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LATTICE FORMULATIONS OF REGGEON INTERACTIONS

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A B S T R A C T

We examine a class of lattice analogues to the Reggeon field theory. First the transition from a continuum to a lattice field theory is discussed, emphasizing the necessity of a Wick rotation and the consideration of symmetry properties. Next the theory is transformed to a discrete system with two spins at each lattice site, and the problems of the triple Reggeon interaction and the Reggeon energy gap are discussed. It is pointed out that transferring the theory from the continuum to a lattice necessarily introduces new relevant operators not normally present in Reggeon field theory.

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1. INTRODUCTION

One of the most powerful languages for discussing high energy scattering is the Reggeon calculus of Gribov¹⁾. In their original work Gribov and his collaborators showed how the calculus was formally identical to the infrared problem in a field theory, and started applying field theoretic techniques to its analysis. This programme was revitalized by the work of Migdal, Polyakov and Ter-Martirosyan²⁾ and of Abarbanel and Bronzan²⁾, who showed how the renormalization group and the ϵ -expansion³⁾ could be used to estimate critical indices describing high energy scattering. Their work was elaborated to include general multi-Reggeon interactions⁴⁾ and modifications⁵⁾ to the "non-relativistic" Pomeron propagator favoured by earlier authors.

In the analysis of high-energy scattering as an infrared problem and the application of the ϵ -expansion a parallel with statistical mechanics⁶⁾ became apparent. A potent method of analysing statistical mechanical systems on a lattice has been to obtain a continuum field theory with similar infrared critical properties, and use the ϵ -expansion to calculate them³⁾. Several authors^{7, 8, 9)} suggested it might be useful to reverse this procedure for the Reggeon field theory, by constructing a lattice analogue with similar critical behaviour, and then trying to calculate its behaviour with lattice techniques. Renormalization group techniques on a lattice have been tried without much success¹⁰⁾: on the other hand a high-temperature expansion has been used to give a preliminary calculation¹¹⁾ of the critical exponents describing high energy scattering.

The primary purpose of this paper is to give a formal analysis of the construction of lattice analogues to the Reggeon Field Theory. The techniques used are standard ones^{3, 6)} used in establishing the analogy between, for example, the Ising model on a lattice and a ϕ^4 continuum field theory. We will describe the procedure in considerably more detail than in the papers of Parisi⁷⁾ and BESZ⁸⁾. A key feature is the classification of lattice models by their symmetry properties under basic symmetry transformations.

In Section 2 we discuss the transfer of a continuum Reggeon field theory to a lattice, stressing the importance of performing a Wick rotation and the symmetry properties of the lattice theory. In Section 3 we transform from an unbounded field theory on a lattice to a system with two discrete spins at each lattice site, using the standard Gaussian integral transformation^{12, 13)}. In this analysis we stress the importance of ensuring similar criticality in both the real and imaginary components of the complex Reggeon field ψ (or equivalently both spins s and t). This necessitates introducing new relevant operators not normally present in the Reggeon field theory, such as $(\psi^2 + \psi^{+2})$ and $(\partial\psi \cdot \partial\psi + \partial\psi^+ \cdot \partial\psi^+)$. On the other hand, the precise form of triple-Reggeon interactions corresponding to the

lattice system is shown to be irrelevant as far as calculating critical indices is concerned. This section concludes with a comparison of the two spin model with different Reggeon field theories and with previously proposed lattice models.

Section 4 summarizes our results and conclusions. The paper closes with two appendices. The first expresses the lattice Reggeon model as a cut-off in momentum space of the path integral formulation. The second discusses the nature of the vacuum in theories with a non-Hermitian interaction.

2. REGGEON FIELDS ON A LATTICE

In this section we discuss the transfer of the Reggeon field theory to a lattice. We start from the usual^{1,2)} field theoretic Lagrangian in 1 time and D space dimensions (in the physical case D = 2):

$$\mathcal{L}(\underline{x}, \tau) = \left. \begin{aligned} & \frac{i}{2} \psi^*(\underline{x}, \tau) \overleftrightarrow{\frac{\partial}{\partial \tau}} \psi(\underline{x}, \tau) - \alpha'_0 \frac{\partial \psi^*(\underline{x}, \tau)}{\partial \underline{x}} \cdot \frac{\partial \psi(\underline{x}, \tau)}{\partial \underline{x}} \\ & - \Delta_0 \psi^*(\underline{x}, \tau) \psi(\underline{x}, \tau) + \frac{ig_0}{2} (\psi^* \psi)(\psi^* + \psi) \end{aligned} \right\} \quad (2.1)$$

which has a complex, continuous, unbounded field ψ in a continuous (space, time) = $(\underline{x}, \tau)^*$.

2.1 Wick rotation

Before beginning the discretization, we want to emphasize one point not previously discussed. The Feynman propagator corresponding to the bilinear terms in the Lagrangian (2.1) has the form

$$\frac{i}{E - \alpha'_0 \underline{k}^2 - \Delta_0 + i\epsilon} \quad (2.2)$$

where we have chosen the conventional $i\epsilon$ prescription, and the E integration is along the contour C shown in Fig. 1a. We could equally well have taken the alternative

$$\frac{i}{E - \alpha'_0 \underline{k}^2 - \Delta_0 - i\epsilon} \quad (2.3)$$

in which case the E integration would be along the contour C' shown in Fig. 1b.

*) Other possible Reggeon field theories⁴⁾ will be discussed in Section (3.4).

The only important thing in actual calculations is not to mix the prescriptions (2.2) and (2.3). In making the transition to a lattice from the Lagrangian (2.1), there is no way of specifying the $i\epsilon$ -prescription. Indeed a simple lattice model obtained from Eq. (2.1) by replacing derivatives with differences would correspond to a field theory calculated using a symmetric mixture of the two $i\epsilon$ prescriptions (2.2) and (2.3). This problem is familiar from studies of lattice analogues to relativistic theories, where the same problem occurs^{3,6)}. The solution is to make a Wick rotation in the continuum field theory. If we do the same to the Reggeon field theory (2.1), we introduce a "new" variable $Y \equiv -i\tau$. The E integral is then along the imaginary axis (contour C'' in Fig. 1c) as in the usual Sommerfeld-Watson transform

$$\int_{-i\infty}^{i\infty} \frac{dE}{2\pi i} e^{-EY} (\dots) \quad (2.4)$$

so that Y is recognized as the usual rapidity variable. The contour C of Fig. 1a can be deformed into the contour C'' of Fig. 1c, so that the Wick-rotated field theory corresponds unambiguously to the $i\epsilon$ prescription of Eq. (2.2). We have indeed checked that this prescription gives the correct answer in the case of 1-dimensional Ising and classical Heisenberg ferromagnet models^{*)}. We will therefore seek a lattice analogue to the Wick-rotated theory

$$\mathcal{L} = \frac{1}{2} \psi^+ \overleftrightarrow{\partial}_Y \psi - \alpha'_0 \frac{\partial \psi^+}{\partial x} \cdot \frac{\partial \psi}{\partial x} - \Delta_0 \psi^+ \psi + \frac{ig_0}{2} (\psi^+ \psi) (\psi^+ + \psi) \quad (2.5)$$

which is in a configuration space related to the $E = 1 - J$ plane by the usual Regge theory Mellin transform, as seen in Eq. (2.4) [see also Appendix A].

2.2 Discretization of space-time

It is convenient in discretizing the theory (2.5) to first re-express ψ and ψ^+ in terms of two components ϕ and χ ^{**) :}

$$\left. \begin{aligned} \psi &= \phi + i\chi \\ \psi^+ &= \phi - i\chi \end{aligned} \right\} \quad (2.6)$$

*) These models, i.e. spin models with complex temperature, have phase transitions even in 1 dimension, analogous to the case of the Ising model with imaginary magnetic field described in Ref. 11.

