Ref.TH.1975-CERN

IDEAL RELATIVISTIC QUANTUM GAS AND INVARIANT PHASE SPACE

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A B S T R A C T

The density of states of an ideal relativistic gas with Bose-Einstein or Fermi-Dirac statistics is written in closed form as a cluster decomposition over usual phase space integrals with Boltzmann statistics. We stress the difference between invariant phase space and invariant momentum space ; only the former gives the correct description of ideal gases. Our results are formulated for the microcanonical, grand microcanonical, canonical and grand canonical ensembles in invariant phase space and invariant momentum space. A whole section is devoted to a numerical study, which reveals that while under normal conditions thermodynamical quantities (derived from the logarithm of the density of states) acquire only very small corrections, the densities (phase space integrals) themselves are significantly changed. This holds in particular for elementary particle applications (statistical models, phase space considerations) where the effects on phase space of Bose-Einstein and/or Fermi-Dirac statistics can almost never be neglected ; however, in these applications, the number of equal particles is so low that the cluster decompositions can be explicitly evaluated.

Ref.TH.1975-CERN

5 February 1975

1. - INTRODUCTION

The general theory of boson and fermion gases has been widely discussed in the literature on the basis of the canonical ¹⁾ and grand canonical ²⁾ ensemble and has already found its way into the text-books. It turns out that the effect of Bose-Einstein and/or Fermi-Dirac statistics can be separated off from the "true" interaction and that, taken by itself (i.e., for an ideal quantum gas), these effects simulate an interaction as compared to a classical ideal Boltzmann gas. Indeed the effect of statistics can be written in form of a cluster decomposition and virial expansion which are formally the same as for interacting gases.

Much less attention has been paid to microcanonical descriptions of ideal quantum gases, for the obvious reason that in macroscopic situations this only leads to unnecessary complications.

In particle physics, however, the number of particles in a system is small - typically of the order of 10 - while total energy and momentum are usually known ; furthermore, the treatment has to be fully relativistic, preferably covariant and it has to include quantum statistics.

The present paper is an extension of the work of Magalinskii and Terletskii ³⁾, in which this problem was attacked for the first time. We shall develop a cluster decomposition for the N particle phase space with exact quantum statistics, i.e., with Bose-Einstein (BE) and/or Fermi-Dirac (FD) statistics. The individual terms of this decomposition involve only the well-known usual phase space integrals without BE and/or FD statistics, i.e., with Boltzmann (BO) statistics only. Such usual phase space integrals with BO statistics will be called "ordinary".

All our formulae will be written such that they easily allow specialization to BE, FD and BO statistics and in each of these cases to two possible formulations ; namely, to what usually is called "non-covariant phase space" and we shall call "Invariant Phase Space" (IPS), and to what usually is called "Invariant Momentum Space" (IMS) which name we shall keep also here. The first is the correct one when one is sure to deal with a truly ideal gas, while the second one may be used for computational convenience when all particles are non-relativistic and/or when the gas is only approximately ideal and the distortions of phase space introduced by using IMS are believed to be unimportant compared to the unknown residual interaction. More about that in Section 1.2.

In Section 2 we derive closed formulae (cluster decomposition) with correct BE/FD statistics and compare them to BO statistics ; we give a formal discussion of our results and illustrate them by means of the black body radiation.

In Section 3 we discuss the numerical effects of BE and FD and show that they are so large that only in exceptional cases can the BO term be considered as an approximation.

1.1. - Units, notation and terminology

We take units such that $\hbar = c = k = 1$; h is then an abbreviation for 2π ; the Lorentz metric is $g_{00} = -g_{ii} = 1$. A "system" is a set of particles enclosed in a box of rest volume V . For an arbitrary observer its four-velocity is u ($u^2 = 1$). If, in the Gibbs sense, infinitely many equal such systems form an ensemble with total four-velocity v ($v^2 = 1$), then an inverse four-temperature β can also be defined. Thus

$$\text{"inverse four-temperature"} : \quad \beta = \frac{1}{T} \cdot v \quad (1.1)$$

$$\text{"four-volume"} : \quad \Omega = V \cdot u \quad (1.2)$$

$$\text{"quantum four-volume"} : \quad \omega = \Omega/h^3 \quad (1.3)$$

Some usual words acquire a narrower meaning in an invariant description :

- "microcanonical" : total four-momentum and particle number fixed ;
- "canonical" : inverse four-temperature and particle number fixed ;
- "grand microcanonical" : total four-momentum and chemical potential fixed ;
- "grand canonical" : inverse four-temperature and chemical potential fixed.

1.2. - Which is the "right" (ordinary BO) phase space ?

In what follows we employ the usual technique of quantizing plane wave states in a cubic box at rest (volume V) and then replacing sums over discrete momenta by integrals over a density. Thus we encounter from the outset the question : which is the correct invariant measure for such integrations ?

We start by counting allowed plane wave states of a single particle with statistical weight g [for spin J and isospin I we have $g = (2J+1)(2I+1)$] in a cubic box with rest volume V . From text-books :

$$\text{number of states in } \{\vec{p}, d^3p\} = g \frac{V}{(2\pi)^3} d^3p \quad (1.4)$$

In field theory the invariant generalization of (1.4), using spacelike surface elements, has been used since long ; its application to relativistic statistical thermodynamics with given volume was first exploited by Touschek ⁴⁾, whom we follow here by introducing the invariant measure for phase space integration :

$$d\sigma_{IPS}(p, m) = g \cdot 2\omega \cdot p \delta_0(p^2 - m^2) d^4p = g \frac{\omega p}{p_0} d^3p$$

$$g = \text{statistical weight, } m = \text{particle mass, } \omega p = \omega_\mu p^\mu \quad (1.5)$$

subscript 0 on δ function indicates $p_0 \geq 0$

This is the correct integration measure for ideal gases. Unfortunately it is little known among particle physicists, who prefer the measure proposed by Srivastava and Sudarshan ⁵⁾, which makes the momentum space separately Lorentz invariant by writing

$$d\sigma_{IMS}(p, m) = g B \delta_0(p^2 - m^2) d^4p = g B \frac{d^3p}{2p_0} \quad (1.6)$$

$$B = \text{constant [length}^2\text{]} \text{ of order } (V/h^3)^{2/3}.$$

The difference is that while IPS depends on p^2 and $\omega \cdot p$, IMS depends only on p^2 , because the volume available to the particle is Lorentz contracted according to its velocity. This, however, leads to an incorrect counting of states for the ideal gas and is equivalent to a dynamical assumption ; the simplest proof is that for the light quantum gas it yields the wrong Stefan-Boltzmann law [see Eqs. (2.45), (2.46)].

Of course, IMS is correct if in the integral a squared S matrix element appears, provided it is evaluated between states normalized as $\langle p|p' \rangle = 2p_0 \delta^3(\vec{p}-\vec{p}')$. In that case, when S describes elastic scattering of hard spheres with vanishing radius, the denominators cancel and IPS results. We can also say that then for $S \rightarrow \underline{1}$ an initial IPS distribution will remain an IPS distribution. This is not so if, instead of letting the operator S tend to the unit operator, one puts all squared matrix elements equal to the same constant. In that case the IMS distribution results, which, because of the measure $gBd^3p/2p_0$, does not correspond to "non-interacting" particles.

