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MASS SPECTRA OF DUAL STRINGS

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ABSTRACT

Dual strings seem to correspond uniquely to certain subgroups of the modular group. Above space-time dimension 8, this approach yields just the known models, plus one additional candidate for 10 dimensions. Further models exist probably for dimensions 8, 6, and 4, and possibly for dimensions 5 and 3. The mass spectra of the models can be determined directly from the corresponding subgroup. The mass spectrum of one candidate for the "correct model" is determined. It yields only mesons of charge zero, but otherwise the agreement with experiment is excellent.

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1. INTRODUCTION

Certainly, matter does not consist of one-dimensional objects. Nevertheless, dual strings may approximate a more fundamental theory of hadronic properties. For the moment they are much easier to handle than, say, soliton solutions of non-Abelian gauge theories. However, up to now only models for 26 and 10 space-time dimensions exist ¹⁾⁻³⁾. The Bardakçi-Halpern modification of the ordinary model ⁴⁾ is rather ad hoc, and Frampton's model ^{3),5)} cannot yet be judged because it is not yet fully factorized.

One promising approach to the "correct model" centres on the local properties of the string which manifest themselves in the gauge algebra ⁶⁾. Here I shall try a different approach, which centres on the global properties of the string, such as periodicities and over-all behaviour of the mass spectrum. I hope to show that certain subgroups of the modular group Γ are as characteristic for the global behaviour of the model as the gauge algebra is for the local properties. The choice of subgroups of Γ seems to be as restricted as the choice of supersymmetries. Moreover, this approach allows a direct comparison with the experimental mass spectrum. Section 2 contains the basic features of this method. In Section 3 the relation of the modular group to the periodicities of the string models is discussed and the mass spectra and critical dimensions of the known models are re-derived. Section 4 contains an educated guess for the mass spectrum of the correct string model. In Section 5 the results are discussed, and possibilities for further developments of this approach are pointed out.

2. MODULAR FUNCTIONS AND MASS SPECTRA

The mass spectrum of a dual string should have the following properties. Particles should lie on linear trajectories with common slope α' , thus differences of $\alpha'm^2$ should be quantized. We may have to admit tachyons, but certainly at most a finite number of them. The mass spectrum should rise exponentially,

$$d(m) \sim \exp(m\beta_0), \quad (1)$$

where $d(m)$ is the degeneracy at mass level m . In fact

$$\tau(E) = \sum_m \int d(m) \delta(E - \sqrt{k^2 + m^2}) d^R k \quad (2)$$

should be, to a very good approximation, a sum of pure exponentials in E ⁷⁾. Here R is the space dimension, and the sum goes over all mass levels of the model.

These properties can be expressed in terms of the Fourier transform of the mass spectrum

$$A(w) = \sum_m d(m) \exp(2\pi i \alpha' m^2 w). \quad (3)$$

They can be derived from the working hypothesis

WH 1: $A(w)$ is a modular function for a subgroup G of finite index in the modular group Γ .

What does this mean? The modular group ⁸⁾ Γ is the group of transformations

$$w \rightarrow \frac{aw+b}{cw+d}, \quad ad-bc=1, \quad a,b,c,d \in \mathbb{Z}. \quad (4)$$

$\mathbb{Z}$ denotes the integers, and w is a complex number. Γ transforms the upper half plane onto itself. A particularly important orbit of Γ consists of the rational points of the real axis plus the point at infinity. The points of this orbit are called cusps. We restrict w to the domain H which is formed by the upper half plane plus the cusps.

The transformation φ defined by

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \rightarrow \frac{aw+b}{cw+d} \quad (5)$$

yields the canonical homomorphism of the matrix group $SL(2, \mathbb{Z})$ onto Γ . Its kernel is $\pm I$.

Invariance of dual models under Γ has been studied by Shapiro ⁹⁾ and Minami ¹⁰⁾, invariance under the special transformation $w \rightarrow -1/w$ by Brink and Nielsen ¹¹⁾ and Minami ¹²⁾. However, these investigations were concerned with amplitudes and propagators, whereas the mass spectrum is a much more convenient point to start with.