**) Note that in the presence of the non-Hermitean interaction of Eq. (2.5) the equations of motion would make components ϕ and χ complex in general.

in terms of which the Lagrangian (2.5) can be written

$$\mathcal{L} = \left. \begin{aligned} & i \left(\phi \frac{\partial \chi}{\partial Y} - \chi \frac{\partial \phi}{\partial Y} \right) - \alpha'_0 \left(\frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi}{\partial x} + \frac{\partial \chi}{\partial x} \cdot \frac{\partial \chi}{\partial x} \right) \\ & - \Delta_0 (\phi^2 + \chi^2) + i g_0 (\phi^2 + \chi^2) \phi \end{aligned} \right\} \quad (2.7)$$

The lattice theories discussed in this paper will have two spins at each site analogous to the continuous variables ϕ and χ .

The first step now is to make the transition to a discrete space-time structure (Y_i, x_i) , while retaining continuous fields (ϕ_i, χ_i) with the ranges $(-\infty, \infty)$. To do this we replace^{13, 14)} the derivatives in Eq. (2.7) by differences:

$$\frac{\partial \phi}{\partial Y} \rightarrow \frac{\phi_{i+\hat{Y}} - \phi_i}{a}, \quad \frac{\partial \chi}{\partial Y} \rightarrow \frac{\chi_{i+\hat{Y}} - \chi_i}{a} \quad (2.8)$$

where a is the lattice spacing in the Y -direction, and

$$\frac{\partial \phi}{\partial x} \rightarrow \frac{\phi_{i+\hat{x}} - \phi_i}{b}, \quad \frac{\partial \chi}{\partial x} \rightarrow \frac{\chi_{i+\hat{x}} - \chi_i}{b} \quad (2.9)$$

where b is the lattice spacing in the D transverse x directions.

We recall that this discretization procedure only alters the ultra-violet (short distance) behaviour of the theory. In fact the Regge field theoretic propagator is presumably not accurate at short rapidity separations either, both for the theoretical reasons of direct channel thresholds and the related phenomenological belief that talking of Pomeron exchange does not make sense for rapidity separations $\Delta Y < \text{some finite value } a_0 \sim 2$. It is tempting to relate the cut-off a_0 in the phenomenological propagator

$$\Theta(\Delta Y - a_0) A_{\text{Regge}}(\Delta Y, \Delta \underline{b})$$

to a lattice spacing, but recall that Pomeron propagation can take place over any $\Delta Y > a_0$, while lattice interactions are limited to spacings na : n an integer. It may also be tempting to believe that some mean hadronic "size" b_0 may introduce a granularity in impact parameter which could be related to the space-like spacing b , but we do not see how to realize this idea.

Substituting formulae (2.8) and (2.9) into the Lagrangian (2.7) we get (for more details see Appendix A):

$$\begin{aligned} \mathcal{L} = & \frac{i}{a} (\phi_i \chi_{i+\hat{y}} - \chi_i \phi_{i+\hat{y}}) \\ & - \frac{\alpha'_0}{b^2} \left[0(\phi_i^2 + \chi_i^2) - 2 \sum_{\underline{\hat{x}}} (\phi_i \phi_{i+\underline{\hat{x}}} + \chi_i \chi_{i+\underline{\hat{x}}}) + \sum_{\underline{\hat{x}}} (\phi_{i+\underline{\hat{x}}}^2 + \chi_{i+\underline{\hat{x}}}^2) \right] \\ & - \Delta_0 (\phi_i^2 + \chi_i^2) + i g_0 (\phi_i^2 + \chi_i^2) \phi_i \end{aligned} \quad (2.10)$$

The space-time integral over $(Y, \underline{x})$ of the original Lagrangian $\mathcal{L}$ to get the field-theoretic action now becomes a sum over all lattice sites of the Lagrangian (2.10). In this summation the terms $(\phi_{i+\underline{\hat{x}}}^2 + \chi_{i+\underline{\hat{x}}}^2)$ and $(\phi_i^2 + \chi_i^2)$ become equivalent. The Lagrangian (2.10) becomes the lattice theory

$$\begin{aligned} \mathcal{L} = & iE (\phi_i \chi_{i+\hat{y}} - \chi_i \phi_{i+\hat{y}}) + K \sum_{\underline{\hat{x}}} (\phi_i \phi_{i+\underline{\hat{x}}} + \chi_i \chi_{i+\underline{\hat{x}}}) \\ & + M (\phi_i^2 + \chi_i^2) + iG (\phi_i^2 + \chi_i^2) \phi_i \end{aligned} \quad (2.11)$$

where the parameters E , K , M and G are related simply to the parameters of Eq. (2.10):

$$E = \frac{i}{a}, \quad K = \frac{2\alpha'_0}{b^2}, \quad M = -\Delta_0 - \frac{2D\alpha'_0}{b^2}, \quad G = g_0 \quad (2.12)$$

Notice that in Eqs. (2.11) and (2.12) the parameter iE is purely imaginary, reflecting the Wick rotation.

We should observe at this point that while the discretized field theory (2.11) should have the same critical indices as the theory (2.1), there are additional non-zero Green's functions

$$G^{(n,0)} \equiv \langle 0 | T(\psi(y_1, \underline{x}_1) \psi(y_2, \underline{x}_2) \dots \psi(y_n, \underline{x}_n)) | 0 \rangle$$

and $G^{(0,n)}$. In Appendix A we demonstrate this by studying the path integral formalism in $(E, \underline{k})$ space. Here, the lattice theory is shown to be equivalent to a cut-off $a^{-1}(b^{-1})$ in $E(|\underline{k}|)$. The Green's functions $G^{(n,0)}$ arise because of the disruption of causality due to the cut off a . Clearly the large Y behaviour (critical exponents) should be independent of a . The discussion of appendix A shows that extra terms are induced in the lattice Lagrangian (2.11) in perturbation

theory: we refer the reader there for details. Related terms will recur in Section 3, and will be discussed there.

2.3 Symmetries

For systems exhibiting critical behaviour in the infrared region, the critical exponents are expected to depend only on the dimensionality and symmetries of the system. In discussing the lattice model (2.11) we introduce the following symmetry operations on the lattice:

$$\left. \begin{aligned} R_\phi : \phi_i &\leftrightarrow -\phi_i, & R_\chi : \chi_i &\rightarrow -\chi_i \\ T : (\phi_{i+n\hat{y}}, \chi_{i+n\hat{y}}) &\leftrightarrow (\phi_{i-n\hat{y}}, \chi_{i-n\hat{y}}), & C : i &\leftrightarrow -i \\ I : \phi_i &\leftrightarrow \chi_i, & P : (\phi_{i+n\hat{x}}, \chi_{i+n\hat{x}}) &\leftrightarrow (\phi_{i-n\hat{x}}, \chi_{i-n\hat{x}}) \end{aligned} \right\} (2.13)$$

We observe that each of the transformations (2.13) [with the exception of P], when applied to the nearest-neighbour interactions in Eq. (2.11), has the effect of changing $iE \leftrightarrow -iE$. All the transformations (2.13) leave K invariant. Hence all the nearest-neighbour interactions in Eq. (2.13) are left invariant by the product of any pair of the transformations (2.13). The mass term $M(\phi_1^2 + \chi_1^2)$ is invariant under all the transformations (2.13), whereas the triple interaction $iG(\phi_1^2 + \chi_1^2)\phi_1$ is invariant under R_χ , T and P, but changes sign under R_ϕ and C, and has no simple properties under the I transformation. Thus I is not a good invariance of the Lagrangian (2.11). Indeed, the only invariances of the entire Lagrangian (2.11) are CR_ϕ , TR_χ and P. More symmetric Lagrangians are expected to fall into different universality classes with different critical indices, and some will be discussed in Section 3.4.