Finally, it is of course correct to simulate an interaction by some fake squared matrix element, e.g., $f(P, p_1 \dots p_N) \delta^4(P - \sum p_i) \prod d^3p_i / 2p_{i0}$ - as long as one keeps in mind that for "vanishing" interaction $f(P, p_1 \dots p_N) \rightarrow \prod_{i=1}^N 2\omega p_i$ and not $\rightarrow 1$.

An extensive discussion of "which is the right phase space" has been given by Jabs ⁶⁾.

Thus, for the truly ideal gas, IPS is correct ; for certain applications (non-relativistic limit, unknown interaction) IMS is permissible and more convenient for computation - we then speak of a quasi-ideal gas :

invariant measure for ideal gas :

$$d\sigma_{IPS}(p, m)$$

invariant measure for quasi-ideal gas :

$$d\sigma_{IMS}(p, m)$$

(1.7)

henceforth : $d\sigma_\ell(p, m)$ with $\ell = 1$ for IMS, $\ell = 2$ for IPS

Note that both measures are dimensionless.

2. - DERIVATION OF CLOSED FORMULAE FOR INVARIANT PHASE SPACE WITH STATISTICS

2.1. - Posing the problem

Consider an ideal gas contained in a box of quantum four-volume ω and consisting of

$$\begin{aligned} N_1 & \text{ identical particles of mass } m_1 \\ N_2 & \text{ identical particles of mass } m_2 \\ & \vdots \\ N_\nu & \text{ identical particles of mass } m_\nu \end{aligned} \quad (2.1)$$

Adopting the usual plane wave description, the possible four-momenta in the box constitute a discrete ensemble. Let the set

$$\{q_j\} = \{q_{j1}, q_{j2}, \dots, q_{j\alpha}, \dots\} \quad (2.2)$$

characterize all possible four-momenta of a particle of kind j and let the occupation number

$$n(q_{j\alpha}) = n_{j\alpha} \quad (2.3)$$

indicate how many particles of kind j have momentum $q_{j\alpha}$. We account for BE/FD statistics by

$$n_{j\alpha} = \begin{cases} 0, 1 & \text{for FD} \\ 0, 1, 2, \dots, \infty & \text{for BE} \end{cases} \quad (2.4)$$

To each set $\{n\}$ of occupation numbers $n_{j\alpha}$ corresponds exactly one state of our system. The allowed states with total four-momentum Q and particle numbers $N = \{N_1 \dots N_\nu\}$ form a discrete set. We define the density of these states by

$$\sigma_{N_1 \dots N_\nu}(Q, \omega) \stackrel{\text{def}}{=} \sum_{\{n\}} \delta^4(Q - \sum_{j,\alpha} q_{j\alpha} n_{j\alpha}) \prod_{j=1}^{\nu} \delta_K(N_j - \sum_{\alpha} n_{j\alpha}) \quad (2.5)$$

Here, the Kronecker δ_K fixes the particle numbers and the δ^4 function selects the allowed four-momenta. As it stands, σ is a distribution, having δ^4 peaks on the set of allowed Q values ; but as we shall introduce a continuum description, it will in fact be a smooth function. Note that $\sigma_{N\dots}(Q,\omega)d^4Q$ is dimensionless.

2.2. - Evaluation via Laplace transform

Equation (2.5) has already been given in Ref. 3) where evaluation proceeds by writing a Fourier representation for the δ^4 function. We shall use a different technique, namely Laplace transforming the whole equation, separating off the N_j particle part and finally inverting explicitly the Laplace transform. Both methods have advantages and disadvantages ; we choose the Laplace transform because it has an obvious physical interpretation : for $v=u$, i.e., for β and ω parallel, it is the canonical partition function with all its immediate relations to thermodynamics. We define the canonical partition function $Z_{N_1\dots N_v}$ which is a dimensionless function of the invariant quantities $\Omega^2 = v^2$, $\beta^2 = 1/T^2$ and $\beta\omega$ (we omit these arguments throughout) :

$$Z_{N_1\dots N_v} = \int e^{-\beta Q} \sigma_{N_1\dots N_v}(Q, \omega) d^4Q \quad (2.6)$$

which, with (2.5) inserted, yields

$$Z_{N_1\dots N_v} = \sum_{\{n\}} \prod_{j=1}^v \exp\left(-\sum_{\alpha} n_{j\alpha} \beta q_{j\alpha}\right) \cdot \delta_K\left(N_j - \sum_{\alpha} n_{j\alpha}\right) \quad (2.7)$$

Introducing the shorthand notation

$$x_{j\alpha} = \exp(-\beta q_{j\alpha}) ; \quad \beta q_{j\alpha} = \beta_{\mu} q_{j\alpha}^{\mu} \quad (2.8)$$

and a Fourier representation for the Kronecker δ_K we obtain

$$Z_{N_1\dots N_v} = \sum_{\{n\}} \prod_{j=1}^v \int_0^{2\pi} e^{i\lambda N_j} \prod_{\alpha} \left(e^{-i\lambda} x_{j\alpha}\right)^{n_{j\alpha}} \frac{d\lambda}{2\pi} \quad (2.9)$$

the condition $\sum_{\alpha} n_{j\alpha} = N_j$ being taken care of by the λ integral, the sum over $\{n\}$ is subject only to the restriction (2.4) implied by BE/FD statistics.

We introduce

$$z = e^{-i\lambda} \quad (2.10)$$

and obtain

$$Z_{N_1 \dots N_v} = \prod_{j=1}^v Z_{N_j} \quad (2.11)$$

where

$$Z_{N_j} = \frac{1}{2\pi i} \oint \frac{dz}{z^{N_j+1}} \prod_{\alpha} \sum_{n_{j\alpha}} (z x_{j\alpha})^{n_{j\alpha}} \quad (2.12)$$

As $Z_{N_1 \dots N_v}$ factorizes, we consider henceforth only one of its factors corresponding to N_j particles of one kind, being either all bosons or all fermions. We then omit the subscript j .

The sum in (2.12) yields $(1 - zx_{\alpha})^{-1}$ for bosons and $1 + zx_{\alpha}$ for fermions. We now introduce the grand canonical partition function

$$Z(z, \gamma) = \prod_{\alpha} (1 + \gamma z x_{\alpha})^{\gamma} \quad \begin{cases} \gamma = 1 & \text{FD} \\ \gamma = -1 & \text{BE} \end{cases} \quad (2.13)$$

so that

$$Z_N(\gamma) = \frac{1}{2\pi i} \oint \frac{dz}{z^{N+1}} Z(z, \gamma) = \frac{1}{N!} \left[\frac{d^N}{dz^N} Z(z, \gamma) \right]_{z=0} \quad (2.14)$$

