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COLOUR AND MAGNETIC MONOPOLES

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A B S T R A C T

If the exact gauge symmetry of nature consists of the $U(1)_{EM}$ generated by the electric charge operator Q and the colour group K , with Q a colour singlet, then, if g is a possible magnetic charge, $\exp(4\pi igQ)$ must equal an element of the colour group. For colour singlet particles this reduces to Dirac's condition $eg = n/2$. In general possible monopoles correspond to points of intersection of the colour and electromagnetic groups. If the colour group is semi-simple and compact there can at most be a finite number p of such points ($p = N$ if $K = SU(N)$). The existence of non-trivial (not equal to unity) solutions to our condition means that there must be fractionally charged (with p the fraction) coloured particles and magnetic monopoles emanating colour magnetic flux as well as electromagnetic flux.

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1. - INTRODUCTION

Many years ago Dirac showed that if g and e are the magnetic and electric charges of any two particles then ¹⁾

$$e g = n/2 \quad n = 0, \pm 1, \pm 2, \dots \quad (1.1)$$

It has recently been learnt that there are interesting situations in which Dirac's argument does not apply and (1.1) can be violated ^{2),3)}. Dirac considered a situation with just a $U(1)$ gauge symmetry, namely electromagnetism, whereas we now believe that the true unbroken gauge symmetry of nature may well be

$$H = U(1) \otimes K \quad (1.2)$$

where we shall talk of K as the "colour gauge group", possibly $SU(3)$, possibly larger with factors not yet known. The direct product symbol in (1.2) is to be understood as referring to the local structure of H , near the identity (i.e., the Lie algebra which determines the Lagrangian) : Q , the electric charge operator and generator of $U(1)_{EM}$, is a colour singlet. Globally H is not necessarily a direct product since, as explicit examples will show, there may be elements, other than the identity, common to both $U(1)$ and K . These points of intersection will correspond to possible monopoles : we shall show that, if g is a possible magnetic charge,

$$\exp(4\pi i g Q) = k \quad (1.3)$$

where k is an element of the colour group K . If we consider only particles which are colour (K) singlets $k|s\rangle = |s\rangle$ and so

$$e^{4\pi i g Q_s} = 1 \quad (1.4)$$

where Q_s is the charge operator restricted to the singlet subspace. This is equivalent to (1.1). In particular it relates the minimum possible magnetic charge g_0 to the minimum possible colour singlet charge e_0 : $e_0 g_0 = \frac{1}{2}$. Thus our condition (1.3) is the generalization of (1.1) which applies also to quarks and other coloured states.

What (1.3) does imply about the element k of K is that k must commute with all the elements of K (since $e^{4\pi i g Q}$ does) and so lie in what is called the centre of K . Physically it is reasonable to assume that K is compact and semi-simple [or else we might see a second photon associated with a $U(1)$]. Then the centre of K is a finite group. For example if $K = SU(N)$ we must have

$$k = \exp(2\pi i n/N) \cdot I_N \quad n = 0, 1, 2, \dots, (N-1)$$

where I_N is the $N \times N$ unit matrix. Then (1.3) implies that $\exp(4\pi i N g Q) = 1$, i.e.,

$$e g = n/2N$$

Thus a fractionally charged quark (with charge e_0/N) may be compatible with the previously mentioned monopole $g_0 = \frac{1}{2}e_0$, just because it is coloured (and so may be confined).

In Section 3 we shall discuss this $SU(N)$ example at greater length showing in detail how our condition (1.3) can be satisfied non trivially. In Section 4 we shall discuss what can be said if K is compact and semi-simple. It will turn out that the possible structure is very like the $SU(N)$ example. The possible monopoles must correspond to some cyclic subgroup of the centre of the colour group. If p is the number of elements in this subgroup, i.e., the number of points at which the electromagnetic and colour groups intersect, then in general, (1.1) is replaced by

$$e g = n/2p \quad n = 0, \pm 1, \pm 2, \dots$$

We shall show that if there exist an electric charge e and a magnetic charge g not satisfying Dirac's condition (1.1) for any integer n , then e must be coloured and fractional (with p the fraction) and g must emanate magnetic colour flux as well as electromagnetic flux.