In setting up a lattice analogue model we could *a priori* allow all interactions which have the same symmetry properties. It is therefore important to ask what other simple local or nearest neighbour interactions are allowed. We then see that the interactions

$$\phi_i \chi_i, \quad (\phi_i \chi_{i+\hat{x}} + \phi_{i+\hat{x}} \chi_i), \quad (\phi_i \chi_{i+\hat{y}} + \phi_{i+\hat{y}} \chi_i)$$

are not invariant under for example the transformation $R_\chi T$ which is a symmetry of Eq. (2.11). The simplest nearest neighbour interactions with similar invariance properties to the lattice Lagrangian (2.11) are the time-like nearest neighbour interactions

$$L \phi_i \phi_{i+\hat{y}} \quad \text{and} \quad L' \chi_i \chi_{i+\hat{y}} \quad (2.14a)$$

which are invariant under all the transformations (2.13) with the exception of I. Invariance under I requires $L = L'$ but we have already seen that I is not a symmetry of the trilinear interaction. In the absence of I-symmetry we may also introduce different couplings

$$\Delta'(\phi_i^2 - \chi_i^2) \quad \text{and} \quad \kappa'(\phi_i \phi_{i+\hat{x}} - \chi_i \chi_{i+\hat{x}}) \quad (2.14b)$$

The interaction (2.14a) with $L = L'$ would correspond to a term $\lambda[(\partial\psi^+/\partial\tau)(\partial\psi/\partial\tau)]$ in a continuous space Lagrangian, and hence to an E^2 term in the continuum field theory propagator

$$\frac{i}{\lambda E^2 + E - \alpha'_0 k^2 - \Delta_0 + i\epsilon} \quad (2.15)$$

Since in the infrared region $E^2 \ll E$, we expect that $\lambda[(\partial\psi^+/\partial\tau)(\partial\psi/\partial\tau)]$ will be an irrelevant variable, and so such a term should not change the infrared behaviour of the continuum or lattice theory (see also Appendix B).

3. TWO COMPONENT SPINS ON A LATTICE

We now consider how the lattice theory (2.11) can be reduced to a spin system by restricting the ranges of the variables ϕ_i, χ_i . Our explicit discussion will be of discrete spins $= \pm 1$, but other restricted ranges such as $[-1, 1]$ can be studied similarly.

3.1 Gaussian integral transformation

In making a first discrete spin analogue to the lattice Lagrangian (2.11) we will introduce two spins $s_i, t_i = \pm 1$ at each lattice site, and demand that their interaction Hamiltonian $\mathcal{H}$ have similar symmetry properties to $\mathcal{L}$ (2.11) under the transformations corresponding to Eq. (2.13) [in an obvious notation]: R_s, R_t, T, C and I . It will be given by

$$\begin{aligned} \mathcal{L} = & i \sum_{i, \hat{y}} E_0 (s_i t_{i+\hat{y}} - t_i s_{i+\hat{y}}) \\ & + \sum_{i, \hat{x}} K_0 (s_i s_{i+\hat{x}} + t_i t_{i+\hat{x}}) + \sum_{i, \hat{y}} L_0 (s_i s_{i+\hat{y}} + t_i t_{i+\hat{y}}) \\ & + i \sum_i C_0 s_i \end{aligned} \quad (3.1)$$

The term $iG_0 s_i$ in Eq. (3.1) is the natural discrete spin analogue of the triple field interaction $iG(\phi_i^2 + \chi_i^2)\phi_i$. To establish a connection between $\mathcal{H}$ (3.1) and the lattice field theory $\mathcal{L}$ (2.11), we follow standard procedures^{3,6,12,13} by writing the nearest neighbour interactions $\mathcal{H}_0$ in Eq. (3.1) in terms of a matrix K_{kl} over a space $\{u_k\}$ of lattice spins (where $\{u_k\} = \{s_i, t_i\}$):

$$\beta \mathcal{H}_0 = \frac{1}{2} \sum_{k,l} K_{kl} u_k u_l \quad (3.2)$$

Then by Gaussian integration we can write (where $G_k = \beta G_0$ if u_k is a s_i , and $G_k = 0$ if u_k is a t_i):

$$\begin{aligned} \exp(-\beta \mathcal{H}) &= \exp\left(-\frac{1}{2} \sum_{k,l} K_{kl} u_k u_l - i \sum_k G_k u_k\right) \\ &= (2\pi)^{-N} D^{-\frac{1}{2}} \prod_{k=1}^{2N} \int_{-\infty}^{\infty} d\tilde{z}_k \exp\left[\frac{1}{2} \sum_{k,l} K_{kl}^{-1} \tilde{z}_k \tilde{z}_l + \sum_k (\tilde{z}_k - i G_k) u_k\right]. \end{aligned} \quad (3.3)$$

In Eq. (3.3), N is the number of lattice sites, $D \equiv \det(K)$, and K^{-1} is the inverse of K . Just as half of the u_k are s_i and half t_i , so half of the $\tilde{z}_k$ can be taken as ϕ_i and half as χ_i . Then the symmetries of the matrix K will be carried over as symmetries of the matrix K^{-1} . In particular, because of our choice of symmetries of $\mathcal{H}$, K^{-1} will have the same transformation properties under R_s , R_t , T , C and I (2.13) as do the kinetic terms in the Lagrangian $\mathcal{L}$ under the related transformations R_ϕ , R_χ , T , C and I . In general K^{-1} may have non-nearest neighbour interactions: these would be related to higher order derivative interactions in the continuum field theory Lagrangian. They are therefore expected to be irrelevant in the infrared limit because they have the same basic symmetry properties as the interactions in $\mathcal{L}$. As discussed earlier, the simplest nearest neighbour interactions are those in $\mathcal{L}$ (2.11), and the L interaction (2.14) which, as we argued before, is itself believed to be irrelevant in the infrared limit.

We now examine the local terms in Eq. (3.3), by introducing the partition function Z of the spin system:

$$\begin{aligned} Z &= \sum_{s_i, t_i = \pm 1} \exp(-\beta \mathcal{H}) \\ &= \pi^{-N} D^{-\frac{1}{2}} \prod_{k=1}^{2N} \int_{-\infty}^{\infty} d\tilde{z}_k \exp\left[\frac{1}{2} \sum_{k,l} K_{kl}^{-1} \tilde{z}_k \tilde{z}_l + \sum_k (\tilde{z}_k - i G_k) u_k\right] \end{aligned} \quad (3.4)$$

Hence, the partition function for the spin system looks like that for fields ranging over $(-\infty, \infty)$ with a local potential term

$$V(\phi, \chi) \equiv \frac{\mathcal{K}}{2}(\phi^2 + \chi^2) + \ln \cosh(\phi - i\zeta_0) + \ln \cosh \chi \quad (3.5)$$

(where $\mathcal{K} \equiv \sum_{k,1} K_{k1}^1$) in place of the terms

$$-\Delta_0(\phi^2 + \chi^2) - i g_0(\phi^2 + \chi^2)\phi$$

in the original Lagrangian $\mathcal{L}$ (2.1, 2.11). To compare Eq. (3.5) with the Lagrangian (2.11) it is first necessary to shift to a stable vacuum. The nature and meaning of the vacuum in a theory with a non-Hermitean Hamiltonian is discussed in detail in Appendix B. For the moment, however, we may continue with our naïve argument. From Eq. (3.5) we have

$$\left. \begin{aligned} \frac{\partial V}{\partial \phi} &= \mathcal{K}\phi + \tanh(\phi - i\zeta_0) = 0 \\ \frac{\partial V}{\partial \chi} &= \mathcal{K}\chi + \tanh \chi = 0 \end{aligned} \right\} \quad (3.6)$$

It is apparent from Eq. (3.6) that a pure imaginary shift $\phi \rightarrow \phi'$ is necessary, but that no shift in χ is needed. Suppose this shift is made: then V will take the form

$$V(\phi', \chi) = -\Delta_\phi \phi'^2 - \Delta_\chi \chi^2 - V_\phi(\phi') - V_\chi(\chi) \quad (3.7)$$

where in general $\Delta_\phi \neq \Delta_\chi$, and even terms in $V_\phi(\phi')$ are real while odd terms are pure imaginary (V_χ is even in χ , and real). The difference between Δ_ϕ and Δ_χ reflects the violation of the symmetries IR_s , IR_t , IT and IC [see Eq. (2.13)] by the imaginary magnetic field term $iG_0 s_i$ in the spin Hamiltonian (3.1). It is also worth pointing out that this difference is one of the reasons that $G_{\psi\psi}$ and $G_{\psi+\psi+}$ are non-zero. The other source of this effect is the discretization of space-time as discussed in Appendix A.