As a non-relativistic formula, this is familiar from the text-books ; z is the fugacity and $T \ln z = \mu$ is the chemical potential per particle. However, both z and μ differ by normalization from the quantities used in non-relativistic situations (NR) : $\mu^{NR} = \mu - m$. Note that $z=1$ or $\mu=0$ corresponds to undetermined particle number ; indeed from (2.14)

$$Z(\gamma) \stackrel{\text{def}}{=} \sum_{N=0}^{\infty} Z_N(\gamma) = Z(z=1, \gamma) \quad (2.15)$$

is the grand canonical partition function for free particle creation. We evaluate (2.13) as usual by writing $\exp(\ln z)$ for z and expanding the logarithm in powers of $\gamma z x$; reinserting $x = \exp(-\beta q)$ we obtain

$$\begin{aligned} \ln Z(z, \gamma) &= \sum_{\alpha} \gamma \ln(1 + \gamma z x_{\alpha}) \\ &= \sum_{\alpha} \sum_{k=1}^{\infty} \frac{z^k}{k} (-\gamma)^{k-1} e^{-\beta q_{\alpha} \cdot k} \end{aligned} \quad (2.16)$$

Here we have changed $(-\gamma)^{k+1}$ into $(-\gamma)^{k-1}$ in order to include the BO case which is represented by $\gamma = 0$ ($0^0 = 1$). We then interchange the summations and replace the discrete sum over states by an integral (thereby excluding a proper treatment of Bose condensation, which needs separate consideration)

$$\sum_{\alpha} e^{-k\beta q_{\alpha}} \Rightarrow \int e^{-\beta q \cdot k} d\sigma_e(q, m) \begin{cases} l=1 & \text{IMS} \\ l=2 & \text{IPS} \end{cases} \quad (2.17)$$

Hence

$$\ln Z^{(e)}(z, \gamma) = \sum_{k=1}^{\infty} \frac{z^k}{k} (-\gamma)^{k-1} \int e^{-k\beta q} d\sigma_e(q, m) \quad (2.18)$$

$Z^{(1)}$ is an invariant function of β^2 and B , $Z^{(2)}$ of β^2 , ω^2 and $\beta\omega$.

As far as Z stands for the mathematical procedure of Laplace transforming, β and ω are independent four-vectors and $Z^{(2)}$ will indeed depend on all three invariants, thereby allowing to invert the Laplace transform to find back $\sigma^{(2)}(Q, \omega)$ which itself is an invariant function depending on Q^2 , V and $Q \cdot \omega$. This will be used later. However, as soon

as we wish to interpret $Z^{(2)}$ as the physical partition function of the system we have to require that β and ω are parallel ^{*}) and then $Z^{(2)}$ depends on T and V only.

In this latter case, Eq. (2.17) can be evaluated in the common rest frame of β and ω with the result (K_ℓ are the usual modified Hankel functions)

$$\sum_{\alpha} e^{-k\beta q_{\alpha}} \xRightarrow{\beta \parallel \omega} C_\ell \frac{T}{mk} K_\ell\left(\frac{mk}{T}\right)$$

$$C_1 = g \cdot 2\pi B m^2$$

$$C_2 = g \cdot V m^3 / (2\pi^2)$$
(2.19)

and

$$Z^{(\ell)}(z, \gamma) \xRightarrow{\beta \parallel \omega} \exp \left\{ C_\ell \frac{T}{m} \sum_{k=1}^{\infty} \frac{z^k}{k^2} (-\gamma)^{k-1} K_\ell\left(\frac{mk}{T}\right) \right\}$$
(2.20)

This is, for $z=1$, the grand canonical partition function for free creation of bosons ($\gamma=-1$), boltzmannions ($\gamma=0$) or fermions ($\gamma=1$) with statistical weight g . If there are several different kinds of particles one has to sum in the exponent over their masses or, in the most general case, to integrate over a mass spectrum.

The N particle canonical partition function $Z_N^{(\ell)}(\gamma)$ follows now from Eq. (2.14). If N is fixed, Z_N is the partition function of the N particle system; if N is not fixed, then $W(N) = Z_N/Z(z)$ yields the particle number distribution at temperature T and chemical potential $\mu = T \ln z$. If the sum in the exponent is restricted to the first term, BO statistics results ($\gamma=0$).

^{*}) In some cases β and Q have to be parallel.

2.3. - Evaluation of the canonical partition function

The rule for calculating the Nth derivative of a compound function ⁷⁾ yields for (2.14)

$$Z_N^{(e)}(\delta) = \sum_{\{n, N\}} \prod_{k=1}^N \frac{1}{n_k!} \left[C_\ell \frac{T}{m} \frac{(\delta)^{k-1}}{k^2} \cdot K_\ell \left(\frac{mk}{T} \right) \right]^{n_k} \quad (2.21)$$

$\{n, N\} =$ sets of non-negative integers with $\sum_{k=1}^N k n_k = N$

The sets $\{n, N\}$ over which the sum is extended are called "partitions". The sum represents formally a cluster decomposition ; this is easily seen by comparing (2.21) with the Boltzmann case ($\gamma = 0$) in which only one partition, $\{n, N\} = \{N, 0, 0, \dots\}$, survives ; for $N = n$ it reads

$$Z_n^{(e)}(v, \tau)_{BO} = \frac{1}{n!} \left[C_\ell \frac{T}{m} K_\ell \left(\frac{m}{T} \right) \right]^n \quad (2.22)$$

Now it becomes obvious that for each given partition $\{n, N\} = \{n_1, n_2, \dots, n_N\}$ we obtain one term of the sum consisting of a product of N Boltzmann partition functions (2.22), the k th of which has n_k "particles" of mass km ; these "particles" may be called " k cluster", although they do not physically stick together.

Equation (2.21) will be used for a convenient discussion of the numerical effects of BE/FD as compared to BO statistics (Section 3) ; for the derivation of our main result it is, however, of no use.

2.4. - Cluster decomposition of IPS and IMS

We obtain a complete and unique inversion of the Laplace transform Eq. (2.6) by going back to Eqs. (2.16), (2.17) and following the same steps as before except that the integrations (2.17) shall not be carried out. Then in all subsequent formulae $C_\ell T / (km) K_\ell(km/T)$ is to be replaced by its original meaning $\int \exp(-k\beta q) d\sigma_\ell$. In place of Eq. (2.21), we now have

$$Z_N^{(l)} = \sum_{\{n, N\}} \prod_{k=1}^N \frac{1}{n_k!} \left[\frac{(-\gamma)^{k-1}}{k} \int e^{-k\beta q} d\sigma_l(q, m) \right]^{n_k} \quad (2.23)$$

where the q integral is of the form [see Eqs. (1.5), (1.6), (1.7)] :

$$\begin{aligned} \int e^{-k\beta q} q \cdot 2\omega q \delta_0(q^2 - m^2) d^4 q & \quad (l=2; \text{IPS}) \\ \int e^{-k\beta q} q_B \delta_0(q^2 - m^2) d^4 q & \quad (l=1; \text{IMS}) \end{aligned} \quad (2.24)$$

putting $kq = p$ results in

$$\int e^{-k\beta q} d\sigma_l(q, m) = \left(\frac{1}{k}\right)^{l+1} \int e^{-\beta p} d\sigma_l(p, km) \quad (2.25)$$

where the masses km , already encountered in (2.21), reappear. Now (2.23) becomes

$$Z_N^{(l)} = \sum_{\{n, N\}} \prod_{k=1}^N \frac{1}{n_k!} \left(\frac{(-\gamma)^{k-1}}{k^{l+2}} \right)^{n_k} \left[\int e^{-\beta p} d\sigma_l(p, km) \right]^{n_k} \quad (2.26)$$