The framework of our proof consists of considering a larger gauge group G (assumed to be semi-simple and compact) which is spontaneously broken to leave the $U(1) \times K$ as the residual exact gauge symmetry. Thus the magnetic charge is seen as a topological quantum number and there

is no need for Dirac strings 4)-6). Our result is proved, in the next section, by considering path dependent phase factors 7), which seem to be the most fundamental quantities for describing a gauge theory and can be thought of as a sort of exponentiated non-Abelian flux if the path is closed. The phase factor associated with a closed loop is considered as this loop is dropped over a sphere S_2 (large enough for the Higgs' field to be in its ground state) surrounding the potential monopole. Since the loop is infinitesimally small at the north and south poles and the fields non-singular, the phase factor describes a closed path in group space [actually $U(1) \times K$] starting and finishing at the identity. By looking at a special gauge the phase factor is evaluated in terms of the electromagnetic flux through S_2 to the north of the loop when it is at an intermediate stage. The result (1.3) follows in the limit as the loop shrinks to the south pole.

2. - PROOF OF THE GENERALIZED QUANTIZATION CONDITION

We consider the Yang-Mills gauge theory of a compact semi-simple group G , and suppose that the little group of the Higgs' field ψ when in its ground state is $U(1)_{EM} \times K$. Because of the assumed structure of G we can choose its generators $\underline{T}$ such that the structure constants are totally antisymmetric and the metric is the unit matrix. The electric charge which generates the $U(1)_{EM}$ is then

$$Q(x) = e \underline{\phi}(x) \cdot \underline{T} \quad (2.1)$$

where $\underline{\phi}$ is a unit vector (in the self-adjoint representation) constructed out of ψ such that $\mathcal{D}^\mu \underline{\phi}$ vanishes if $\mathcal{D}^\mu \psi$ does. ($\mathcal{D}^\mu$ is the relevant covariant derivative.) The colour generators are

$$K^\alpha(x) = \underline{K}^\alpha \cdot \underline{T} \quad \alpha=1,2,\dots \dim K. \quad (2.2)$$

Without loss of generality we can assume (see the Appendix) that

$$\underline{\phi} \cdot \underline{K}^\alpha = 0 \quad (2.3)$$

Now consider a sphere S_2 surrounding a potential monopole sufficiently large that on it the Higgs' field is in its ground state. This means that if D is the representation matrix relevant for ψ then

$$D(\underline{x} \cdot \underline{T}) \psi = 0$$

implies that

$$\underline{\chi} = A \underline{\phi} + B_{\alpha} \underline{K}^{\alpha} \quad (2.4)$$

for some A and B_{α} . Further the covariant derivative of ψ vanishes :

$$\mathcal{D}^{\mu} \psi \equiv \partial^{\mu} \psi - ie D(\underline{W}^{\mu} \underline{I}) \psi = 0 \quad (2.5)$$

where $\underline{W}^{\mu}$ is the gauge potential.

Let $\underline{W}_0$ be a particular solution to this equation constructed out of ψ and $\partial^{\mu} \psi$. Then it follows from (2.4) that the general solution is

$$\underline{W}^{\mu} = \underline{W}_0^{\mu} + A^{\mu} \underline{\phi} + B_{\alpha}^{\mu} \underline{K}^{\alpha} \quad (2.6)$$

Further the gauge field tensor has the form

$$\underline{G}^{\mu\nu} = \underline{G}_0^{\mu\nu} + (\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}) \underline{\phi} + B_{\alpha}^{\mu\nu} \underline{K}^{\alpha} \quad (2.7)$$

where $\underline{G}_0^{\mu\nu}$ is constructed out of $\underline{W}_0^{\mu}$ and the structure of $B_{\alpha}^{\mu\nu}$ need not concern us. Equation (2.7) is proved in the Appendix.

The magnetic charge g contained inside the sphere S_2 is given by the magnetic flux out of S_2 of the $\underline{\phi}$ component of $\underline{G}^{\mu\nu}$

$$g = \frac{1}{4\pi} \int_{S_2} d\sigma_{\mu\nu}^* \underline{\phi} \cdot \underline{G}^{\mu\nu} \quad (2.8)$$

where $d\sigma_{\mu\nu}^*$ is the dual of the surface element.