3.2 The triple interaction

Let us first discuss the trilinear interactions in Eqs. (3.7) and (2.11). Looking at the potential (3.7) we observe that it has no local interactions

between ϕ and χ , in particular it has an imaginary ϕ'^3 interaction, but no $\phi'\chi^2$ as in the original Lagrangian (2.7, 2.11).

Suppose for the moment that we can modify Eq. (3.1) in such a way that the corresponding continuum theory when renormalized has equal energy gaps Δ_ϕ, Δ_χ for the real and imaginary parts, ϕ and χ . Should it then worry us that the trilinear interaction in Eq. (3.7) does not correspond directly to the Reggeon field theory? First we note that any combination of ϕ^3 and $\phi\chi^2$ interactions can be written in terms of interactions

$$\frac{ig}{2}(\psi^+\psi)(\psi^+\psi) = ig(\phi^2 + \chi^2)\phi \quad \text{and} \quad \frac{ih}{6}(\psi^+ + \psi^3) = \frac{ih}{3}\phi(\phi^2 - 3\chi^2) \quad (3.8)$$

It is not difficult to study the renormalization group fixed points of a theory of the type (2.1) but with additional interactions as in Eq. (3.8). The relevant diagrams up to 3rd order in the coupling constants are shown in Fig. 2. Calculating the contributions to the Callan-Symanzik β -functions for g and h yields

$$\left. \begin{aligned} \beta_g(g, h) &= -\frac{\epsilon}{4}g + 6Ag^3 + 8Ag^2h + 6Ag h^2 \\ \beta_h(g, h) &= -\frac{\epsilon}{4}h + 22Ag^2h - 2Ah^3 \end{aligned} \right\} \quad (3.9)$$

where $\epsilon = 4-D$, $A = 1/128\pi^2$, and we have set $\epsilon = 0$ in the coefficients of the third order terms in Eq. (3.9). The corresponding expression for the anomalous dimension, γ , to second order is

$$\gamma(g, h) = A(g^2 + h^2) \quad (3.10)$$

We now study the infrared zeroes of the Eqs. (3.9), and their stability. The zeroes are found from the equations:

$$\left. \begin{aligned} B_g \equiv \frac{\epsilon}{4A}g &= 6g^3 + 8g^2h + 6gh^2 \\ B_h &= 22g^2h - 2h^3 \end{aligned} \right\} \quad (3.11)$$

and are:

$$\left. \begin{aligned} g=0, h=0 & \quad ; \quad g^2 = \frac{B}{6}, h=0 \quad ; \quad g=0, h^2 = -\frac{A}{2} \\ g^2 = h^2 = \frac{B}{20} & \quad ; \quad g^2 = \frac{B}{14}, h = -2g \end{aligned} \right\} \quad (3.12)$$

To analyse the stability of these zeroes, we need to examine the matrix $\partial(\beta_g, \beta_h)/\partial(g, h)$ in the neighbourhood of each of the fixed points (3.12). We find

$$\frac{\partial(\beta_g, \beta_h)}{\partial(g, h)} = \begin{pmatrix} -B + 18g^2 + 16gh + 6h^2 & 8g^2 + 12gh \\ 44gh & -B + 22g^2 - 6h^2 \end{pmatrix} \quad (3.13)$$

For the zero at $g^2 = B/6, h = 0$ we find that

$$\frac{\partial(\beta_g, \beta_h)}{\partial(g, h)} = \begin{pmatrix} 2B & \frac{4B}{3} \\ 0 & \frac{8B}{3} \end{pmatrix} \quad (3.14)$$

which has two positive eigenvalues, so that this zero is completely infrared stable. On the other hand, all the other zeroes have at least one negative eigenvalue and so are unstable. We conclude that in a theory with both g and h interactions, at least in the ϵ expansion, the only stable infrared zero is the conventional Reggeon field theory²⁾ one with [from Eq. (3.10)] $\gamma = \epsilon/24$ and a cross-section rise $\alpha(\log s)^{2\gamma} \approx (\log s)^{\epsilon/12+\dots}$. Hence a theory with any combination of g and h interactions, in particular, the lattice theory (3.1), should, insofar as the triple couplings are concerned, behave like the usual Reggeon field theory in the infrared region of interest.

3.3 Unequal energy gaps

In contrast to the previous analysis, problems do arise if ϕ and χ have unequal "energy gaps" Δ_ϕ and Δ_χ ^{*)}. Consider a field theory Lagrangian with the bilinear terms ^{**) :}

$$-(\phi \frac{\partial \chi}{\partial \tau} - \chi \frac{\partial \phi}{\partial \tau}) - \alpha'(\partial \phi \cdot \partial \phi + \partial \chi \cdot \partial \chi) - \Delta_\phi \phi^2 - \Delta_\chi \chi^2 \quad (3.15a)$$

*) Strictly speaking, Δ_ϕ and Δ_χ do not have the interpretation of energy gaps -- see later in this subsection.

**) This analysis can be extended to the case where the coefficients of $\partial \phi \cdot \partial \phi$ and $\partial \chi \cdot \partial \chi$ are unequal, but for simplicity we suppress that complication for now.

or, equivalently

$$\frac{i}{2} \psi^\dagger \overleftrightarrow{\partial}_\tau \psi - \alpha' \partial_\tau \psi^\dagger \partial_\tau \psi - \Delta \psi^\dagger \psi - \frac{\delta}{2} (\psi^{\dagger 2} + \psi^2) \quad (3.15b)$$

where

$$\Delta + \delta = \Delta_\phi, \quad \Delta - \delta = \Delta_\chi \quad (3.16)$$

We now wish to construct the (free) propagators for this Lagrangian. This is most easily done by constructing the generating functional⁶⁾ of the one-particle irreducible Green's functions. Since we have only bilinear terms in Eq. (3.15) we need simply to perform a Gaussian functional integral. Dropping non-essential total derivatives, the Lagrangian density can be written in the form

$$\mathcal{L} = \frac{1}{2} \bar{\mathcal{Z}}^+ A \bar{\mathcal{Z}} \quad (3.17)$$

where

$$\bar{\mathcal{Z}} \equiv \begin{pmatrix} \psi^\dagger \\ \psi \end{pmatrix} \quad (3.18)$$

and the matrix A is

$$A = \begin{pmatrix} -i\partial_\tau + \alpha' \partial_\tau^2 - \Delta & -\delta \\ -\delta & i\partial_\tau + \alpha' \partial_\tau^2 - \Delta \end{pmatrix} \quad (3.19)$$

The propagators $G^{(2,0)} \equiv G_{\psi\psi}$, $G^{(0,2)} \equiv G_{\psi^+\psi^+}$ and $G^{(1,1)} \equiv G_{\psi^+\psi}$ are the appropriate matrix elements of A^{-1} . Inverting A, Fourier transforming τ and $\underline{x}$ to energy E and momentum $\underline{k}$, we get

$$G_{\psi\psi} = G_{\psi^+\psi^+} = \frac{\frac{1}{2} (P_\phi(\underline{k}) - P_\chi(\underline{k}))}{-E^2 + P_\phi(\underline{k}) P_\chi(\underline{k})} \quad (3.20a)$$

$$G_{\psi^+\psi} = \frac{-E - \frac{1}{2} (P_\phi(\underline{k}) + P_\chi(\underline{k}))}{-E^2 + P_\phi(\underline{k}) P_\chi(\underline{k})} \quad (3.20b)$$

where

$$P_{\phi\chi}(\underline{k}) = \alpha' \underline{k}^2 + \Delta_{\phi\chi} \quad (3.21)$$

and the $i\epsilon$ -prescription $E \rightarrow E - i\epsilon$ is understood. The propagators $G_{\phi\phi}$, $G_{\chi\chi}$ and $G_{\phi\chi}$ can be constructed similarly from Eq. (3.15a), or from linear combinations of Eq. (3.19), and are:

$$G_{\phi\phi} = \frac{-\frac{1}{2} P_{\chi}(\underline{k})}{-E^2 + P_{\phi}(\underline{k}) P_{\chi}(\underline{k})} \quad (3.22a)$$

$$G_{\chi\chi} = \frac{-\frac{1}{2} P_{\phi}(\underline{k})}{-E^2 + P_{\phi}(\underline{k}) P_{\chi}(\underline{k})} \quad (3.22b)$$

$$G_{\phi\chi} = \frac{\frac{i}{2} E}{-E^2 + P_{\phi}(\underline{k}) P_{\chi}(\underline{k})} \quad (3.22c)$$

It should be noted that $G_{\chi\phi}$ has the opposite sign for E in the numerator, so that $G_{\phi\chi}$ and $G_{\chi\phi}$ cancel in $G_{\psi\psi}$ and $G_{\psi^+\psi^+}$ but not in $G_{\psi^+\psi}$. [This follows from the single time derivative in Eq. (3.15)]. Notice that there are poles at $E = \pm \sqrt{(\alpha' \underline{k}^2 + \Delta_{\phi})(\alpha' \underline{k}^2 + \Delta_{\chi})}$ in the propagators, the negative energy state restoring causality. Despite the two "energy gaps" Δ_{ϕ} and Δ_{χ} there is really only one intercept

$$\sqrt{\Delta_{\phi}\Delta_{\chi}} = \sqrt{\Delta^2 - S^2}$$

When Δ_{ϕ} or $\Delta_{\chi} \rightarrow 0$ both ϕ and χ fields develop long-range correlations.