We introduce the abbreviations

$$G^{(l)}(\{n, N\}, \gamma) = \prod_{k=1}^N \frac{1}{n_k!} \left(\frac{(-\gamma)^{k-1}}{k^{l+2}} \right)^{n_k} \quad (2.27)$$

$$H^{(l)}(\{n, N\}) = \prod_{k=1}^N \left[\int e^{-\beta p} d\sigma_l(p, km) \right]^{n_k} \quad (2.28)$$

and rewrite $H^{(l)}(\{n, N\})$ by a series of trivial manipulations :

$$\begin{aligned}
 H^{(l)}(\{n, N\}) &= \prod_{k=1}^N \prod_{i_k=1}^{n_k} \int e^{-\beta p_{i_k}} d\sigma_l(p_{i_k}, km) \\
 &= \int e^{-\beta \sum_{k=1}^N \sum_{i_k=1}^{n_k} p_{i_k}} \prod_{k=1}^N \prod_{i_k=1}^{n_k} d\sigma_l(p_{i_k}, km) \\
 &= \int d^4 Q e^{-\beta Q} \int \delta^4(Q - \sum_{k=1}^N \sum_{i_k=1}^{n_k} p_{i_k}) \prod_{k=1}^N \prod_{i_k=1}^{n_k} d\sigma_l(p_{i_k}, km)
 \end{aligned} \tag{2.29}$$

This is already the complete solution because each term of

$$Z^{(l)} = \sum_{\{n, N\}} G^{(l)}(\{n, N\}, \gamma) H^{(l)}(\{n, N\})$$

is now a four-dimensional Laplace transform ; inserting (2.27) and (2.29) into (2.26) and comparing with the definition (2.6) of $Z_N^{(l)}$ as the Laplace transform of $\sigma_N^{(l)}$, yields immediately our main result for N identical particles of mass m and statistical weight g :

$$\left. \begin{aligned}
 \sigma_N^{(IMS)}(\gamma, Q^2, B) &= \text{momentum space integral} \\
 \sigma_N^{(IPS)}(\gamma, Q^2, \omega^2, Q\omega) &= \text{phase space integral}
 \end{aligned} \right\} = \text{invariant microcanonical densities of states} \tag{2.30}$$

$$= \sum_{\{n, N\}} G^{(l)}(\{n, N\}, \gamma) \int \delta^4(Q - \sum_{k=1}^N \sum_{i_k=1}^{n_k} p_{i_k}) \prod_{k=1}^N \prod_{i_k=1}^{n_k} d\sigma_l(p_{i_k}, km)$$

where

$$G^{(\ell)}(\{n, N\}, \gamma) = \prod_{k=1}^N \frac{1}{n_k!} \left(\frac{(-\gamma)^{k-1}}{k^{\ell+2}} \right)^{n_k}$$

with $\{n, N\}$ = non-negative integers $n_1, n_2, \dots, n_N$ such that $\sum_{k=1}^N k n_k = N$;
 $\{n, N\}$ goes over all such partitions. Specializations :

$\gamma = +1$ Fermi-Dirac statistics

$\gamma = 0$ Boltzmann statistics

$\gamma = -1$ Bose-Einstein statistics

$\ell = 1$ corresponds to invariant momentum space (IMS)

$$d\sigma_{\text{IMS}}(p, m) = g B_0^6 (p^2 - m^2) d^4 p$$

$\ell = 2$ corresponds to invariant phase space (IPS)

$$d\sigma_{\text{IPS}}(p, m) = g 2\omega p_0^6 (p^2 - m^2) d^4 p.$$

In the Boltzmann case, $\gamma = 0$, only one partition survives, namely
 $\{n, N\} = \{N, 0, 0, \dots, 0\}$ and ordinary IMS and IPS results.

Equation (2.30) is the closed solution of a recurrence relation given by Nahm⁸⁾. This settles the microcanonical densities. The grand microcanonical ones would follow from (2.30) by summing over all N , which at this stage is not simple. We obtain them easily by going back to Eq. (2.15) which tells us that $\sigma^{(\ell)}$ is obtained from $Z^{(\ell)}(z=1)$ by inverting the Laplace transform. From Eq. (2.16) we obtain

$$Z^{(\ell)}(z=1) = \exp \left\{ \sum_{k=1}^{\infty} \frac{(-\gamma)^{k-1}}{k} \int e^{-k\beta q} d\sigma_{\ell}(q, m) \right\} \quad (2.31)$$

Expanding the exponential function and changing $kq \Rightarrow p$ yields

$$Z^{(\ell)}(z=1) = \sum_{N=0}^{\infty} \frac{1}{N!} \left[\int e^{-\beta p} \sum_{k=1}^{\infty} \frac{(-\gamma)^{k-1}}{k^{\ell+2}} d\sigma_{\ell}(p, km) \right]^N \quad (2.32)$$

Defining the effective integration measure

$$d\sigma_l^{\text{eff}}(p, m, \gamma) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{(-\gamma)^{k-1}}{k^{l+2}} d\sigma_l(p, km) \quad (2.33)$$

and carrying out the same manipulations as between Eqs. (2.28) and (2.30) we obtain

$$\left. \begin{aligned} \sigma^{(\text{IMS})}(\gamma, Q^2, B) &= \text{sum of momentum space integrals} \\ \sigma^{(\text{IPS})}(\gamma, Q^2, \omega^2, Q\omega) &= \text{sum of phase space integrals} \end{aligned} \right\} = \text{grand micro-canonical densities of states } (z=1)$$

$$= \sum_{N=0}^{\infty} \frac{1}{N!} \int \delta^4(Q - \sum_{i=1}^N p_i) \prod_{i=1}^N d\sigma_l^{\text{eff}}(p_i, m, \gamma) \quad (2.34)$$

where

$l = 1$ corresponds to invariant momentum space integration with

$$d\sigma_{\text{IMS}}^{\text{eff}}(p, m, \gamma) = gB \left[\sum_{k=1}^{\infty} \frac{(-\gamma)^{k-1}}{k^3} \delta_0(p^2 - (km)^2) \right] d^4p$$

$l = 2$ corresponds to invariant phase space integration with

$$d\sigma_{\text{IPS}}^{\text{eff}}(p, m, \gamma) = g2\omega p \left[\sum_{k=1}^{\infty} \frac{(-\gamma)^{k-1}}{k^4} \delta_0(p^2 - (km)^2) \right] d^4p$$

and where $\gamma = -1, 0, 1$ stands for BE, BO, FD statistics.

While the N^{th} term in the above sum (2.34) [as the expression (2.30)] involves an integration over N four-momenta with resultant Q , it is not true that the N^{th} term of Eq. (2.34) equals σ_N of Eq. (2.30).

The effective measures $d\sigma^{\text{eff}}$ may be interpreted as integration over an effective mass spectrum with masses at all integer multiples of the original mass m and with weight k^{-3} or k^{-4} respectively ; all positive for BE, with alternating sign for FD and with one single contribution ($k=1$) in the Boltzmann case.

