent" pictures respectively - the first involving relatively simple forms for currents and the latter involving relatively few constituents. The model operator F , which transforms between what we call constituent and current pictures, is something quite other than what we call the "Melosh transformation" - which is a way of exhibiting the approximate spin independence of interactions. This distinction has been insisted on also by G. Morpurgo ⁶⁾.

5. SPIN INDEPENDENCE

Interaction matrix elements tend to become spin independent when all the particles involved are slow. That is to say that we have invariance under rotation of the spin states, leaving the momenta fixed. This follows from ordinary rotation invariance; we can omit the rotation of momenta when these are negligibly small. However, it may not be realistic, in some case of interest, to regard all particles as slow. There may be a light meson (or even photon) involved, which has to be considered as relativistic. It may be enough, however, that other particles are slow, as we will see in some examples.

We work here in interaction representation, so that we can exploit knowledge of free field operators (Appendix) and pass readily to null plane dynamics.

Note first the spin independence of the free Dirac Hamiltonian

$$\begin{aligned} \int d^4x \delta(\epsilon) \psi^\dagger (\beta m + \vec{\alpha} \cdot \vec{p}) \psi \\ = \int d^4x \delta(\epsilon) \phi^\dagger \beta \sqrt{m^2 + \vec{p}^2} \phi \end{aligned}$$

where ϕ is the Foldy-Wouthuysen field operator. This is invariant under

$$\delta \phi = -i \Lambda \phi$$

provided

$$[\beta, \Lambda] = 0$$

i.e.,

$$\Lambda = \frac{1}{2} (1 \pm \beta) (1, \vec{\sigma})$$

This symmetry will be denoted by

$$(U(2) \times U(2))_{FW} \quad (17)$$

Moreover, it will be convenient to separate the generators into two sets, defining two subgroups

$$\left. \begin{array}{ll} U(2)_{FW} & \Lambda = (1, \vec{\sigma}) \\ U(2)_{FWW} & \Lambda = (1, \beta \sigma_x, \beta \sigma_y, \sigma_z) \end{array} \right\} \quad (18)$$

Given more than one fermion field, say three, of equal mass, these groups generalize in an obvious way to $(U(6) \times U(6))_{FW}$, etc.

Add now an interaction

$$\mathcal{H}_I = g \bar{\psi} A \psi \quad (19)$$

where A is a neutral scalar gluon. In terms of the Foldy-Wouthuysen field ϕ ,

$$\psi = \left(1 - \frac{i \vec{\gamma} \cdot \vec{p}}{2m} + \dots \right) \phi \quad (20)$$

so that to first order in small quantities $(\vec{p}/m)$, $(\vec{p}'/m)$,

$$g^{-1} \mathcal{H}_I = \phi^\dagger \beta A \phi + \phi^\dagger \frac{i\vec{r} \cdot \vec{p}'}{2m} \beta \phi - \phi^\dagger \beta \frac{i\vec{r} \cdot \vec{p}}{2m} \phi \quad (21)$$

where $\vec{p}'$ denotes a final momentum, as distinct from an initial momentum $\vec{p}$, or alternatively a differential operator on $\phi^\dagger$ rather than ϕ . The leading term in (21) is invariant under the full $(U(2) \times U(2))_{FW}$. The subsequent terms are not invariant, and it looks as if symmetry breaking sets in at first order. But it can be argued that for many purposes this is effectively not so. Suppose that we do not have sources or sinks of gluons other than the interaction Hamiltonian (19). Then in the course of a perturbation calculation such terms have to be paired off with one another. Moreover, when all fermions are slow, leading and subleading parts of (21) cannot pair together. This is because the subleading parts necessarily create or absorb a fermion-antifermion pair (because $i\vec{\gamma}\beta$ anticommutes with β) and therefore (in null plane dynamics) a gluon of large η ($\sim 2m$). For the leading part the reverse is true. So the symmetry breaking terms contribute in twos, and the symmetry breaking is effectively of second rather than first order.