To satisfy ourselves that the theory with $\Delta_{\phi} \neq \Delta_{\chi}$ will not have the same infrared behaviour as the usual Reggeon field theory, we should examine the small $E, \underline{k}^2$ behaviour of the propagators (3.20) or (3.22). We consider separately the two possibilities (a) $\Delta_{\phi}, \Delta_{\chi} \neq 0$, $\Delta_{\phi} \neq \Delta_{\chi}$, and (b) $\Delta_{\phi}(\Delta_{\chi}) = 0$ but $\Delta_{\chi}(\Delta_{\phi}) \neq 0$. The third case $\Delta_{\phi} = \Delta_{\chi} = 0$ is the free Lagrangian version of the usual Reggeon field theory with $\alpha_p(0) = 1$.

First consider (a): $\Delta_{\phi}, \Delta_{\chi} \neq 0$. Then neither $G_{\psi^+\psi^+}$ nor $G_{\psi\psi}$ have singularities when $E, \underline{k}^2 \rightarrow 0$, so that they should cause no trouble in the infrared

limit and exhibit no long-range forces or critical behaviour. There are singularities in these propagators at finite E and $\underline{k}^2$, but even if there were a zero mass-gap somewhere in the theory, these singularities would not be expected to influence the leading infrared behaviour.

Now consider possibility (b): suppose $\Delta_\phi = 0$, $\Delta_\chi \neq 0$. The propagator $G_{\psi^+\psi}$ has the correct infrared behaviour in the limit $E = 0$, $\underline{k}^2 \rightarrow 0$, i.e. $G_{\psi^+\psi} \sim 1/\underline{k}^2$. But it does not have the correct behaviour in the limit $\underline{k}^2 = 0$, $E \rightarrow 0$, as then $G_{\psi^+\psi} \sim 1/E^2$, not $1/E$ as in the usual Regge theory with $\alpha_p(0) = 1$. Similar infrared behaviour for $G_{\psi\psi}$ and $G_{\psi^+\psi^+}$ can be read off from Eq. (3.20a): when $E = 0$, $\underline{k}^2 \rightarrow 0$, $G_{\psi\psi}$ and $G_{\psi^+\psi^+} \sim 1/\underline{k}^2$ and when $\underline{k}^2 = 0$, $E \rightarrow 0$ both $G_{\psi\psi}$ and $G_{\psi^+\psi^+} \sim 1/E^2$. Neither result is as it should be, since in the standard continuum field theory there should be no infrared singularities at all in these propagators: indeed they are identically zero.

To circumvent this problem one must use more general scaling relations for $G_{\phi\phi}$, $G_{\chi\chi}$ and $G_{\phi\chi}$ in the theory with $P_\phi \neq P_\chi$. There will be two new critical exponents (β_1, ν_1) corresponding to the new relevant operators $(\psi^{+2} + \psi^2)$ and $(\partial\psi \cdot \partial\psi + \partial\psi^+ \cdot \partial\psi^+)$. It will in general be necessary to determine the combinations of the two point functions (3.22) necessary to disentangle the physically relevant indices (η, ν, β) . This will only be possible in calculations with a lattice analogue model if it can be made to approach the condition corresponding to $\Delta_\phi \approx \Delta_\chi \approx 0$.

3.4 Models and comparisons

We saw in the previous section that it was important to ensure that the spin Hamiltonian could approximate a field theory with $\Delta_\phi = \Delta_\chi$. Also, it was evident from Appendix A that this constraint is non-trivial on a lattice, and from Eq. (3.6) that $\Delta_\phi \neq \Delta_\chi$ with the naïve Hamiltonian $\mathcal{H}$ (3.1) whenever $G_0 \neq 0$. To have $\Delta_\phi \approx \Delta_\chi$ as a possibility it is in general necessary to have another interaction in the spin system which violates the symmetries IR_s , IR_t , IT and IC . The simplest possibility is to allow different nearest neighbour interactions L_0 for the s and t spins, as in Eq. (2.14a). (We observe that the symmetries under $R_s T$, $R_t T$, $R_s C$ and $R_t C$ all require retaining the relative minus sign between the $s_i t_{i+\hat{t}}$ terms in Eq. (3.1).) We therefore obtain

$$\begin{aligned} \mathcal{H} = & i \sum_i E_0 (s_i t_{i+\hat{y}} - t_i s_{i+\hat{y}}) + \sum_{i,\hat{x}} K_0 (s_i s_{i+\hat{x}} + t_i t_{i+\hat{x}}) \\ & + \sum_i L_0 (s_i s_{i+\hat{y}} + t_i t_{i+\hat{y}}) + \sum_i L'_0 (s_i s_{i+\hat{y}} - t_i t_{i+\hat{y}}) \\ & + i \sum_i G_0 s_i \end{aligned} \quad (3.23)$$

In general the Hamiltonian (3.23) will also correspond to $\Delta_\phi \neq \Delta_\chi$, but now at least it is possible to make a judicious choice of variables to pick out the critical indices in a field theory with $\Delta_\phi \approx \Delta_\chi$. In other words, the model (3.23) in general corresponds to a continuum field theory with terms of the form

$$(\psi^2 + \psi^{+2}), \quad (\partial\psi \cdot \partial\psi + \partial\psi^+ \cdot \partial\psi^+), \quad \text{etc}$$

but the lattice model (3.23) now has enough parameters in principle to enable these operators to be eliminated, and the usual Regge eigenvalues to be determined, as well as the new ones related to these operators. The main problem would be to locate among all the fixed points of the model the one corresponding to Reggeon field theory. It is not clear this problem can be solved.

For example, one could try a high temperature expansion¹¹⁾ for Eq. (3.23) calculated for $L_0, L'_0 \neq 0$, and then selecting values which gave series for the lattice analogue of $G_{\psi^+\psi}$ which had different critical behaviour from those corresponding to $G_{\psi\psi}$ and $G_{\psi^+\psi^+}$. According to the analysis of subsection 3.3, these have similar critical indices only if just one of $\Delta_\phi, \Delta_\chi \neq 0$.

The theoretical possibility of making a correct choice of L_0 and L'_0 is described in Appendix B. Introducing $L'_0 \neq 0$ corresponds in the continuum field theory to introducing different values λ and λ' for the weights of the second time derivatives of the ϕ and χ fields. It is easy to see using the generating functional formalism that if $\lambda \neq \lambda'$, but if the intercepts associated with ϕ and χ are both 1 (corresponding to the correct choice of the ratio L'_0/L_0) then the infrared behaviour of the propagators $G_{\psi\psi}$, $G_{\psi^+\psi^+}$ and $G_{\psi\psi^+}$ will be the same as in the standard Reggeon field theory.