Modular functions for subgroups G of Γ fulfil the equation

$$A\left(\frac{aw+b}{cw+d}\right) = (cw+d)^{-D/2} A(w) \quad \text{for } \varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \quad (6)$$

where $-D/2$ is the so-called weight of $A(w)$. We shall only be concerned with integer $D/2$, although generalization to half odd values is possible. If $D/2$ is odd, Eq. (6) is not yet consistent when a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ is replaced by $-M$. In this case, one introduces a character $\chi(M)$, i.e., a representation of the corresponding subgroup of $\text{SL}(2, \mathbb{Z})$ in the multiplicative group of complex numbers, with property $\chi(-M) = -\chi(M)$, and writes

$$A\left(\frac{aw+b}{cw+d}\right) = \chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} (cw+d)^{-D/2} A(w). \quad (7)$$

In the cases which are considered below such characters exist. For the moment we shall not care about this complication.

A function $A(w)$ with property (6) is called weakly modular for G . Modular functions have to fulfil further conditions. They must be meromorphic in the upper half plane, and in a certain sense also at the cusps.

Now $A(w)$ is even holomorphic in the upper plane because we admit only mass spectra which do not increase faster than exponentially. To understand the behaviour at the cusps, we first note that each subgroup G of finite index in Γ contains translations of w , i.e., transformations of the form $w \rightarrow w+k$, as otherwise all translations would belong to different co-sets of G . Let the subgroup of these translations in G be generated by $w \rightarrow w+r$, $r \in \mathbb{Z}$. Using Eq. (6), $A(w)$ can be written in the form

$$A(w) = \sum_{n=-N_0}^{\infty} d(n) q^n, \quad q = \exp(2\pi i w/r), \quad (8)$$

which has to be compared with Eq. (3). We admit at most a finite number of tachyons, so that A is a meromorphic function of $q = \exp(2\pi i w/r)$ at $q=0$.

Note that not only differences of squares of masses are quantized in units of $(\alpha' r)^{-1}$, but also the masses themselves. In particular, all meson Regge intercepts must be multiples of $1/r$.

Now, by the same argument as before, G also contains a subgroup of translations of $-1/w$. Let this subgroup be generated by $w \rightarrow w/(sw+1)$, $s \in \mathbb{Z}$. Then $A(w)$ can be written in the form

$$w^{-D/2} A(w) = (-i)^{D/2} \sum_{n=-N_0}^{\infty} d_0(n) q^n, \quad q = \exp(-2\pi i /ws). \quad (9)$$

We have inserted the factor $(-i)^{D/2}$ to obtain real coefficients $d_0(n)$. We show in the Appendix that N_0 is finite if the mass spectrum does not increase faster than exponentially. Thus $A(w)$ is a meromorphic function of $q = \exp(-2\pi i/ws)$ at $q=0$.

A similar q expansion can be derived for any other cusp. One finds

$$(cw+d)^{-D/2} A(w) = \sum_{n=-N_{cd}}^{\infty} d_{cd}(n) q^n, \quad q = \exp(2\pi i \frac{aw+b}{cw+d} / t_{cd}) \quad (10)$$

for the cusp at $w = -c/d$, where a, b are chosen in such a way that $ad - bc = 1$, and t_{cd} determines the subgroup of translations of $(aw+b)/(cw+d)$ in G . Again N_{cd} must be finite, if the mass spectrum does not increase faster than exponentially.

We characterize the behaviour of $A(w)$ at the cusps by the corresponding q expansions, saying that A is meromorphic, holomorphic, zero, etc. at a cusp, if the corresponding q expansion has these properties as a function of q at $q=0$. $A(w)$ has identical q expansions for all cusps which can be transformed into each other by G (but remember that q denotes a different function of w for each cusp).

A weakly modular function which is everywhere meromorphic in H (including the cusps) is called a modular function. We have seen that it is physically very plausible that $A(w)$ is modular, if it is weakly modular.