By (2.3), (2.7) and Stokes theorem this is

$$g = \frac{1}{4\pi} \int_{S_2} d\sigma_{\mu\nu}^* \underline{\phi} \cdot \underline{G}_0^{\mu\nu} \quad (2.9)$$

Thus g is determined entirely by the Higg's field on S_2 since $\underline{\phi}$, $\underline{W}_0^{\mu}$ and hence $\underline{G}_0^{\mu\nu}$ are. In fact it is a function of the homotopy class of ψ , $\pi_2(G/H)$ (6).

Now we come to prove our quantization condition (1.3). Consider the path dependent phase factor ⁷⁾

$$U(r, \Gamma) = T \exp \left(i e \int_{\Gamma} \underline{W}^r \cdot \underline{I} dx_{\mu} \right) \quad (2.10)$$

denoting some sort of exponentiated non-Abelian flux through the closed curve Γ lying on the sphere S_2 and starting and ending at the point r . The total flux is studied by letting Γ sweep over the sphere S_2 from the north pole to the south pole. Since Γ shrinks to a point at these poles and $\underline{W}^{\mu}$ is continuous and differentiable on the sphere (there are no Dirac strings) the factor (2.10) traces a closed path in G [actually $\bar{H}$] starting and finishing at the identity.

We shall evaluate (2.10) at an intermediate stage when Γ is a circle of longitude and show that

$$U(r, \Gamma) = \exp \left(i Q(r) \int_{\hat{S}_2} \underline{\phi} \cdot \underline{G}^{\mu\nu} d\sigma_{\mu\nu}^* \right) \cdot k^{-1}(r) \quad (2.11)$$

where $k(r)$ is an element of K the colour group at r and $\hat{S}_2$ is the surface of S_2 to the north of Γ . Then as Γ tends to the south pole $U(r, \Gamma) \rightarrow 1$, $\int_{\hat{S}_2} \underline{\phi} \cdot \underline{G}^{\mu\nu} d\sigma_{\mu\nu}^* \rightarrow 4\pi g$ by (2.8) and we have our result (1.3).

First choose a gauge in which $\psi(r)$ is constant in $\hat{S}_2$ by considering

$$\psi(z) \rightarrow \psi'(z) = g(z) \psi(z) \equiv T \left(\exp \left(i e \int_z^N dx_{\mu} \underline{W}^r \cdot \underline{I} \right) \right) \psi(z)$$

z is any point on S_2 to the north of Γ and the contour of integration runs along the meridian line from z to the north pole N (with "T" ordering point nearer N to the left). If n^{μ} is the tangent to this meridian line

$$n^{\mu} \partial_{\mu} \psi'(z) = T \exp \left(i e \int_z^N dx_{\mu} \underline{W}^r \cdot \underline{I} \right) n^{\mu} \partial_{\mu} \psi(z)$$

by (2.5). Integrating along the meridian line

$$\psi'(z) = \psi'(N) = \psi(N)$$

so that ψ is indeed now constant (and similarly for ϕ). In this gauge the $\underline{W}_0^\mu$ and $\underline{G}_0^{\mu\nu}$ in (2.6) and (2.7) vanish and the gauge transformed (2.10) factorizes :

$$U'(r, \Gamma) = \exp(i e \phi' \cdot \int_{\Gamma} A'^{\mu} dx_{\mu}) \cdot T \exp(i e \int_r^r B_a^{\mu} K'^{\mu} dx_{\mu})$$

But in this gauge

$$\begin{aligned} \int_{\hat{S}_2} A'^{\mu} dx_{\mu} &= \int_{\hat{S}_2} d\sigma_{\mu\nu}^* (\partial^{\mu} A'^{\nu} - \partial^{\nu} A'^{\mu}) \\ &= \int_{\hat{S}_2} d\sigma_{\mu\nu}^* \phi' \cdot \underline{G}^{\mu\nu} \\ &= \int_{\hat{S}_2} d\sigma_{\mu\nu}^* \phi \cdot \underline{G}^{\mu\nu} \end{aligned}$$

since $\phi \cdot \underline{G}^{\mu\nu}$ is gauge invariant. Hence

$$U'(r, \Gamma) = \exp(i e Q'(r) \int_{\hat{S}_2} d\sigma_{\mu\nu}^* \phi \cdot \underline{G}^{\mu\nu}) \cdot K'^{-1}(r)$$

But since $U'(r, \Gamma) = g(z) U(r, \Gamma) g^{-1}(z)$ we just need to remove the dashes to return to the original gauge thereby obtaining (2.11) and hence the quantization condition (1.3).