2.5. - Formal discussion of the results

Our results (2.30) and (2.34) can be made more transparent by noting that they are sums over convolutions of ordinary IMS and IPS integrals (with BO statistics), in which "cluster particles" of mass km appear. Denoting the ordinary integrals by $R_N^{(\ell)}$; $\ell=1,2$ for IMS and IPS :

$$\begin{aligned}
 R_N^{(\ell)}(P, m) &= \int \delta^4(P - \sum_{i=1}^N p_i) \prod_{i=1}^N d\sigma^{(\ell)}(p_i, m) \\
 R_1^{(\ell)}(P, m) &= \int \delta^4(P - p) d\sigma^{(\ell)}(p, m) = \begin{cases} gB \delta_0(P^2 - m^2) \\ g2\omega P \delta_0(P^2 - m^2) \end{cases} \\
 R_0^{(\ell)}(P, m) &= \delta^4(P)
 \end{aligned} \tag{2.35}$$

we can write Eq. (2.30) as follows :

$$\sigma_N^{(\ell)} = \sum_{\{N, N\}} G^{(\ell)}(\{N, N\}, \gamma) \cdot \int \delta^4(Q - \sum_{k=1}^N p_k) \prod_{k=1}^N R_N^{(\ell)}(p_k, km) d^4 p_k \tag{2.36}$$

while Eq. (2.34) becomes

$$\sigma^{(\ell)} = \sum_{N=0}^{\infty} \frac{1}{N!} R_N^{(\ell), \text{eff}}(Q, m)$$

with

$$R_N^{(\ell), \text{eff}}(Q, m) = \sum_{k_1, k_2, \dots} \left(\prod_{i=1}^N \frac{(-\gamma)^{k_i-1}}{k_i!} \right) R_N^{(\ell)}(Q, \underbrace{k_1 m, k_2 m, \dots, k_N m}_{N \text{ masses}}) \tag{2.37}$$

We stress again that the N^{th} term of (2.37) is not equal to $\sigma_N^{(\ell)}$. All R_N have dimension $[P^{-4}]$.

Although the ordinary IMS and IPS integrals $R_N^{(\ell)}$ are still complicated objects, they are at least familiar ; for the non-relativistic as well as for the ultrarelativistic case closed analytical expressions are known ⁹⁾ while for intermediate cases efficient computer programs exist ¹⁰⁾, which cover the cases unsuitable for applying asymptotic expressions ¹¹⁾.

2.6. - First example : black body radiation

In order to show that our results give back well-known expressions and also that IMS does not correspond to ideal gases we treat the $m=0$ case ; as the number of zero mass particles cannot be fixed, we consider the grand microcanonical case Eq. (2.37). For $m=0$ also the k fold masses vanish and we obtain

$$R_N^{(\ell), \text{eff}}(Q, 0) = R_N^{(\ell)}(Q, 0) \cdot \left(\sum_{k=1}^{\infty} \frac{(-\gamma)^{k-1}}{k^{\ell+2}} \right)^N \quad (2.38)$$

The sum is a well-known function of γ and ℓ :

$$f(\gamma, \ell) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{(-\gamma)^{k-1}}{k^{\ell+2}} = [1 - (\gamma+1)^{1-(\ell+2)}] \cdot \zeta(\ell+2) \quad (2.39)$$

where $\zeta(x)$ is the Riemann ζ function. Hence

$$\sigma^{(\ell)}(\gamma, Q^2, Q\omega \propto B) = \sum_{N=0}^{\infty} \frac{1}{N!} [f(\gamma, \ell)]^N R_N^{(\ell)}(Q, 0) \quad (2.40)$$

As the $R_N^{(\ell)}$ are still rather complicated [in particular $R_N^{(2)}$ if $\vec{Q} \neq 0$]; we do not insert their explicit forms. Instead we Laplace transform :

$$\begin{aligned} Z^{(\ell)}(\gamma) &= \sum_{N=0}^{\infty} \frac{1}{N!} \int d^4Q e^{-\beta Q} \delta^+(Q - \sum q_i) \prod_{i=1}^N d\sigma_{\ell}(q_i, 0) \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} [f(\gamma, \ell) \int e^{-\beta q} d\sigma_{\ell}(q, 0)]^N \end{aligned} \quad (2.41)$$

Thus

$$Z^{(MS)} = \exp \left\{ g 2\pi B T^2 \zeta(3) \cdot \left[\begin{array}{l} 1 \text{ for BE} \\ 1/\zeta(3) \text{ for BO} \\ 3/4 \text{ for FD} \end{array} \right] \right\} \quad (2.42)$$

$$Z^{(PS)} = \exp \left\{ g \frac{V T^3}{\pi^2} \zeta(4) \cdot \left[\begin{array}{l} 1 \text{ for BE} \\ 1/\zeta(4) \text{ for BO} \\ 7/8 \text{ for FD} \end{array} \right] \right\} \quad (2.43)$$

As the mean energy $\langle E \rangle$ of a system is given by

$$\langle E \rangle = T^2 \frac{\partial}{\partial T} \ln Z \quad (2.44)$$

it follows with $\zeta(4) = \pi^4/90$ that

$$\langle E \rangle^{MS} = T^3 \cdot \left[g 4\pi B \zeta(3) \left\{ \begin{array}{l} 1 \text{ BE} \\ 1/\zeta(3) \text{ BO} \\ 3/4 \text{ FD} \end{array} \right\} \right] \quad (2.45)$$

$$\langle E \rangle^{PS} = T^4 \cdot \left[g \frac{V \pi^2}{30} \cdot \left\{ \begin{array}{l} 1 \text{ BE} \\ 90/\pi^4 \text{ BO} \\ 7/8 \text{ FD} \end{array} \right\} \right] \quad (2.46)$$

with $g=2$ and BE, invariant phase space yields the correct Stefan-Boltzmann law (in our units the usual Stefan constant becomes $\pi^2/15$) ; compared to BO, quantum statistics only gives rise to a slight change of the Stefan constant ; in contrast to this, invariant momentum space yields a wrong functional dependence (T^3 instead of T^4).

3. - NUMERICAL CONSEQUENCES

3.1. - Second example : three particles (N=3)

The condition $\sum_k n_k = 3$ allows three partitions $\{1\}, \{2\}, \{3\}$.

	n_1	n_2	n_3
$\{1\}$	3	0	0
$\{2\}$	1	1	0
$\{3\}$	0	0	1

(3.1)

The weight factors $G^{(\ell)}(\{n, N\}, \gamma)$ become [see (2.30)]

	IMS	IPS
$G^{(\ell)}(\{1\}, \gamma)$	$\frac{1}{3!}$	$\frac{1}{3!}$
$G^{(\ell)}(\{2\}, \gamma)$	$-\frac{\gamma}{2^3}$	$-\frac{\gamma}{2^4}$
$G^{(\ell)}(\{3\}, \gamma)$	$\frac{1}{3^3}$	$\frac{1}{3^4}$

(3.2)

We thus obtain for three particles of mass m :

$$\sigma^{(IPS)} = \frac{1}{3!} R_3^{(IPS)}(P, m) \pm \frac{1}{2^4} R_2^{(IPS)}(P, m, 2m) + \frac{1}{3^4} R_1^{(IPS)}(P, 3m)$$

$$\sigma^{(IMS)} = \frac{1}{3!} R_3^{(IMS)}(P, m) \pm \frac{1}{2^3} R_2^{(IMS)}(P, m, 2m) + \frac{1}{3^3} R_1^{(IMS)}(P, 3m) \quad (3.3)$$

Here P is the total four-momentum ; $+$ stands for BE and $-$ for FD statistics ; in the Boltzmann case only the first term remains.