Insertion of Eq. (9) into Eq. (2) shows ⁷⁾ that $A(w)$ has just the correct form to yield a $\tau(E)$ which is a sum of pure exponentials. In addition, to obtain this behaviour of $\tau(E)$, D must be given by

$$D = R - 1, \quad (11)$$

where $R+1$ is the space-time dimension. Then we obtain

$$\tau(E) = (2\alpha')^{-D/2} \sum_{n=1}^{N_0} d_0(n) \exp(4\pi \sqrt{\frac{\alpha' n}{s}} E) + o(\exp(\epsilon E)) \quad (12)$$

for any $\epsilon > 0$. In terms of the level degeneracies

$$d(m) \sim m^{-(D+3)/2} \exp(m\beta_0), \quad (13)$$

with

$$\beta_0 = 4\pi \sqrt{\frac{\alpha' N_0}{S}}. \quad (14)$$

Obviously, if A has a zero or a pole at some point, it has zeros and poles of the same order at all points of the corresponding orbit of G . Thus we identify points of H (the upper half plane plus the cusps) which can be transformed into each other by G . This yields the fundamental domain H/G of G .

For each non-zero modular function of weight $-D/2$ one has the important theorem

$$(\text{number of poles in } H/G) - (\text{number of zeros in } H/G) = (\Gamma:G)D/24, \quad (15)$$

where $(\Gamma:G)$ is the index of G in Γ and the zeros and poles are counted with appropriate weight. For some points in the upper half plane and some groups G , zeros and poles have to be counted with a weight factor $1/2$ or $1/3$. We need not care, however, because we are going to exclude zeros and poles everywhere, except at the cusps.

Now for each finite index Γ has only a finite number of subgroups. Thus the field of modular functions is already rather restricted, especially if we use the additional information that the $d(n)$ in Eq. (8) must be non-negative integers. However, possible mass spectra of string models must certainly fulfil much tighter restrictions. Guided by the known models, we guess

WH 2: $A(w)$ has no zeros and only one simple pole in the fundamental domain H/G .

Of course, $A(w)$ has no poles in the upper half plane, but nevertheless WH 2 is a strong assumption which may have to be relaxed. It fixes the location of all poles of $A(w)$, once G is given, as we must have a pole at $w=0$. According to WH 2 this pole must be simple which yields $\beta_0 = 4\pi(\alpha'/S)^{1/2}$.

Note that for $w=\infty$ we get only two possibilities. If this cusp is in the same orbit as 0, we have exactly one tachyon mass level. If not, the q expansion (8) starts with $n=0$ and we have a massless particle. Thus if we want to have a model with only massive particles, we have to weaken WH 2 at least for the cusp at infinity. WH 1+2 fix $A(w)$ up to a normalization constant once G is given: Eq. (15) yields

$$D = 24 / (\Gamma : G). \quad (16)$$

Now, take two functions A_1 and A_2 which fulfil WH 1+2. Normalize them in such a way that their difference has no pole. By Eq. (15) a modular function of negative weight without poles must vanish, thus $A_1 - A_2 = 0$. Even the normalization is almost fixed because the level degeneracies must be integers.

3. PHYSICAL INTERPRETATION OF SUBGROUPS OF Γ . A RE-DERIVATION OF THE MASS SPECTRA OF KNOWN DUAL STRING MODELS.

Equation (15) shows that D must be a divisor of 24 if WH 1+2 are fulfilled. As for higher indices there are more subgroups, it is convenient to proceed from $D=24$ to smaller values.

Take $(\Gamma:G)=1$, i.e., $G=\Gamma$. Then $D=24$, and by (11) we have space-time dimension 26. As we require that A has no zeros, $A(w)^{-1}$ is a modular form, i.e., a modular function holomorphic in H . The modular forms for Γ with weight 12 form a two-dimensional vector space⁸⁾. Its only elements with a zero at $w=0$ are the multiples of

$$A_V(w)^{-1} = \eta(w)^{24} = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \quad q = \exp(2\pi i w), \quad (17)$$

which yields the mass spectrum of the conventional string. H/Γ contains only one cusp, thus ∞ is in the same orbit as 0 and a tachyon is inevitable, even if only WH 1 is required. The squares of masses are quantized in units of $1/\alpha'$. For β_0 one obtains $\beta_0 = 4\pi\sqrt{\alpha'}$.

Γ contains only one subgroup of index 2 which is called G_2 , and four subgroups of index 3, namely $G_3, \Gamma^0(2), \Gamma_0(2)$, and G_θ . G_2 and G_3 are normal subgroups, whereas the other three subgroups are conjugate to each other.