We consider Eq. (3.23) to be a plausible candidate for a lattice analogue to the Reggeon field theory. In obtaining it we considered explicitly the case $s_i, t_i = \pm 1$. However, a similar analysis of systems with s_i and t_i confined to the finite range $[-1, 1]$ would also yield Eq. (3.23) as a possible lattice analogue^{8, 11)}. Another possibility in this case would be to replace the linear $iG_0 s_i$ interaction with the more familiar interaction

$$iG_0 (s_i^2 + t_i^2) s_i \tag{3.24}$$

as was considered in Refs. 8 and 11.

It is instructive to consider to what the Hamiltonian (3.23) corresponds in various different special cases which have additional symmetries. Since they are more symmetric they belong to different universality classes and have different values for their critical indices.

A) $G_0 = 0, L_0 \neq 0, L'_0 = 0, E_0 \neq 0, K_0 \neq 0.$

In this case the theory corresponds to a Reggeon field theory with a linear trajectory [corresponding to the E_0 piece in Eq. (3.23)] but only with interactions even in the Pomeron fields ψ and ψ^+ . This theory should therefore have the same infrared behaviour as the Reggeon field theory with fourth-order interactions discussed by Abarbanel and Bronzan⁴⁾.

B) $G_0 \neq 0, L_0, L'_0 \neq 0, E_0 = 0, K_0 \neq 0:$

In this case the s and t spins decouple completely and the new symmetry is T. The t spin sector is an asymmetric Ising model with no magnetic field, whereas the spin sector has an imaginary magnetic field. An analysis similar to the previous section then shows that the s spin sector

$$d\mathcal{L}_s = \sum_{i, \hat{x}} K_0 S_i S_{i+\hat{x}} + \sum_i L_0 S_i S_{i+\hat{y}} + i G_0 S_i \quad (3.25)$$

is a valid analogue to a Reggeon field theory with a "relativistic" propagator $\propto 1/E^2 - \bar{\alpha}' k^2 + i\epsilon$ corresponding to a pair of Schwarz¹⁵⁾ trajectories.

C) $G_0 = 0, L_0, L'_0 \neq 0, E_0 = 0, K_0 \neq 0.$

Both the s and t sectors are now anisotropic Ising models, and by the usual analysis they would correspond to Regge models with Schwarz¹⁵⁾ trajectories and no interactions odd in the Pomeron fields.

We finish this section by observing that on general kinematic grounds^{1, 9)}, independent of the Reggeon calculus, one should expect high energy scattering to be a long-distance (infrared) problem in rapidity y and transverse distance $\underline{x}$. If it is describable as a critical phenomenon involving the exchanges and interactions of many Pomerons, then universality suggests these may be mimicked by a lattice system. Then basic properties of hadronic interactions such as parity and translation invariance in Y and $\underline{x}$ greatly restrict the form of interactions of spin systems. In fact the models described in this section are nearly all the ones allowed for simple systems with one or two discrete spins ± 1 at each lattice site. Thus their relevance may be more general than suggested by their derivation from the very specific and technical framework of the Reggeon calculus, cut discontinuities imposed by t-channel unitarity, and the Reggeon field theory.

4. DISCUSSION

In this paper we have analyzed many of the problems arising in the construction of lattice analogues to the usual Reggeon field theory (2.1). Thus we found in Section 2 that when making the transition from a continuum field theory to a lattice field theory it is important to make a Wick rotation first. This is to ensure that the lattice theory corresponds to a consistent and unambiguous choice of $i\epsilon$ -prescription for the continuum theory. In the same section we also discussed the symmetry properties of the Reggeon field theory on a lattice. Since universality suggests that theories with the same symmetry properties should have the same critical exponents, a characterization of these symmetries is needed for classifying different lattice models.

In Section 3 we discussed the transition from a lattice field theory to a lattice system with two spins at each site, using the standard Gaussian integral transformation. We found that the most natural choice of spin interactions corresponded to a triple-Reggeon interaction different from that in the standard Reggeon field theory, and with different "energy gaps" (bilinear non-derivative terms) for the real and imaginary parts of the complex Reggeon field, ψ . We were able to show that the theory with non-standard triple Reggeon interactions but with equal "energy gaps" should have the same infrared behaviour as the standard model (2.1), since only the conventional fixed point is completely infrared stable. On the other hand, having different "energy gaps" would change the infrared behaviour of the theory, and we have discussed what prescription could be made to obtain results corresponding to their coincidence.

In addition to the insight which lattice models give about the infrared structure of the Reggeon calculus, they enable one to use alternative techniques for the calculation of critical exponents for high energy diffractive processes. This is unfortunately very difficult -- renormalization group calculations¹⁰⁾ on a lattice have great problems coping with the configurations of models with two spins at each site. The large number of different interactions allowed by the symmetry properties of Reggeon field theory gives other problems. A preliminary estimate of Reggeon exponents using a high-temperature expansion has been given¹¹⁾. However, reference to Section 3 shows that the model calculated may not correspond to the model with a unique "energy gap". On the other hand, given the low order and approximate nature of this calculation, it is probable that these estimates still fall in the right ball-park. This conjecture is reinforced by the observation that the optimum values of the coupling constants reported in Ref. 11 are all of the same order of magnitude. The addition of a term with $L_0 \neq L'_0$ is likely to result in a change in the estimates of the exponents of order one. It remains to be seen whether the model (3.23) could be used in a realistic calculation, but some of us (J.E., R.S.) have been trying.

APPENDIX A

GREEN'S FUNCTIONS OF THE LATTICE THEORY

When we write the Reggeon field theory in terms of the real and imaginary parts of $\psi = \phi + i\chi$ (as in Section 2.2) we have a theory for two real fields ϕ and χ . The universality class of the theory is defined by all the symmetries of $\mathcal{L}(\phi, \chi)$. As discussed in Section 2.3, the complete set of symmetries of the full $\mathcal{L}$ is

$$CR_\phi, TR_\chi, P \quad (A.1)$$

together with $SO(D)$ in the transverse plane for the continuum theory. This allows, in addition to the terms in Eq. (2.10), new terms like

$$(\phi_i^2 - \chi_i^2), (\phi_i \phi_{i+1} - \chi_i \chi_{i+1}), \\ i \phi_i (\phi_i^2 - 3\chi_i^2), \text{ etc} \quad (A.2)$$

which are related to the relevant operators

$$(\psi^2 + \psi^{*2}), (\partial\psi \cdot \partial\psi + \partial\psi^* \cdot \partial\psi^*), i(\psi^3 + \psi^{*3}), \text{ etc} \quad (A.3)$$

in the continuum field theory. We now show that these new terms are generated in perturbation theory when a cut-off is introduced.

All Green's functions for the Reggeon field theory may be expressed in the path integral formalism as functional derivatives of

$$Z(J, J^*) = \int \mathcal{D}\psi \mathcal{D}\psi^* \exp \left[- \int dY d^D x (\mathcal{L} + J\psi^* + J^*\psi) \right] \quad (A.4)$$

where

$$\mathcal{L} = \frac{1}{2} \psi^* \overleftrightarrow{\partial_Y} \psi - \alpha'_0 \partial\psi^* \cdot \partial\psi - \Delta_0 \psi^* \psi + \frac{ig_0}{2} \psi^* \psi (\psi^* + \psi)$$

is the Lagrangian of Eq. (2.5) with $Y = -i\tau$ the Wick-rotated "time". We can now Fourier transform to momentum space $(\omega, \underline{k})$

$$\psi(\gamma, \underline{k}) = \frac{1}{(2\pi)^{D+1}} \int_{-\infty}^{\infty} d\omega d^D \underline{k} \psi_{\omega, \underline{k}} \exp(i(\omega \gamma - \underline{k} \cdot \underline{x})) \quad (\text{A.5})$$

where $\omega = iE = i(1-J)$. Note that the inverse transformation

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega \gamma} (\dots) = \frac{1}{S} \int \frac{dJ}{2\pi i} S^{-J} (\dots) \quad (\text{A.6})$$

is the familiar Sommerfeld-Watson transformation of Regge theory, so that the Wick-rotated field theory is the more fundamental analogue field theory from the outset (see Section 2.1).