Consider now an interaction

$$\mathcal{H}_I = g \bar{\Psi} i\gamma_\mu A_\mu \Psi \quad (22)$$

with a neutral vector gluon. To first order

$$\begin{aligned} g^{-1} \mathcal{H}_I = & -\phi^\dagger A_0 \phi \\ & + \phi^\dagger \vec{\alpha} \cdot \vec{A} \phi \\ & - \phi^\dagger \left(\frac{i\vec{r} \cdot \vec{p}'}{2m} A_0 - A_0 \frac{i\vec{r} \cdot \vec{p}}{2m} \right) \phi \\ & + \phi^\dagger \left(\frac{i\vec{r} \cdot \vec{p}'}{2m} \vec{\alpha} \cdot \vec{A} - \vec{\alpha} \cdot \vec{A} \frac{i\vec{r} \cdot \vec{p}}{2m} \right) \phi \end{aligned} \quad (23)$$

The discussion of the first and third terms here goes along the lines just given for the scalar gluon, with the same conclusions. The second term here goes along the lines just given for the scalar gluon, with the same conclusions. The second term is more troublesome; it breaks the $(U(2) \times U(2))_{FW}$ symmetry, and looks of similar order to the leading term. The subgroup

$$U(2)_{FW}$$

can be restored by augmenting the infinitesimal operation

$$\delta \phi = -i \vec{\omega} \cdot \vec{\sigma} \phi \quad (24)$$

by

$$\delta \vec{A} = \vec{\omega} \times \vec{A} \quad (25)$$

This latter is an invariance of the free Hamiltonian if we adopt the Gupta-Bleuler formalism. More generally, note that the second term in (18), because α anticommutes with β , is a fermion-antifermion hard gluon type. Again for many purposes such terms have to be taken in pairs. The emission and absorption matrix elements of the hard gluon each involve a factor $1/\sqrt{\eta}$ and the propagation involves an energy denominator of order $\eta \approx m$ (assuming $\mu \ll m$, where μ is the gluon mass). So for such contributions there is a suppression factor $\approx m^{-2}$, which has to be compared with μ^{-2} or $|\vec{p}|^{-2}$ (where $\vec{p}$ is a typical fermion momentum) for leading contributions. In this sense, whose precision clearly leaves much to be desired, $(U(2) \times U(2))_{FW}$ again holds good to first order.

We mention only briefly two other kinds of gluon ⁴⁾. With a pseudo-scalar gluon and

$$\mathcal{H}_I = g \bar{\Psi} i \gamma_5 A \Psi$$

the leading terms are already fermion-antifermion-hard-gluon-type and correspondingly suppressed. With a pseudovector gluon and

$$\mathcal{H}_I = g \bar{\Psi} i \gamma_5 \gamma_\mu A_\mu \Psi$$

the leading terms already break $(U(2) \times U(2))_{FW}$, leaving only $U(2)_{FW}$ in the sense of (24), (25). Moreover, the associated Stueckelberg pseudoscalar has gradient coupling with strength (g/μ) , so that the "small" corrections involve factors $(\vec{p}/\mu)$ as well as $(\vec{p}/m)$.

6. NULL PLANE CHARGES

The free Hamiltonian can be expressed in terms of the good components $\hat{\psi}_g$ of the null plane field (see Appendix)

$$H_0 = 2 \int d^4x \delta(t) \hat{\psi}_g \left(\frac{\eta}{2} + \frac{m^2 + \vec{p}_\perp^2}{2\eta} \right) \hat{\psi}_g \quad (26)$$

It is then clearly invariant under a symmetry

$$U(2)_w \quad \delta \hat{\psi}_g = -i \Lambda \hat{\psi}_g \quad \Lambda = (1, \sigma_x, \sigma_y, \sigma_z) \quad (27)$$