G_2 is generated by all the squares of elements of Γ , G_3 by all the cubes. For G_2 one obtains

$$A_2(w) = \eta(w)^{-12} \quad (18)$$

and for G_3

$$A_3(w) = \eta(w)^{-8}. \quad (19)$$

No string models have been found which yield these mass spectra, and probably none can be found. To obtain the subgroups which really may correspond to string models, we use:

WH 3: Let L be the period system of a dual string model and $\hat{G}$ its invariance group. Then $G = \varphi(\hat{G})$, where φ denotes the canonical homomorphism of $SL(2, \mathbb{Z})$ onto Γ .

Consider, e.g., the conventional string ^{13), 14)}. Its states are described by functions $x_\mu(\sigma, \tau)$, $\mu = 0, \dots, R$, $0 \leq \sigma \leq \sigma_0/2$, which give a parametrization of the corresponding world sheet. After quantization, these functions become operators. It is natural to choose an orthonormal gauge, where

$$\left(\frac{\partial x}{\partial \sigma} \pm \frac{\partial x}{\partial \tau} \right)^2 = \sigma. \quad (20)$$

Usually one sets $\sigma_0 = 2\pi$, but sometimes it is more convenient to identify σ_0 with the energy of the string. In any case one can continue x_μ as a function of σ in such a way that $\partial x_\mu / \partial \sigma$ and $\partial x_\mu / \partial \tau$ are periodic functions of (σ, τ) , with basic periods $(\sigma_0, 0)$ and $(0, \sigma_0)$. These periods form a lattice, i.e., a vector space over $\mathbb{Z}$. The invariance group $\hat{G}$ of this lattice is just $SL(2, \mathbb{Z})$ where we identify $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the transformation

$$\begin{aligned} \sigma &\rightarrow a\sigma + b\tau, \\ \tau &\rightarrow c\sigma + d\tau. \end{aligned} \quad (21)$$

Thus we obtain $G = \varphi[SL(2, \mathbb{Z})] = \Gamma$. To understand this result better, take any two-dimensional lattice in the (σ, τ) plane with basic periods (σ_1, τ_1) and (σ_2, τ_2) . This pair of basic periods is not unique, e.g., (σ_1, τ_1) , $(\sigma_1 + \sigma_2, \tau_1 + \tau_2)$ generate the same lattice. For any pair of basic periods we form

$$W = \frac{\sigma_1 + i\tau_1}{\sigma_2 + i\tau_2}. \quad (22)$$

We restrict ourselves to pairs with $\text{Im}(w) > 0$. Then pairs which generate the same lattice yield w 's that are in the same orbit of Γ . Conversely, each orbit, i.e., each point of H/Γ , determines the lattice up to rotations and dilations.

Now let us consider the Neveu-Schwarz string. Again we have co-ordinate functions $x_\mu(\sigma, \tau)$, the derivatives of which have periods $(\sigma_0, 0)$ and $(0, \sigma_0)$. But this time we also have fermionic co-ordinates which change sign under $\sigma \rightarrow \sigma + \sigma_0$, $\tau \rightarrow \tau$ and under $\sigma \rightarrow \sigma$, $\tau \rightarrow \tau + \sigma_0$. Thus we have a generalized lattice L_{NS} of periods $(m\sigma_0, n\sigma_0)$, $m+n$ even, and "anti-periods" $(m\sigma_0, n\sigma_0)$, $m+n$ odd (see Fig. 1). We define a generalized lattice as a lattice where each element is marked by an element of a finite set. We admit only automorphisms which do not change this mark.

The invariance group of L_{NS} is the subgroup of $SL(2, \mathbb{Z})$

$$\hat{G}_{NS} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid b+c \text{ even} \right\}. \quad (23)$$

The corresponding subgroup of Γ is $\varphi(\hat{G}_{NS}) = G_\theta$.