In $q \equiv (\omega, \underline{k})$ space we have

$$\begin{aligned} Z(J_q, J_q^+) = & \int \mathcal{D}\psi_q \mathcal{D}\psi_q^+ \exp \left\{ - \int \frac{d^{D+1}q}{(2\pi)^{D+1}} \times \left[(E + \alpha' \underline{k}^2 - \Delta_0) \psi_q^+ \psi_{-q} \right. \right. \\ & \left. \left. + J_q \psi_{-q}^+ + J_q^+ \psi_{-q} + \mathcal{L}_{\text{int}}(\psi_q, \psi_q^+) \right] \right\} \end{aligned} \quad (\text{A.7})$$

The effect of going on a lattice in rapidity space is equivalent to introducing a cut-off in ω at ω_N :

$$\psi_j = \psi(\gamma_j, \underline{x}) = \int_{-\omega_N}^{\omega_N} \frac{d\omega}{2\pi} e^{i\omega \gamma_j} \int d^D \underline{k} \psi_q e^{-i\underline{k} \cdot \underline{x}} \quad (\text{A.8})$$

where $\gamma_j = \pi_j / \omega_N$, $j = 0, \pm 1, \pm 2, \dots$. Only specifying the field at discrete lattice sites, we can invert the transformation (A.8), getting

$$\psi(\omega) = \sum_{j=-\infty}^{\infty} e^{-i\omega \gamma_j} \psi_j \quad (\text{A.9})$$

The energy term of the action (suppressing $\underline{k}$ integrals) is

$$\int_{-\omega_N}^{\omega_N} d\omega i\omega \chi_{\omega}^{\dagger} \chi_{-\omega} = i\omega_N^2 \sum_{i \neq j} \Delta_{ij} \phi_i \chi_j \quad (\text{A.10})$$

where

$$\Delta_{ij} = \frac{4}{\pi} \frac{(-1)^{i-j}}{(i-j)}$$

In terms of $\psi_i = \phi_i + i\chi_j$ we have

$$\int_{-\omega_N}^{\omega_N} d\omega i\omega \chi_{\omega}^{\dagger} \chi_{-\omega} \approx \frac{-4i\omega_N^2}{\pi} (\phi_i \chi_{i+\hat{y}} - \chi_{i+\hat{y}} \phi_i) + \dots \quad (\text{A.11})$$

which is the same as Eq. (2.10) for the nearest neighbour part. The non-nearest neighbour couplings do not affect the universality class (i.e. the critical exponents). Similarly a cut-off for $|\underline{k}_i| \leq k_N$ will be equivalent to a lattice theory in the impact parameter space, but this is not germane to our present discussion.

To study the effect of the cut-off in ω , we can do the Feynman perturbation expansion by standard functional techniques. The factor with $\mathcal{L}_{\text{int}}$ is pulled outside the functional integral as

$$\exp \int \frac{d^{D+1}q}{(2\pi)^{D+1}} \mathcal{L}_{\text{int}} \left(\frac{\delta}{\delta J_q}, \frac{\delta}{\delta J_q^+} \right) \quad (\text{A.12})$$

Then we do the Gaussian integrals over ψ_q, ψ_q^+ getting

$$\begin{aligned} \mathbb{Z}(J, J^+) &= \exp \int d^{D+1}q \mathcal{L}_{\text{int}} \left(\frac{\delta}{\delta J_q}, \frac{\delta}{\delta J_q^+} \right) \times \\ &\quad \exp \int \frac{d^{D+1}q}{(2\pi)^{D+1}} \frac{J_q J_{-q}^+}{(E - \alpha' k^2 - \Delta_0)} \end{aligned} \quad (\text{A.13})$$

Expanding the functional derivatives in g for

$$\mathcal{L}_{int}(\psi_q, \psi_q^+) = \frac{ig}{3!} \int d^{D+1}q_1 d^{D+1}q_2 \psi_q^+ \psi_{q_2} (\psi_{q_2-q_1}^+ + \psi_{q_1-q_2}) \quad (A.14)$$

we recover the standard Feynman rules of the Reggeon field theory^{1,2)}. Consider now the effect of the cut-off on the $O(g^2)$ contribution to $G_{\psi\psi}$, shown in Fig. 3. The Feynman integral

$$g^2 \int_{-\omega_N}^{\omega_N} d\omega \int d^D k \left(\frac{i}{i\omega - \alpha' k^2 - \Delta_0} \right) \left(\frac{i}{i(\omega + \omega_1) - \alpha' (k + k_1)^2 - \Delta_0} \right) \quad (A.15)$$

is zero for $\omega_N \rightarrow \infty$ because both singularities are in the lower half of the ω -plane and the contour may be closed through the upper semi-circle at infinity^{*)}. Hence as a consequence $G_{\psi\psi} = 0$ to order g^2 . But for ω_N finite causality is destroyed and we generate all Green's functions $G^{(n,0)} = \langle 0 | \psi_1 \dots \psi_n | 0 \rangle$ and $G^{(0,n)} = \langle 0 | \psi_1^+ \dots \psi_n^+ | 0 \rangle$. Hence the general lattice theory must include new terms corresponding to

$$(\psi^2 + \psi^{+2}), \quad (\psi^3 + \psi^{+3}), \quad \text{etc}$$

No lattice scheme can avoid this difficulty, and to understand fully the transition from lattice to continuum we need to define a more general theory, with all the relevant operators:

$$\left. \begin{aligned} \mathcal{L}_0 &= \frac{1}{2} \psi^+ \overleftrightarrow{\partial}_\gamma \psi - \alpha'_0 \partial_\gamma \psi^+ \partial_\gamma \psi - \frac{\alpha'_1}{2} \left[\partial_\gamma \psi^+ \partial_\gamma \psi^+ + \partial_\gamma \psi \partial_\gamma \psi \right] \\ &\quad - \Delta \psi^+ \psi - \frac{\delta}{2} (\psi^2 + \psi^{+2}) \\ \mathcal{L}_{int} &= \frac{ig}{2} \psi^+ \psi (\psi^+ + \psi) + \frac{ih}{6} (\psi^3 + \psi^{+3}) \end{aligned} \right\} \quad (A.16)$$

A linear term $(\psi + \psi^+)$ need not be included, since $\psi \rightarrow \psi + i\lambda$, $\psi^+ \rightarrow \psi^+ + i\lambda$ ^{**) will cancel $\langle 0 | (\psi + \psi^+) | 0 \rangle$ with only a reshuffling of the terms in $\mathcal{L}$ (see section 3.1 and Appendix B).}

*) In terms of the $i\epsilon$ prescription of the theory before Wick rotation, both singularities are in the lower half of the E plane.

**) Note that the translations $\psi \rightarrow \psi + i\lambda$, $\psi^+ \rightarrow \psi^+ + i\lambda$ preserve the special anti-hermiticity given by the CS_ϕ symmetry, as they correspond to $\phi \rightarrow \phi + i\lambda$, $\chi \rightarrow \chi$.

APPENDIX B

THE NATURE OF THE VACUUM IN THE REGGEON CALCULUS

In traditional physical theories possessing a Hermitean or real interaction potential, the definition of the vacuum is quite straightforward. Classically, it is simply the minimum of the potential, or the lowest value of the potential such that $\partial V/\partial\phi = 0$, and $\partial^2 V/\partial\phi^2 \geq 0$, where ϕ is some real field in the theory. This definition can be cast into a more general form by letting the fields take on complex values. Consider then the generating functional⁶⁾ of some theory:

$$Z = \int \prod_i d\phi_i e^{A(\phi_i)} \quad (B.1)$$

where i labels the species of fields in the theory, and the functional integral, which is over real values of the ϕ_i , can be thought of as running along the real axis in the ϕ_i -plane. In the mean-field approximation the action will just be proportional (assuming no derivative couplings in the interaction potential) to $-V(\phi_i)$, where V is the potential energy.

Now, suppose we approximate Z by evaluating the integral in the region where the integrand is largest. This will just be in the region of the vacuum since $V(\phi_i)$ will be a minimum. Correction terms to this zeroth order approximation can be obtained by considering the second derivatives of V at the minimum. If V and the ϕ_i are complex, this same procedure will work, and is called the method of steepest descent.