We see that the second terms, containing clusters of mass $2m$, will increase (BE) or decrease (FD) the weight of these states already contained in R_3 , in which two particles are in the same state. It seems that the third term would favour, even for FD statistics, a state in which all three particles occupy the same momentum state ; we know that this must not be so. In fact the last term has the explicit form [see (2.35)]

$$R_1^{(e)}(P, 3m) = \begin{cases} gB \delta_0(P^2 - 9m^2) \\ 2g\omega P \delta_0(P^2 - 9m^2) \end{cases} \quad (3.4)$$

and therefore is zero except in the pathological case where $P^2 = 9m^2$ and where the first two terms are zero. This term would not be present if we had kept the states discrete instead of introducing the continuum. This becomes more illuminating if we consider a numerical case, where we chose the less trivial invariant phase space. For three masses a, b, c , total four-momentum Q and quantum four-volume ω we obtain from the definition (2.35)

$$R_2^{IPS}(Q^2, \omega Q, a, b) = \frac{2\pi\sqrt{\lambda}}{Q^2} \left\{ (\omega Q)^2 \frac{3(Q^2)^2 - 3(a^2 - b^2)^2 - \lambda}{12(Q^2)^2} + \frac{\lambda \omega^2}{12Q^2} \right\} \quad (3.5)$$

where

$$\lambda(Q^2, a^2, b^2) = (Q^2)^2 + a^4 + b^4 - 2(Q^2 a^2 + Q^2 b^2 + a^2 b^2)$$

and

$$R_3^{IPS}(Q^2, \omega Q, a, b, c) = \int 2\omega p \delta_0(p^2 - c^2) R_2^{IPS}((Q-p)^2, (Q-p)\omega, a, b) d^4p \quad (3.6)$$

The integral can be evaluated in terms of elliptic integrals ¹²⁾ but it is simpler to do it numerically after specializing to $\vec{Q} = 0$ in the ω rest frame and remaining with a single integral. Figures 1 and 2 show the result for the phase space integral of three particles [first line of Eq. (3.3) as a function of $M = \sqrt{Q^2}$ and with $g = \omega_0 = (v/h^3) = m = 1$].

We see that the corrections are very important and, in the FD case, even make $\sigma < 0$ below $M \approx 3.27$. We attribute this to the fact that three fermions cannot be in the ground state and that therefore the lowest possible three-fermion state must lie above $M=3$. Thus the negative part of IPS as well as the δ_0 function at $M=3$ are the consequences of replacing discrete states by a continuum. We check this interpretation by a little calculation with discrete states : consider a cubic box of side-length L ; periodic boundary conditions yield for the possible momenta

$$\vec{p}_{0,0,0} = (0, 0, 0)$$

$$\vec{p}_{1,0,0} = \frac{2\pi}{L} (1, 0, 0) ; \text{ furthermore } \vec{p}_{0,1,0} \text{ and } \vec{p}_{0,0,1}$$

$$\vec{p}_{-1,0,0} = \frac{2\pi}{L} (-1, 0, 0) ; \text{ furthermore } \vec{p}_{0,-1,0} \text{ and } \vec{p}_{0,0,-1}$$

Thus, there are three different states (differing only by direction of $\vec{p}$, $-\vec{p}$) with the same total energy. We do not gain anything by introducing spin, because if we put two spin states (up and down) into the same momentum state, we cannot achieve (with three particles) total $\vec{p}=0$. The total energy is [remember $V/h^3 = (L/2\pi)^3 = 1$]

$$M_0 = \sum_{i=1}^3 \sqrt{\vec{p}_i^2 + m^2} = 1 + 2\sqrt{2} = 3.83$$

Thus the number of states as function of M rises in one step from 0 to 3 at $M=M_0=3.83$. The number of states in the continuum representation is obtained from σ_3^{IPS} by integrating with respect to M , i.e., by integrating the full curve in Fig. 2 ; the result is shown in Fig. 3. If we compare a step function with a smoothly rising approximation to it, we would expect the half-value of the approximation to lie at the step. In our case, we look at which M value our continuous curve in Fig. 3 reaches half the magnitude of the step, i.e., 1.5. The corresponding M value is 3.93 in very reasonable agreement with the step position at 3.83 in the discrete case. This confirms that our interpretation of "negative phase space" is correct ; it is a forbidden region, but in integrating its contribution to the number of states must be taken into account. The negative part of $\int \sigma_3^{\text{IPS}} dM$ in Fig. 3 is then really unphysical.

No forbidden region exists in the BE case where we indeed can reach the condensation point at $M=3m$ with weight $\delta_0(M^2-9m^2)$. Also here however, the continuous curves are a poor approximation to the sequence of further δ functions which would appear in a discrete calculation. This does not really matter, because in most applications we are in regions where the continuous approximation is good (as the FD example shows, one integration smoothes out sufficiently to obtain already reasonable approximations at the very lowest state).

Note that the figures depend on our choice of $g=V/h^3=m=1$. This happens to be realistic for elementary particle applications : choosing the mass unit to be 1 GeV, and considering L as the Compton wave length of a particle with mass m_0 , it follows from $L/2\pi=1/(2\pi m_0)=1$ that $m_0 \approx 160$ MeV. Our figures refer then to three fermions of mass $m=1$ GeV confined to an elementary volume $L^3=m_0^{-3}$. In macroscopic applications, however, $L \gg m_0^{-1}$.

Our example shows one important feature : at least in the mass region considered the effects of BE and/or FD statistics are not small corrections to the Boltzmann term ; it might therefore be wrong to assume that in general the Boltzmann term is leading. Indeed, we shall see in the next section that in almost all cases the effects of BE and/or FD statistics amount to correction terms which are much bigger than the Boltzmann term.

3.2. - Orders of magnitude : a general discussion of the effects of BE/FD statistics

The structure of the cluster decomposition is the same in the microcanonical and in the canonical ensemble ; furthermore the differences between IPS and IMS are small and finally BE and FD are distinct only by the alternating sign in the latter case while the magnitude of corresponding BE and FD cluster terms is the same. We thus simplify (without loss of generality) our discussion by limiting it to BE in canonical IPS. We then obtain from Eq. (2.21)

$$Z_N = \sum_{\{n_k\}} \prod_{k=1}^N \frac{1}{n_k!} \left[\frac{1}{k^4} \frac{QV}{2\pi^2} (km)^2 T K_2\left(\frac{mk}{T}\right) \right]^{n_k} \quad (3.8)$$

We shall consider three further examples (examples No 3, 4, 5)

- i) $m=0$
 ii) $m \gg T$
 iii) $m=T$
- } in a macroscopic situation
- in an elementary particle situation

We leave case iii) for later and discuss i) and ii) now. Using the behaviour of K_2 for very large and very small argument ⁷⁾, we write

$$\frac{gV}{2\pi^2} (km)^2 T K_2\left(\frac{mk}{T}\right) \xrightarrow{m=0} \frac{gV}{\pi^2} T^3 \quad (3.9a)$$

$$\xrightarrow{m \gg T} gV \left(\frac{mT}{2\pi}\right)^{3/2} k^{3/2} e^{-\frac{km}{T}} \quad (3.9b)$$