As we have said, G_θ has index 3 in Γ . Its fundamental domain H/G contains two cusps, corresponding to the orbits of $w=0$ and $w=1$. The vector space of modular forms for G , with weight 4, is two-dimensional⁸⁾. Its elements, which have a zero for $w=0$, are multiples of

$$A_{NS}(w)^{-1} = q^{1/2} \prod_{n=1}^{\infty} (1-q^n)^8 (1+q^{n-1/2})^{-8}, \quad q = \exp(2\pi i w). \quad (24)$$

This can be deduced from the properties of Jacobi's theta functions¹⁵⁾. One finds

$$A_{NS}(w)^{-1} = \left(\frac{1}{2} \theta_2(w) \theta_4(w) \right)^4 = \left(\frac{1}{2} \theta_2\left(\frac{1}{4} \mid \frac{w}{2}\right) \right)^8, \quad (25)$$

where we have written

$$\theta_i(w) = \theta_i(0|w). \quad (26)$$

Equation (24) yields the mass spectrum of the Neveu-Schwarz model. The global properties of this spectrum can be obtained directly from simple features of G_0 : infinity lies in the same orbit as G , thus we must have a tachyon. The translations of w in G_0 are generated by $w \rightarrow w+2$, those of $-1/w$ by $w \rightarrow w/(2w+1)$. Thus intercepts are quantized in units of $\frac{1}{2}$, and $\beta_0 = 2\pi(2\alpha')^{\frac{1}{2}}$.

The bosonic and fermionic co-ordinates of the Ramond string have both periods $(\sigma_0, 0)$ and $(0, \sigma_0)$. However, $(0, \sigma_0)$ is strictly speaking a kind of "anti-period", because the string changes direction for $\sigma \rightarrow \sigma, \tau \rightarrow \tau + \sigma_0$, and for a fermionic string this must correspond to a change of state. Thus we obtain the generalized lattice L_R of periods $(m\sigma_0, n\sigma_0)$, n even, and anti-periods $(m\sigma_0, n\sigma_0)$, n odd (Fig. 1). The invariance group of L_R is the subgroup of $SL(2, \mathbb{Z})$

$$\hat{G}_R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c \text{ even} \right\}, \quad (27)$$

which yields the subgroup $\Gamma_0(2) = \varphi(\hat{G}_R)$ of Γ .

As $\Gamma(2)$ is conjugate to G_0 , its modular functions can be obtained from the modular functions of G_0 using the transformation

$$A(w) \rightarrow (w+1)^{-k} A\left(\frac{w}{w+1}\right) \quad (28)$$

where k is the weight of $A(w)$. Thus the space of modular forms for $\Gamma_0(2)$ with weight 4 is also two-dimensional. As the transformation (28) leaves $w=0$ invariant, the transform of A_{NS} has a pole for $w=0$, too. Applying (28) to A_{NS} we obtain

$$A_R(w) = 2^4 \theta_4(2w)^8 = 2^4 \prod_{n=1}^{\infty} (1-q^n)^{-8} (1+q^n)^8. \quad (29)$$

$A_R(w)$ yields the mass spectrum of the Ramond string. $H/\Gamma_0(2)$ has two cusps which correspond to the orbits of 0 and ∞ . Thus we have no tachyon, but of course a massless particle. The translations of w in $\Gamma_0(2)$ are generated by $w \rightarrow w+1$, those of $-1/w$ by $w \rightarrow w/(2w+1)$. Thus the squares of the masses are quantized in units of $1/\alpha'$, and $\beta_0 = 2\pi(2\alpha')^{\frac{1}{2}}$. That A_{NS} and A_R can be transformed into each other by modular transformations, corresponds obviously to the fact that they are obtained from the two sectors of the spinning string model.

The subgroup $\Gamma^0(2)$ of Γ , which is conjugate to $\Gamma_0(2)$ and G_θ , corresponds to the invariance group of a lattice consisting of periods $(m\sigma_0, n\sigma_0)$, m even, and antiperiods $(m\sigma_0, n\sigma_0)$, m odd. One can obtain the modular functions of $\Gamma^0(2)$ from the modular functions of G_θ by the transformation $A(w) \rightarrow A(w+1)$. However, $A_{NS}(w+1)$ has its pole at $w=1$. Instead WH 1+2 yield

$$A^0(w) = \theta_4(w)^{-8} = \prod_{n=1}^{\infty} (1-q^n)^{-8} (1-q^{n-\frac{1}{2}})^{-16}. \quad (30)$$

This function cannot be obtained from A_{NS} and A_R by modular transformations. Perhaps it corresponds to an additional dual string model with 10 space-time dimensions.