In view of the null-plane canonical anticommutators, this can be considered to be generated by the good null plane charges

$$2 \int d^4x \delta(\epsilon) \hat{\psi}_g^+ \wedge \hat{\psi}_g \quad (28)$$

A term in the interaction Hamiltonian, for example

$$\psi^+ \psi A_0 \quad (29)$$

can be expressed in terms of $\hat{\psi}_g$ by eliminating the bad components

$$\hat{\psi}_b = \eta^{-1} (m\sigma_3 - \vec{\sigma}_1 \cdot \vec{p}_1) \hat{\psi}_g \quad (30)$$

Then (29) becomes

$$\begin{aligned} & \hat{\psi}_g^+ A_0 \hat{\psi}_g + \\ & \hat{\psi}_g^+ (m\sigma_3 - \vec{\sigma}_1 \cdot \vec{p}_1') \eta^{-1} A_0 \eta (m\sigma_3 - \vec{\sigma}_1 \cdot \vec{p}_1) \hat{\psi}_g \end{aligned} \quad (31)$$

When the small quantities $(\vec{p}_1/m, \vec{p}_1'/m)$ are neglected, the interaction respects the symmetry (27). But this symmetry is broken at first order.

The Foldy-Wouthuysen symmetry of the last section is (for many purposes, we argued) still good at first order. Moreover, there is no reason why it should not be expressed on the null plane. Writing the 4x4 Dirac rotation matrices σ in the good representation (A.7) and eliminating reference to bad components by

$$\phi_b = \epsilon \sigma_3 \phi_g$$

[from (A.9)] where

$$\epsilon = \text{sign } p_0 = \text{sign } \eta = \sqrt{\eta^2} / \eta$$

we have the symmetry operations

$$\begin{aligned} U(2)_{FW} \quad \delta \hat{\phi}_g &= -i \Lambda \hat{\phi}_g & \Lambda &= (1, \epsilon \sigma_x, \epsilon \sigma_y, \sigma_z) \\ U(2)_{FWW} \quad \delta \hat{\phi}_g &= -i \Lambda \hat{\phi}_g & \Lambda &= (1, \sigma_x, \sigma_y, \sigma_z) \end{aligned} \quad (32)$$

These are generated by the null plane charges

$$2 \int d^4x \delta(t) \phi_g^+ \wedge \phi_g \quad (33)$$

These differ from the corresponding constructions, including (28), from the fields $\hat{\psi}$, by

$$\hat{\psi}_g = \frac{m + \sqrt{\eta^2} + \sigma_3 \vec{\sigma}_1 \cdot \vec{p}_1}{\sqrt{(m + \sqrt{\eta^2})^2 + \vec{p}_1^2}} \hat{\phi}_g \quad (34)$$

- the "second Melosh" transformation of (A.37).

One could of course, using the inverse of (34), rewrite the symmetry operations (32) and (33) in terms of the fields $\hat{\psi}_g$. They would then be rather complicated. On the other hand the operator for a physical quantity like a current

$$\bar{\psi} i \gamma_\mu \psi$$

is rather complicated in terms of the $\hat{\phi}_g$. So we propose calling $\hat{\psi}_g$ and $\hat{\phi}_g$ "current" and "classification" operators respectively, according to whether they simplify the expression of currents or symmetry operations.

As always in quantum mechanics there is the option, instead of changing operators while holding state vectors fixed, of holding the operators fixed and changing the state vectors. In this latter version ⁴⁾ the Melosh transformation is between "current" and "classification" pictures. It has nothing to do, in our terminology, with the relation between "current" and "constituent" pictures.