Introducing the abbreviations

$$\frac{gV}{\pi^2} T^3 \stackrel{\text{def.}}{=} \omega_0 \quad (3.10a)$$

$$gV \left(\frac{mT}{2\pi}\right)^{3/2} \stackrel{\text{def.}}{=} \omega_1 \quad (3.10b)$$

and using the condition $\sum_k n_k = N$ in (3.8) we obtain

$$Z_N = e^{-N \frac{\omega_1}{T}} \sum_{\{n_k\}} \prod_{k=1}^N \frac{1}{n_k!} \left(\frac{\omega_1}{k^{5/2}}\right)^{n_k} ; \quad m \gg T \quad (3.11)$$

$$Z_N = \sum_{\{n_k\}} \prod_{k=1}^N \frac{1}{n_k!} \left(\frac{\omega_0}{k^4}\right)^{n_k} ; \quad m=0 \quad (3.12)$$

As the BO term is the one with $n_1 = N, n_2 = n_3 = \dots = 0$, it is tempting to single out this term and to consider the rest as corrections which we would expect to be negligible for dilute gases ; this expectation is wrong.

We write down a few partitions :

	n_1	n_2	n_3	n_4	n_5	n_6	...
{1}	N	0	0	0	0	0	
{2}	N-2	1	0	0	0	0	
{3}	N-3	0	1	0	0	0	
{4}	N-4	2	0	0	0	0	
{5}	N-4	0	0	1	0	0	
:	:	:	:	:	:	:	

(3.13)

and read off from (3.11) and (3.12)

$$e^{\frac{N}{T}} Z_N = \frac{\omega_1^N}{N!} \left[1 + \frac{N(N-1)}{2^{5/2} \omega_1} + \frac{N(N-1)(N-2)}{3^{5/2} \omega_1^2} + \frac{N(N-1)(N-2)(N-3)}{2^5 \omega_1^2} \left(\frac{1}{\omega_1} + \frac{1}{2} \right) + \dots \right] \quad (3.14)$$

$$Z_N = \frac{\omega_0^N}{N!} \left[1 + \frac{N(N-1)}{2^4 \omega_0} + \frac{N(N-1)(N-2)}{3^4 \omega_0^2} + \frac{N(N-1)(N-2)(N-3)}{2^5 \omega_0^2} \left(1 + \frac{1}{8\omega_0} \right) + \dots \right] \quad (3.15)$$

Here the 1 represents BO, the other terms are corrections. We first observe that $w_{0,1}$ and N both are extensive quantities and that N/w is intensive. One should think that the corrections depend only on intensive quantities, but, in fact, for $N \gg 1$, the first terms look like

$$1 + C_1 \left(\frac{N}{w} \right) \cdot N + C_2 \left(\frac{N}{w} \right)^2 \cdot N + C_3 \left(\frac{N}{w} \right)^2 \cdot N^2 + \dots \quad (3.16)$$

and it seems that, no matter how dilute a gas may be, the corrections can be made arbitrarily large. Things become worse if we evaluate (3.14), (3.15) numerically for a "normal" situation, which we define as follows :

$m \gg T$ normal situation : 1 mol of H_1 gas at atmospheric pressure and $T = 273$ K ; $N = L = 6.022 \times 10^{23}$,
 $V = 22.4 \times 10^3 \text{ cm}^3$.

$m = 0$ normal situation : $22.4 \times 10^3 \text{ cm}^3$ black body radiation at $T = 273$ K.
 $N = \bar{N}_{BO} = w_0$.

We have chosen the mean particle number $\bar{N}_{BO}$ of a massless BO gas for convenience ; factors like 2, $\zeta(3)$, etc., will turn out to be irrelevant. We obtain from (3.10)

$$w_0 = 7.70 \times 10^{12} \cdot \left[\frac{V}{22.4 \ell} \cdot \left(\frac{T}{273 \text{ K}} \right)^3 \cdot \frac{g}{2} \right] \quad (3.17a)$$

$$w_1 = 1.92 \times 10^{28} \cdot \left[\frac{V}{22.4 \ell} \cdot \left(\frac{m}{939 \text{ MeV}} \cdot \frac{T}{273 \text{ K}} \right)^{3/2} \cdot g \right] \quad (3.17b)$$

where the "normal" case is obtained by putting the square brackets = 1 ; this we do from now on. Hence, under normal conditions with $N = L$ and $\bar{N}$, respectively,

$$\bar{N}/w_0 = 1 \quad (3.18a)$$

$$L/w_1 = 3 \times 10^{-5} \quad (3.18b)$$

From (3.14), (3.15) we now obtain

$$e^{L \frac{m}{T}} Z_L = Z_{BO,L} \cdot \left(1 + \frac{L}{2^{5/2}} \cdot \frac{L}{w_1} + \dots \right) \approx Z_{BO,L} \cdot (1 + 10^{18} + \dots) \quad (3.19)$$

$$Z_{\bar{N}} = Z_{B0, \bar{N}} \cdot \left(1 + \frac{\bar{N}}{2^4} + \dots\right) \cong Z_{B0, \bar{N}} \cdot (1 + 10^{11} + \dots) \quad (3.20)$$

Thus the "first correction term" is 10^{18} and 10^{11} times larger than the respective B0 term ; even in the best vacuum which is technically feasible, the "first correction" exceeds the "leading B0 term" by many orders of magnitude ; higher terms definitely behave worse.

On the other hand, text-books on statistical thermodynamics assure us (and prove !) that for dilute gases BE corrections are negligible. The apparent paradox is solved by noting that text-books always consider the corrections to the logarithm of Z , because all relevant thermodynamical quantities are derived from $\ln Z$. Then, indeed, the situation changes in a way which is characteristic for the large numbers appearing in statistical thermodynamics : anticipating the result of the following calculation we state that - by order of magnitude -

$$e^{L \frac{m}{T}} Z_L \approx \exp(10^{25}) \cdot [\exp(10^{18})]$$

where the first factor is Z_{B0} and the second factor "the correction" due to BE. Nobody would dare to neglect a factor $\exp(10^{18})$; however, if we consider the logarithm, we have

$$\ln(e^{L \frac{m}{T}} Z_L) \approx 10^{25} + 10^{18} = 10^{25} \cdot (1 + 10^{-7})$$

Thus, if the physically interesting quantity is not Z , but $\ln Z$, the correction is really a correction, and a small one for that matter.