G_2 and G_3 are excluded by WH 3 because any two points of a lattice which can be transformed into each other by $SL(2, \mathbb{Z})$ can also be transformed into each other by the subgroups of $SL(2, \mathbb{Z})$, corresponding to G_2 or G_3 . Thus any generalized lattice, which is invariant under one of these subgroups, actually has the larger invariance group $SL(2, \mathbb{Z})$.

The Bardakçi-Halpern modification of the ordinary model satisfies WH 1, but not WH 2. As no string interpretation of this model is known, WH 3 cannot be checked. In this model we have $D=24-N$, $N=1, 2, \dots$, or 24. For $N=8, 16, 24$ the corresponding subgroup of Γ is G_θ . The Fourier transform of the mass spectrum has a double pole at $w=0$ and at $w=1$ a normal point, a simple zero, and a double zero, respectively.

For $N=4, 12, 20$, the relevant subgroup of Γ is $\Gamma(2)$. The invariant subgroups $\Gamma(n)$ of Γ consist of the transformations

$$w \rightarrow \frac{aw+b}{cw+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n}. \quad (31)$$

We shall need $\Gamma(2)$, $\Gamma(3)$, $\Gamma(4)$, and $\Gamma(8)$, which have indices 6, 12, 24, and 192, respectively. Their fundamental domains have 3, 4, 6, and 24 cusps (see Fig. 2).

The Bardakçi-Halpern mass spectrum for $N=4, 12, 20$ yields double poles at the cusps 0 and ∞ of $H/\Gamma(2)$, and at the cusp 1 a simple pole, a simple zero, and a triple zero, respectively. For $N=2, 6, 10, 14, 18, 22$ one needs $G=\Gamma(4)$, four quadruple poles at the cusps 0, ∞ , 2, and $\frac{1}{2}$ of $H/\Gamma(4)$, and at the remaining two cusps 1 and -1 a simple pole, a simple

zero, a triple zero, etc. For odd values of N one needs $G=\Gamma(8)$ and octuple poles at 16 cusps of $H/\Gamma(8)$, plus poles or zeros at the other cusps. Probably such models have not much to do with reality, even if they yield amplitudes which are consistent with space-time dimension 4.

4. GUESSING THE MASS SPECTRUM OF THE "CORRECT MODEL"

According to Eqs. (11) and (16) we need a subgroup of index 12 in Γ to get $D=2$ and space-time dimension 4. It would be rather tedious to study all of them, so we start by constraining the generalized lattice. If the model is reasonable, the fundamental parallelogram of this lattice should fit into a square of not too large size. For the Neveu-Schwarz and Ramond models 2×2 is enough. However, if the fundamental parallelogram fits into a $n \times n$ square, its invariance group contains $\Gamma(n)$. As $\Gamma(2)$ has index 6 in Γ , 2×2 is too small. $\Gamma(3)$ has index 12, thus this group could conceivably yield a model with $D=2$. However, intercepts would be multiples of $1/3$, which is not the way nature looks.

Therefore we choose 4×4 . Thus the subgroup G we are looking for should contain $\Gamma(4)$. As $\Gamma(4)$ is an invariant subgroup of Γ , $G/\Gamma(4)$ must be a subgroup of $\Gamma/\Gamma(4)$. To get $(\Gamma:G)=12$, $G/\Gamma(4)$ should contain two elements. $\Gamma/\Gamma(4)$ is isomorphic to the octahedral group S_4 , which is also the group of permutations of four elements. The fundamental domain $H/\Gamma(4)$ can be visualized as an octahedron with the cusps $0, \infty, 2, \frac{1}{2}, 1, -1$ at its vertices (Fig. 3). Obviously, S_4 has two types of subgroups of order 2: three conjugated subgroups which correspond to rotations through π about an axis joining two vertices, and six conjugated subgroups corresponding to rotations through π about an axis joining two mid-points of edges.