7. BOOST INVARIANCE

The good null plane charges (35) are invariant for boosts in the z direction. The Melosh modified charges (33) are not, if the full form (34) of the second Melosh transformation is used, for this transformation itself is not boost invariant. But our symmetries were good only to first order in small quantities. To this order the transformation (34) is equivalent to

$$\hat{\psi}_g = e^{\frac{\sigma_3 \vec{\sigma}_1 \cdot \vec{p}_1}{2m}} \hat{\phi}_g = \left(1 + \frac{\sigma_3 \vec{\sigma}_1 \cdot \vec{p}_1}{2m} + \dots \right) \hat{\phi}_g \quad (35)$$

- or indeed to any such form with the same first two terms, including the "first Melosh" transformation ⁷⁾

$$\hat{\psi}_g = \frac{m + \sqrt{m^2 + \vec{p}_\perp^2} + \sigma_3 \vec{\sigma}_\perp \cdot \vec{p}_\perp}{\sqrt{(m + \sqrt{m^2 + \vec{p}_\perp^2})^2 + \vec{p}_\perp^2}} \hat{\phi}_g \quad (36)$$

If (35) instead of (34) is used to define $\hat{\phi}_g$ in terms of $\hat{\psi}_g$, the symmetry operations (32) and the corresponding charges (33) become boost invariant. Moreover, this symmetry, which we might call

$$(U(2) \times U(2))_{FWB}, \quad U(2)_{FWB}, \quad U(2)_{FWWB} \quad (37)$$

is just as good (as far as our arguments go) as

$$(U(2) \times U(2))_{FW}, \quad U(2)_{FW}, \quad U(2)_{FWW}$$

Moreover, when one of the symmetries (37) is good for some composite system, because of slow motion of the constituents

$$(\vec{p}_\perp/m)^2 \ll 1 \quad |(h/m) - 1| \ll 1$$

it remains good for that same system arbitrarily boosted in the 3-direction.

8. REGULARIZATION

We return now to one of the subtleties that have been trampled over, to trample less heavily. Consider again the reordering of operators in passing from equal-time to null plane dynamics, and the commutator (5). We have of course commutation outside the light cone, but a singularity on the light cone. It is this singularity that has to be dealt with.

We consider explicitly the Fermi field; the Bose field can be treated similarly. The free field anticommutator is

$$\left. \begin{aligned} \{\psi(x), \bar{\psi}(y)\} &= -i (\gamma_\mu \partial_\mu - m) \Delta(x-y) \\ \Delta(x) &= -\frac{\epsilon(x_0)}{2\pi} \left(\delta(x^2) - \frac{m^2}{2} \frac{\mathcal{J}_1(\sqrt{-m^2 x^2})}{\sqrt{-m^2 x^2}} \theta(-x^2) \right) \\ \epsilon(x_0) &= x_0/|x_0| \quad \theta(-x^2) = \frac{1}{2} + \frac{1}{2} \epsilon(-x^2) \end{aligned} \right\} \quad (37)$$

The Bessel function can be expanded in powers of its argument, beginning with the first power.

The important point here is that the light cone singularities of the anticommutator (37) are either independent of, or proportional to a finite integral power of, the mass m . They can be removed therefore by Pauli-Villars regularization. The field $\psi(m)$ with mass m can be replaced in the interaction Hamiltonian by a combination of two fields

$$\psi(m) \rightarrow \psi(m) + i \psi(m+M) \quad (38)$$

And one can further regularize by again substituting in the regularized interaction

$$\begin{aligned} \psi(m) &\rightarrow \psi(m) + i \psi(m+M) \\ \psi(m+M) &\rightarrow \psi(m+M) + i \psi(m+M+M) \end{aligned} \quad (39)$$

And so on, to obtain the desired degree of regularization. For a sufficiently regulated theory, the formal reasoning leading to (4) is presumably sound, and the problem is shifted to that of subsequently taking the limit $M \rightarrow \infty$.