Suppose, as in our case, the theory is defined with a complex V , but with the ϕ_i real. Then the original integral in z runs along the real ϕ_i axis. We can now deform the $\delta\phi_i$ contour being careful to avoid singularities of the potential, until we find a saddle point of $\text{Re } V$. The smallest saddle point of $\text{Re } V$ will correspond to the largest value of the modulus of the integrand, and so, in analogy with the case of a purely real potential and real values of ϕ_i , will define the vacuum. The direction of steepest descent of the integrand will be the direction of stability of the potential. So, for instance, if we require in our physical theory that the ϕ_i be real, then we will be led to consider as possible candidates for the vacuum only these saddle points such that at least some of the directions of (preferably steepest) descent are along real values of the ϕ_i . Notice that the values of the ϕ_i at the saddle point (vacuum expectation value) are allowed to be complex: we only require that the potential be stable against real variations of the ϕ_i .

As a concrete example, let us consider the potential derived in Section 3.1 from the lattice theory discussed there. While this potential does describe a theory in the same universality class as the standard Reggeon calculus, it is not completely satisfactory as regards the vacuum. Its diseases can, however, be cured by adding terms which do not affect the infrared behaviour. Since it presents features of sufficient complexity to illustrate our points, and because of its intrinsic physical interest it is a good model to examine.

We want to consider the partition function

$$\begin{aligned} Z &= \int d\phi d\chi \exp \left[-\left[K_1 \phi^2 + K_2 \chi^2 + \ln \chi + \ln \chi (\phi + i\zeta_0) \right] \right] \\ &= \int d\chi \exp \left[-\left[K_2 \chi^2 + \ln \chi \right] \right] \int d\phi \exp \left[-\left[K_1 \phi^2 + \ln \chi (\phi + i\zeta_0) \right] \right] \end{aligned} \quad (B.2)$$

where we have allowed the possibility of different values of K for the ϕ and χ fields. Consider first the integral over $d\phi$. The saddle points of $V(\phi)$ satisfy

$$\frac{\partial V}{\partial \phi} = 2K_1 \phi + \text{th}(\phi + i\zeta_0) = 0 \quad (B.3)$$

First we note that we require that the solutions of (B.3) be purely imaginary. The reason is that in the Reggeon calculus N -point coupling constants must be purely imaginary when N is odd. This will not be guaranteed unless the vacuum expectation value of ϕ is purely imaginary¹⁷⁾. Setting $\phi = i\lambda$, we have

$$2K_1 \lambda + \tan(\lambda + \zeta_0) = 0 = 2K_1(\eta - \zeta_0) + \tan \eta \quad (B.4)$$

with the condition that $-\pi/2 < \lambda + \zeta_0 \equiv \eta < \pi/2$, since the original contour had $\lambda = 0$, and we must avoid singularities of V . The solutions to Eq. (B.4) along with a schematic illustration of the potential are shown in Figs. 1b, 2b and 3b. The fact that when $\eta \rightarrow \pm \pi/2$, $V \rightarrow -\infty$ need not worry us here, since we can always add terms singular at $\eta = \pm \pi/2$ to compensate for this divergence without qualitatively changing the nature of the saddle point problem which we wish to study.

Consider first the situation of Fig. 4a where $|K_1|$ is sufficiently large (i.e. $> |K_{1c}|$) so that there are three saddle points. It is straightforward to analyse the stability properties of these saddle points, which can be deduced from Fig. 4a: they are indicated schematically in Fig. 4b. Doing so, we find that the direction of steepest descent of $e^{-V\phi}$ is along the real ϕ -axis for saddle

point A, and along the imaginary ϕ -axis for points B and C. Hence the lowest (and only the lowest) saddle point of V_ϕ is stable against real variations of ϕ , and so a stable perturbation theory evidently exists about the vacuum, A.

Turn now to Fig. 9 in which $|K_1| < |K_{1c}|$. In this case our particular potential has only one saddle point which is not stable with respect to real variations of ϕ . This is a case reminiscent of a negative (mass)² term in the Lagrangian of a normal field theory, or, in the Reggeon calculus, of a Pomeron intercept above 1. For our potential, there is no stable vacuum for the theory to fall into.

Finally, consider the case $K_1 = K_{1c}$, shown in Fig. 6. In this situation, the two extrema (saddle points) A and C of Fig. 4 have coalesced producing an inflection point. A straightforward analysis shows in this case that one of the directions of steepest descent is along the negative real ϕ -axis, while the potential is unstable for small perturbations along the positive ϕ -axis. This situation should not surprise us, since it has the same structure as a potential of the form $m\phi^2 + g\phi^3$, in the limit $m \rightarrow 0$. Aside from the problems of stability of the saddle point, this analogy encourages us to interpret the value K_{1c} for K_1 as a critical value associated with a massless particle, or a Pomeron with intercept 1.

As we have stressed in Section 3.3, it is essential in order to get the required critical behaviour that both the fields ϕ and χ become "massless" at the same critical temperature. Now the part of the potential in Eq. (B.2) which depends on χ , V_χ , is of the same form as V_ϕ with $G_0 = 0$. In this limit, with the help of Figs. 4, 5 and 6, the following statements are easily obtained:

- i) For $|K_2| > |K_{2c}| = \frac{1}{2}$ there are three saddle points symmetric about the origin. The one at $\tilde{\eta} \equiv -i\chi = 0$ is the lowest, and is stable against real variation of χ with the directions of steepest descent of e^{-V_χ} along both directions of the real χ -axis. This corresponds to a positive (mass)² term in the Lagrangian.
- ii) For $|K_2| < |K_{2c}| = \frac{1}{2}$ there is one saddle point at $\tilde{\eta} = 0$ which is unstable against real variations in χ , with the directions of steepest descent of e^{-V_χ} along the positive and negative imaginary χ -axis. This is like a negative (mass)² term in the Lagrangian.
- iii) When $K_2 = K_{2c} = -\frac{1}{2}$, the three saddle points described under (i) coalesce to produce a saddle point at $\tilde{\eta} = 0$ whose second derivative is zero (the first and third are zero since V_χ is an even function of χ), and which is stable against real variations of χ , with at least some of the direction of steepest descent of e^{-V_χ} along both directions of the real χ -axis. This saddle point has the structure of the potential $m^2\chi^2 + \lambda\chi^4$ in the $m \rightarrow 0$ limit, and so

represents a choice of parameters for which the mass of the χ -field is zero. This discussion also illustrates the justification for allowing $K_1 \neq K_2$. $K_{2c} = -\frac{1}{2}$ while K_{1c} is the solution of the equations

$$\begin{aligned} 2K_{1c}(\gamma - G_0) + \tan \gamma &= 0 \\ 2K_{1c} + 1 + \tan^2 \gamma &= 0 \end{aligned} \tag{B.5}$$

which implies $K_{1c} \neq -\frac{1}{2}$ for $G_0 \neq 0$. Hence K_1 and K_2 should be allowed to take on different values in order to ensure that $\Delta_\phi = \Delta_\chi = 0$ at the same critical temperature.

This discussion of the vacuum and shifts in the fields ϕ and χ justifies the discussion after Eq. (3.6) in Section (3.2).

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Figure captions

- Fig. 1a The $i\epsilon$ -prescription (2.2)
1b The $i\epsilon$ -prescription (2.3)
1c Contour used for the Wick-rotated theory

Fig. 2 Graphs contributing to renormalization (a) of the vertex $ih/6(\psi^3 + \psi^{+3})$;
(b) of the vertex $ig/2(\psi^+\psi)(\psi + \psi^+)$; (c) of the propagator.

Fig. 3 A diagram which is zero in the continuum field theory, but non-zero
in a lattice field theory.

- Fig. 4a Saddle points present when $|K_1| > |K_{1c}|$
4b Schematic sketch of the stability of the saddle points when $|K_1| > |K_{1c}|$.

- Fig. 5a Saddle point present when $|K_1| < |K_{1c}|$
5b Schematic sketch of the instability of the saddle point when
 $|K_1| < |K_{1c}|$.

- Fig. 6a Saddle points present when $|K_1| = |K_{1c}|$
6b Schematic sketch of the stability of the saddle points when $|K_1| = |K_{1c}|$.

(a)

(b)

(c)

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6