$H/\Gamma(4)$ has to contain twice the fundamental domain H/G . Thus it should contain two poles of $A(w)$, and the location of the poles should be invariant under $G/\Gamma(4)$. This leaves us with six possibilities:

- 1) double pole at 0; this yields $\beta_0 = 2\pi(2\alpha')^{\frac{1}{2}}$, which is definitely too high;
- 2) simple poles at 0 and ∞ ; this yields a tachyon;
- 3) simple poles at 0 and 1. this is impossible because $d(m)$ in Eq. (3) is real; thus we must have equal behaviour of $A(w)$ for $w=1$ and $w=-1$;
- 4) simple poles at 0 and -1; see case 3);
- 5) simple poles at 0 and 2;

6) simple poles at 0 and $\frac{1}{2}$.

The cases 5) and 6) both yield

$$\beta_0 = 2\pi\sqrt{\alpha'}. \quad (32)$$

If Eq. (11) is true, β_0 can be determined from the experimentally known part of the meson mass spectrum. Equation (32) is fulfilled with surprising accuracy ⁷⁾. The cases 5) and 6) both yield no tachyon. However, in case 5) the Regge intercepts are quantized in units of $\frac{1}{2}$ and in case 6) in units of $1/4$. If we want to have degenerate SU(3) multiplets in zeroth order of approximation, case 6) is excluded. But I prefer Frampton's point of view ⁵⁾, according to which the relations $\alpha'm_K^2 = 1/4$ and $\alpha'm_K^2(892) = 3/4$ should be valid at the tree level.

Thus we choose case 6). The corresponding subgroup G is generated by $\Gamma(4)$ and $\varphi_{12}^{(23)}$. It leaves invariant the generalized lattice of Fig. 3. For the mass spectrum we obtain

$$A(w) = \left(\theta_4(w/2) \theta_3(2w) \right)^{-1} = 1 + 2q^{1/4} + 4q^{1/2} + 8q^{3/4} + 12q + 20q^{5/4} + \dots \quad (33)$$

with $q = \exp(2\pi i w)$, which is to be compared with Eq. (3).

The mesons with $2\alpha'm^2 \in \mathbb{Z}$ are generated by

$$\frac{1}{2} \left(\theta_4\left(\frac{w}{2}\right) \theta_3(2w) \right)^{-1} + \frac{1}{2} \left(\theta_4\left(\frac{w}{2}+1\right) \theta_3(2w+4) \right)^{-1} = \theta_4(w)^{-2}, \quad (34)$$

which is just the function we get in case 5). Thus it turns out that the choice between the cases 5) and 6) is no choice after all. Case 5) just yields a part of the mass spectrum and case 6) the complete spectrum.

One can guess some quantum numbers of the mesons by writing Eq. (33) in the form

$$A(w) = \prod (1-q^n)^{-2} (1+q^r)^3 (1+q^{r/2})^2 (1+q^{2n})^2, \quad (35)$$

$$n = 1, 2, \dots$$

$$r = 1/2, 3/2, \dots$$

In particular, one can try to construct the spectrum from Bose oscillators A_{ni} , $i = 1, 2$, and Fermi oscillators B_{ri} , $i = 1, 2, 3$, $S_{r/2, \alpha}$, $\alpha = \pm$, and $C_{2n, \beta}$, $\beta = \pm$. Comparing (35) with the Neveu-Schwarz model, we see that the index i should be a Lorentz index. But contrary to that model, only one component of the b oscillators is eliminated. This is fortunate because the ρ is now massive. The index α cannot be a Lorentz index since the degeneracy at mass level $\alpha'm^2 = 1/4$ is only two. Thus we interpret α in terms of an internal symmetry (strangeness). For β a similar interpretation suggests itself (charm).

Comparing with the experimental mass spectrum we see that $A(w)$ yields only the states with charge zero. In general, the correspondence is excellent. However, there remain more or less serious problems, e.g.,

- 1) the ω, A_2 trajectory is not degenerate with the ρ, f trajectory, but is shifted one half unit in $\alpha'm^2$; some other states also occur at too high masses;
- 2) charmed mesons have too low masses;
- 3) there is a scalar meson below the $K^*(890)$;
- 4) one obtains exotic mesons with multiple strangeness and charm; the first meson with double strangeness occurs at $\alpha'm^2 = 1$ (note that one meson with double charm should be stable);
- 5) not much is left of $SU(3)$.