The questions of regularization, renormalization, and the limit $M \rightarrow \infty$ arise in ordinary perturbation theory in connection with closed loop diagrams. What is striking here is that the regularization is important even for tree diagrams. Consider for example the interaction [part of the vector interaction (22)]

$$\begin{aligned} g \psi^\dagger A_0 \psi &= g \hat{\psi}_g^\dagger A_0 \hat{\psi}_g \\ &+ g \hat{\psi}_g^\dagger \frac{m\sigma_3 - \vec{\sigma}_1 \cdot \vec{p}_1'}{h'} A_0 \frac{m\sigma_3 - \vec{\sigma}_1 \cdot \vec{p}_1}{h} \hat{\psi}_g \end{aligned} \quad (40)$$

This is explicitly dependent on m , and in the regulated version there are therefore terms which increase with the mass $(m+M)$ of the regulating field. The propagator for $\hat{\psi}_g$ is

$$P = \frac{1/2}{p_0 - \frac{1}{2} \left(h + \frac{m^2 + \vec{p}_1^2}{h} \right)} \quad (41)$$

and

$$\lim_{m \rightarrow \infty} m^2 P = -\eta \quad (42)$$

Thus effectively we have to add to the unregulated interaction (40) a whole series of contact terms. In order g^2 (40) is augmented by, from the regulation of the second term,

$$g^2 \hat{\psi}_g + \frac{m\sigma_3 - \vec{\sigma}_1 \cdot \vec{p}_1}{\eta_I} A_0 \frac{1}{\eta_I} A_0 \frac{m\sigma_3 - \vec{\sigma}_1 \cdot \vec{p}_1}{\eta} \hat{\psi}_g$$

where the reciprocal (η_I^{-1}) of the intermediate η can be interpreted as an integral operator.

Such contact terms are familiar from other approaches to null plane dynamics ^{1),4)}. In the external potential case ⁴⁾ we dismissed them as relativistic corrections of second order (fourth order relative to mc^2). That is again so here, at least if we restore the suppressed factors c and count them.

Is it sensible to separate these particular regularization effects from the renormalization programme as a whole? Is it sensible to speak of non-relativistic approximation, or to order terms by powers of c , when interactions are strong and loop diagrams important? And so on.

A P P E N D I X

Notation : Central European Standard ³⁾.

Plane waves

The plane wave Dirac spinors $u(p,s)$ with $p_0 = \pm \sqrt{m^2 + \vec{p}^2}$, satisfy

$$p_0 u = (\beta m + \vec{\alpha} \cdot \vec{p}) u \quad (A.1)$$

$$\sum_{\epsilon, s} u_i u_j^* = |2p_0| \delta_{ij} \quad (A.2)$$

where the summation is for given $\vec{p}$ over $\epsilon = p_0/|p_0| = \pm 1$ and two spin states s .

The corresponding Foldy-Wouthuysen spinors w satisfy

$$p_0 w = \beta \sqrt{m^2 + \vec{p}^2} w \quad (A.3)$$

$$\sum_{\epsilon, s} w_i w_j^* = |2p_0| \delta_{ij} \quad (A.4)$$

and are related to the u by

$$u = \frac{m + \sqrt{m^2 + \vec{p}^2} - i \vec{\gamma} \cdot \vec{p}}{\sqrt{(m + \sqrt{m^2 + \vec{p}^2})^2 + \vec{p}^2}} w \quad (A.5)$$

where

$$i \beta \vec{\gamma} = \vec{\alpha} \quad (A.6)$$

Good representation

$$\left. \begin{aligned} \alpha_3 &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & \beta &= \begin{pmatrix} & \sigma_3 \\ \sigma_3 & \end{pmatrix} & \vec{\alpha}_1 &= \begin{pmatrix} & -\vec{\sigma}_1 \\ -\vec{\sigma}_1 & \end{pmatrix} \\ \sigma_1 &= \begin{pmatrix} & \sigma_1 \sigma_3 \\ \sigma_1 \sigma_3 & \end{pmatrix} & \sigma_2 &= \begin{pmatrix} & \sigma_2 \sigma_3 \\ \sigma_2 \sigma_3 & \end{pmatrix} & \sigma_3 &= \begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix} \end{aligned} \right\} \quad (A.7)$$