3.2.1. - Third example : $m \gg T$; $N=L$

To show that this is what happens, we evaluate Eqs. (3.11), (3.12) with the usual technique of approximating $\ln Z_N$ by the logarithm of the biggest term of the sum and fulfilling the summation condition $\sum k n_k$ with the help of a Lagrange multiplier. That is, we now take it for granted that neither the Boltzmann term nor the first few corrections

are important, but that the leading terms lie somewhere in the middle of the sum and that they are much bigger than the rest. We justify this by the remark that we thus reproduce exactly the standard results of statistical mechanics. Thus, from (3.11)

$$\frac{\partial}{\partial m} \sum_{k=1}^N \left(-n_k \ln n_k + n_k + n_k \ln \left(\frac{\omega_1}{k^{5/2}} \right) - \alpha k n_k \right) = 0 \quad (3.21)$$

from which the most probable values $\bar{n}_k$ follow :

$$\bar{n}_k = e^{-\alpha k} \frac{\omega_1}{k^{5/2}} \quad (3.22)$$

The value of the Lagrange multiplier α is determined by the summation condition

$$\sum k \bar{n}_k = N = \omega_1 \sum_{k=1}^N k \frac{e^{-\alpha k}}{k^{5/2}} \quad (3.23)$$

With these two equations the logarithm of the biggest term (b.t.) of the sum (3.11) becomes

$$\ln \left(e^{\frac{Nm}{T}} Z_N \right)_{b.t.} = \omega_1 \sum_{k=1}^N \frac{e^{-\alpha k}}{k^{5/2}} + \alpha N \quad (3.24)$$

This [without the normalization factor $\exp(Nm/T)$ and the subscript b.t.] is the standard formula of statistical thermodynamics. The interested reader may prove that one finds the same result if one evaluates Eqs. (2.13), (2.14) by means of the usual saddle-point method (in that case the sum extends to ∞ , but this makes no difference). We note that in that case one finds the saddle-point on the real z axis at a point z_0 which is related to our Lagrange multiplier α by $z_0 \exp(-m/T) = \exp(-\alpha)$; hence the physical significance of α follows from the one of $z_0 = \exp(\mu/T)$:

$$\alpha = - \frac{\mu - m}{T} ; \quad \mu = \text{chemical potential} \quad (3.25)$$

Indeed, α is determined by Eq. (3.23) as a function of N and w_1 .

We now turn to numerical evaluation ; as $N \approx 10^{23}$ and $w_1 \approx 10^{28}$, we conclude from Eq. (3.23) that the sum must be $\approx 10^{-5}$ and hence $\exp(-\alpha) \approx 10^{-5}$. We can then determine α by iteration. We put

$$\frac{N}{w_1} \equiv \varepsilon \approx 10^{-5} \quad (3.26)$$

and find to second order in ε from Eq. (3.23)

$$e^{-\alpha} = \varepsilon \cdot \left(1 - \frac{\varepsilon}{2^{3/2}}\right) \quad (3.27)$$

With this we obtain for Eq. (3.24)

$$\begin{aligned} \ln(e^{\frac{Nw_1}{T}} Z_N) &= \ln\left(\frac{w_1^N}{N!}\right) + \frac{N\varepsilon}{2^{5/2}} + \dots \\ &= \ln\left(\frac{w_1^N}{N!}\right) \cdot \left(1 + \frac{\varepsilon}{2^{5/2} \ln(e/\varepsilon)} + \dots\right) \end{aligned} \quad (3.28)$$

Here the first term is "pure Boltzmann" and, for $N=L$, of order 10^{25} while the second is BE correction and of order 10^{18} only ; the second line of Eq. (3.28) shows that it depends on the intensive quantity (density) $\varepsilon = N/w_1$, as it should (this is because N and w_1 together determine which is the biggest term of the cluster decomposition). We also see that replacing the logarithm of the whole sum (3.11) by the one biggest term is the same as replacing it by many - say $\exp(10^{18})$ - such terms : the correction $N\varepsilon/2^{5/2}$ would remain of the same order ; we can as well omit the subscript b.t. and consider Eq. (3.28) as "exact" ; indeed, Eqs. (3.24) and (3.28) are standard textbook formulae.

3.2.2. - Fourth example : $m=0$; $N \leq \bar{N}_{B0}$

As the calculations follow the same pattern, we only add a few comments : if we take the "normal" case $N = \bar{N}$ then this is just the same as putting $\alpha=0$ and summing over all N (free particle creation). We then obtain

$$\ln Z = \ln Z_{\bar{N}} = \omega_0 \sum \frac{1}{k^4} = \omega_0 \zeta(4) \quad (3.29)$$

while the $B0$ term is

$$\ln Z_{B0} = \omega_0 = \bar{N}_{B0} \quad (3.30)$$

In this case therefore even the correction to the logarithm is significant [$\zeta(4) \approx 1.1$].

If, however, we assume we could fix N (or analyze a posteriori a situation where we know that N particles had been present), then with

$$\varepsilon_0 = \frac{N}{\omega_0} = \frac{N}{\bar{N}_{B0}} \quad (3.31)$$

we obtain a result similar to (3.28) :

$$\ln Z_N = \ln \left(\frac{\omega_0^N}{N!} \right) + \frac{N\varepsilon_0}{16} + \dots \quad (3.32)$$

Thus for all $N \ll \bar{N}_{B0}$ the correction is small.

3.2.3. - Fifth example : $m=T$; N of order 10 ; $V \approx m^{-3}$

This is the typical situation in particle physics when statistical models or phase space considerations are applied. To fix parameters, let us put

$$m = T = m_\pi \quad ; \quad V = \frac{4\pi}{3} m_\pi^{-3} \quad ; \quad g = 1 \quad (3.33)$$

In that case the square bracket in the sum (3.8) becomes

$$\frac{1}{k^4} \frac{qV}{2\pi^2} (km)^2 T K_2\left(\frac{km}{T}\right) = \frac{2}{3\pi} \cdot \frac{K_2(k)}{k^2} \quad (3.34)$$

and the numerical result for Z_N becomes

$$Z_N = \frac{1}{N!} \cdot \left(\frac{2K_2(1)}{3\pi} \right)^N \cdot \left(1 + .113 N(N-1) + \right. \\ \left. + .035 N(N-1)(N-2) + .12 N(N-1)(N-2)(N-3) + \dots \right) \quad (3.35)$$

Again the factor in front is the BO contribution and the rest are corrections, which already for very modest multiplicities ($N > 3$) become significant.

4. - CONCLUSION

Our formulae (2.20), (2.21), (2.30), (2.34), (2.36) and (2.37) allow the evaluation of the effects of BE and FD statistics in closed form. While they do not contain any new information about the usual situations encountered in macroscopic statistical mechanics, they do so for all cases in which a microcanonical description is needed and where (by suppressing any number of integrations) all possible effects on momentum distributions and correlations can now be derived. The most important result - justifying this whole paper - seems to be that the effects of BE and FD statistics are in most practical cases very significant and cannot, in general, be considered as a small correction to Boltzmann behaviour; while this has been established by examples using canonical IPS, it, of course, holds also for microcanonical IPS and IMS, as can easily be exemplified by using the well-known analytic expressions for $m=0$ and for $m \gg$ mean kinetic energy.

ACKNOWLEDGEMENTS

We would like to thank several colleagues, in particular H. Satz and L. Van Hove for criticism and discussions and L. Sertorio for reminding us of the relevance of Touschek's paper for our present work.

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FIGURE CAPTIONS

Figure 1 Three-body invariant phase space with Bose-Einstein statistics (σ_3) compared to Boltzmann statistics (R_3). M is the total energy.

Figure 2 Three-body invariant phase space with Fermi-Dirac statistics (σ_3) compared to Boltzmann statistics (R_3). M is the total energy.

Figure 3 The number of states $\int \sigma dM$ for three particles in invariant phase space with Bose-Einstein (BE) and Fermi-Dirac (FD) statistics. M is the total energy.

- Figure 1 -

- Figure 2 -

- Figure 3 -