5. DISCUSSION

Applying a mathematical technique of the last century, the known dual models have been characterized in a very compact way and a framework for the construction of new models has been found. In particular, one possible model seems to be very interesting from an experimental point of view because it has critical dimension four and no tachyon. Moreover, the agreement with the experimental mass spectrum is excellent, though some serious problems remain. One important step to be done is the introduction of charge (and baryons, if possible). Given the Fock-space representation of the model, it seems to be not too difficult to construct the gauge algebra, and perhaps the vertex operators.

It is interesting to note that the S oscillators in Eq. (35) are necessary for rotational invariance. Insight into the nature of strangeness and charm could be one of the most important yields of the model.

To clarify all these results, one should deduce WH 1-3 from more fundamental principles. Probably this can be done along the lines of Ref. 11), if the meaning of the functional integrals is clarified. In this way it should also be possible to interpret the characters in Eq. (7).

One should also characterize the mass spectra of the Pomeron sectors of the models. Their Fourier transforms are not modular functions, but probably they can be described in an equally compact form. More intriguing is the connection between the subgroups of Γ and the gauge symmetries of the models. Probably smaller groups correspond to larger gauge algebras. In particular, the model with mass spectrum (35) should have a gauge algebra containing the gauge algebra of the Neveu-Schwarz model (with different c numbers), as the corresponding subgroup is contained in G_0 .

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APPENDIX

Let $A(w)$ be a weakly modular function for G with weight k . For its q expansion about $w=\infty$ we assume

$$A(w) = \sum_n d(n) \exp(2\pi i n w / r) \quad (A.1)$$

with

$$|d(n)| = o(\exp(a\sqrt{n})) \quad (A.2)$$

for some constant a .

In order to prove that $A(w)$ is meromorphic at $w=0$, we have to show that

$$|A(w)| = o(\exp(b \ln(-1/w))) \quad (A.3)$$

for some constant b .

Let the subgroup of translations of $-1/w$ in G be generated by $w \rightarrow w/(sw+1)$. We write

$$-1/w = ns - 1/w', \quad n \in \mathbb{Z}, \quad (A.4)$$

where

$$|\operatorname{Re}(1/w')| \leq |s/2|. \quad (A.5)$$

As

$$A(w) = (1 - ns w')^k A(w') \quad (A.6)$$

and

$$\operatorname{Im}(-1/w) = \operatorname{Im}(-1/w'), \quad (A.7)$$

it is sufficient to consider the case $n=0$. Let

$$\operatorname{Im}(-1/w) > 1. \quad (A.8)$$

From (A.5) we obtain

$$\operatorname{Im}(-1/w) > c/\operatorname{Im}(w) \quad (\text{A.9})$$

for some constant c . Thus we only have to show that

$$|A(w)| = o(\exp(d/\operatorname{Im}(w))) \quad (\text{A.10})$$

for some constant d . But this relation follows directly from (A.1) and (A.2).

To prove meromorphic behaviour at the cusp $w=d/c$, we define

$$B(w) = A(w + d/c). \quad (\text{A.11})$$

$B(w)$ is a weakly modular function for some different subgroup G' of finite index in Γ , the Fourier coefficients of which fulfil relation (A.2). Thus we can apply (A.3). We obtain

$$A(w) = o(\exp(f \operatorname{Im}(w)/|cw-d|^2)) \quad (\text{A.12})$$

for some constant f . Q.e.d.

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FIGURE CAPTIONS

Fig. 1 : Period lattices of

- a) the ordinary dual string,
- b) the Neveu-Schwarz string,
- c) the Ramond string,
- d) and e) correspond to possible new models with mass spectra given by Eqs. (30) and (33). The full circle denotes zero.

Fig. 2 : Representations of

- a) H/Γ ,
- b) H/G_Θ ,
- c) $H/\Gamma_\Theta(2)$,
- d) $H/\Gamma^0(2)$,
- e) $H/\Gamma(2)$,
- f) $H/\Gamma(3)$,
- g) $H/\Gamma(4)$.

Edges which carry the same greek letter are to be identified in direction of the arrows.

Fig. 3 : Representation of $H/\Gamma(4)$ on an octahedron. The greek letters agree with Fig. 2. The symmetry group of the octahedron correspond to $\Gamma/\Gamma(4)$.

FIG. 1

FIG. 2

FIG. 3