Good and bad components

The 4-spinors can be separated into pairs of good ($\alpha_3 = +1$) and bad ($\alpha_3 = -1$) 2-spinors. From (A.1)-(A.3)

$$u_b = \eta^{-1} (m\sigma_3 - \vec{\sigma}_1 \cdot \vec{p}_1) u_g \quad (\text{A.8})$$

$$w_b = p_0^{-1} \sqrt{m^2 + \vec{p}^2} w_g \quad (\text{A.9})$$

where

$$\eta = p_0 + p_3 \quad (\text{A.10})$$

For given $\vec{p}$ and $\epsilon (= p_0/|p_0| = \eta/|\eta|)$

$$\sum_s u_{gi} u_{gj}^* = \delta_{ij} |\eta| \quad (\text{A.11})$$

$$\sum_s w_{gi} w_{gj}^* = \delta_{ij} |p_0| \quad (\text{A.12})$$

Free field operators

We associate with each plane wave an operator a , which is a destruction or creation operator for $\epsilon = \pm 1$, respectively. Normalizing covariantly

$$\{a(p, s), a^*(p', s')\} = 2|p_0| (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \delta_{ss'} \delta_{\epsilon\epsilon'} \quad (\text{A.13})$$

The ordinary Dirac field operator is then

$$\psi = \sum_{\epsilon, s} \int \frac{d^3 \vec{p}}{(2\pi)^3 2|p_0|} a(p, s) u(p, s) e^{i\vec{p} \cdot \vec{x} - i p_0 t} \quad (\text{A.14})$$

The Foldy-Wouthuysen field is

$$\phi = \sum_{\epsilon, s} \int \frac{d^3 \vec{p}}{(2\pi)^3 2|p_0|} a(p, s) w(p, s) e^{i\vec{p} \cdot \vec{x} - i p_0 t} \quad (\text{A.15})$$

The null plane Dirac field is

$$\hat{\psi}_g = \sum_{\epsilon, s} \int \frac{d^3 \vec{p}}{(2\pi)^3 |2p_0|} a(p, s) u_g(p, s) e^{i\vec{p} \cdot \vec{x} - i p_0(t-z)} \quad (A.16)$$

The null plane Foldy-Wouthuysen field is

$$\hat{\phi}_g = \sum_{\epsilon, s} \int \frac{d^3 \vec{p}}{(2\pi)^3 |2p_0|} a(p, s) \left| \frac{\eta}{p_0} \right| w_g(p, s) e^{i\vec{p} \cdot \vec{x} - i p_0(t-z)} \quad (A.17)$$

Note that

$$\vec{p} \cdot \vec{x} - p_0(t-z) = \eta z + \vec{p}_\perp \cdot \vec{x}_\perp - p_0 t \quad (A.18)$$

$$\sum_{\epsilon} \int \frac{d^3 \vec{p}}{(2\pi)^3 |2p_0|} = \int \frac{d\eta d^2 \vec{p}_\perp}{(2\pi)^3 |2\eta|} \quad (A.19)$$

$$\sqrt{m^2 + \vec{p}^2} = \pm \eta + (m^2 + \vec{p}_\perp^2)/(2\eta) \quad (A.20)$$

From the basic anticommutator (A.13) and the completeness relations (A.2), (A.4), (A.11) and (A.12), we obtain the canonical anticommutators

$$\{ \psi_i(t, \vec{x}), \psi_j^\dagger(t, \vec{y}) \} = \delta_{ij} \delta(\vec{x} - \vec{y}) \quad (A.21)$$

$$\{ \phi_i(t, \vec{x}), \phi_j^\dagger(t, \vec{y}) \} = \delta_{ij} \delta(\vec{x} - \vec{y}) \quad (A.22)$$

$$2 \{ \hat{\psi}_i(t, \vec{x}), \hat{\psi}_{gj}^\dagger(t, \vec{y}) \} = \delta_{ij} \delta(\vec{x} - \vec{y}) \quad (A.23)$$

$$2 \{ \hat{\phi}_i(t, \vec{x}), \hat{\phi}_{gj}^\dagger(t, \vec{y}) \} = \delta_{ij} \delta(\vec{x} - \vec{y}) \quad (A.24)$$

Free field equations

$$i \frac{\partial \psi}{\partial t} = (\vec{\alpha} \cdot \vec{p} + \beta m) \psi \quad (A.25)$$

$$i \frac{\partial \phi}{\partial t} = \beta \sqrt{m^2 + \vec{p}^2} \phi \quad (\text{A.26})$$

$$i \frac{\partial \hat{\psi}_g}{\partial t} = \left(\frac{\hbar}{2} + \frac{m^2 + \vec{p}_\perp^2}{2\hbar} \right) \hat{\psi}_g \quad (\text{A.27})$$

$$i \frac{\partial \hat{\phi}_g}{\partial t} = \left(\frac{\hbar}{2} + \frac{m^2 + \vec{p}_\perp^2}{2\hbar} \right) \hat{\phi}_g \quad (\text{A.28})$$

From these free field equations, and from the Heisenberg equation of motion

$$i \dot{O} = [O, H] \quad (\text{A.29})$$

can be inferred several forms for the free Hamiltonian

$$H_0 = \sum_{\epsilon, s} \int \frac{d^3 \vec{p}}{(2\pi)^3 2|\vec{p}_0|} p_0 a^\dagger(p, s) a(p, s) \quad (\text{A.30})$$

$$= \int d^4x \delta(t) \psi^\dagger (\vec{\alpha} \cdot \vec{p} + \beta m) \psi \quad (\text{A.31})$$

$$= \int d^4x \delta(t) \phi^\dagger \beta \sqrt{m^2 + \vec{p}^2} \phi \quad (\text{A.32})$$

$$= 2 \int d^4x \delta(t) \hat{\psi}_g^\dagger \left(\frac{\hbar}{2} + \frac{m^2 + \vec{p}_\perp^2}{2\hbar} \right) \hat{\psi}_g \quad (\text{A.33})$$

$$= 2 \int d^4x \delta(t) \hat{\phi}_g^\dagger \left(\frac{\hbar}{2} + \frac{m^2 + \vec{p}_\perp^2}{2\hbar} \right) \hat{\phi}_g \quad (\text{A.34})$$

Foldy-Wouthuysen transformation

From (A.5)

$$\psi = \frac{m + \sqrt{m^2 + \vec{p}^2} - i \vec{\gamma} \cdot \vec{p}}{\sqrt{(m + \sqrt{m^2 + \vec{p}^2})^2 + \vec{p}^2}} \phi \quad (\text{A.35})$$

$$= e^{-iS} \phi e^{iS}$$

$$S = - \int d^4x s(t) \psi^+ \arctan \frac{\vec{r} \cdot \vec{p}}{m + \sqrt{m^2 + \vec{p}^2}} \psi \quad (\text{A.36})$$

Second Melosh transformation

From (A.35)

$$\hat{\psi}_g = \frac{m + \sqrt{y^2} + \sigma_3 \vec{\sigma}_\perp \cdot \vec{p}_\perp}{\sqrt{(m + \sqrt{y^2})^2 + \vec{p}_\perp^2}} \hat{\phi}_g \quad (\text{A.37})$$

$$= e^{-iY} \hat{\phi}_g e^{iY}$$

$$Y = -2 \int d^4x s(t) \hat{\psi}_g^+ \arctan \frac{i\sigma_3 \vec{\sigma}_\perp \cdot \vec{p}_\perp}{m + \sqrt{y^2}} \hat{\psi}_g \quad (\text{A.38})$$

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