Notice that the gauge transformation $g(z)$ is discontinuous at the south pole (in general) and would induce a Dirac string there. This is irrelevant to our argument since we always applied the singular gauge to points to the north of Γ and hence excluded the south pole.

Our proof has been rather general in that it was independent of any equations of motion. We did not even have to specify what fields were present besides the gauge and Higgs's fields. Nor did it depend much on what the group G was nor on the nature of the Higgs' field as long as $U(1) \times K$ was the resultant unbroken gauge symmetry. In fact we suspect that our result (1.3), like Dirac's condition (1.1), is actually independent of this substructure, which is, however, presumably necessary for having concrete models of monopoles.

The element k in (1.3) represents the coloured magnetic flux of the monopole and is related to the electromagnetic flux by our condition. Further it lies in the centre of K since all elements of

K commute with $\exp(4\pi i g Q)$. If we consider two monopoles we can surround them both with a large sphere with a diametral plane separating them into two hemispheres. Our argument can be repeated for each monopole and its surrounding hemisphere or for the sphere surrounding both. We find for the combination

$$\exp(4\pi i g_{12} Q) = k_{12}$$

where

$$g_{12} = g_1 + g_2 \quad \text{and} \quad k_{12} = k_1 k_2$$

k_1 and k_2 commute since both lie in the centre. Thus the combination of monopoles is a commutative operation which preserves our condition (1.3). Thus possible monopole structures form an Abelian group.

3. - NON-TRIVIAL SOLUTIONS FOR $K_{\text{colour}} = \text{SU}(N)$

We shall now exhibit non-trivial solutions to our condition

$$\exp(4\pi i g Q) = k \tag{3.1}$$

when the colour group is $\text{SU}(N)$. Since the left and right-hand sides of (3.1) are elements of two different groups we are looking for points of intersection of two groups whose generators commute. The unit element is an obvious solution but there are other non-trivial ones in general.

The most obvious mechanism for obtaining the structure $U(1) \times K$ as a little group is to take the Higgs' field to be in the self-adjoint representation of G . Then in our previous notation

$$\underline{\phi} = \underline{\psi} / \sqrt{\psi^2}$$

If we consider $G = \text{SU}(N+1)$ and suppose that in the fundamental representation

$$Q = \frac{e}{2} \underline{\lambda} \cdot \underline{\phi} = e_0 \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ & & & -N \end{pmatrix} / N \tag{3.2}$$

when diagonalized and when ϕ is in its ground state. Then the exact gauge symmetry which is the little group of ϕ , consists of a $U(1)$ generated by Q times an $SU(N)$ made up of matrices

$$\begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \quad U \in SU(N)$$

We see from (3.2) that the fundamental representation - the $(N+1)$ of the $SU(N+1)$ - consists of N coloured particles of charge e_0/N and a singlet of charge $-e_0$ [the charges must sum to zero as Q must be traceless, being an $SU(N+1)$ generator]. This is like putting quarks and an electron in the same multiplet or making lepton number the $N+1$ 'th colour, i.e., something like Pati-Salam theory⁸⁾. If g is a possible monopole charge, we saw earlier that, since e_0 is a colour singlet, (3.1) implies that e_0 and g satisfy Dirac's condition (1.1), i.e., $e_0 g = n/2$ for some integer n . Then

$$\exp(4\pi i g Q) = \exp\left(\frac{2\pi i n}{N} \begin{pmatrix} 1 & 0 \\ 0 & I_N \end{pmatrix}\right) = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$$

where $U = I_N \exp(2\pi i n/N)$ and I_N is the $N \times N$ unit matrix. Hence the right-hand side does indeed lie in $K = SU(N)$ and in fact in the centre, and our condition is satisfied in a non-trivial way. n can be any integer between 1 and N , and so there can be distinct monopoles corresponding to each element of the centre of K .

The cases $N=1, 2$ and 3 correspond to Refs. 4), 2) and 3) respectively. The two solutions discussed in Ref. 2) for the case $N=2$ corresponded to the two elements of the centre of $SU(2)$, the monopole to I_2 and the "semi-pole" to $-I_2$.

In the case $N=3$ discussed by 't Hooft³⁾ the Higgs' field was not in the self-adjoint representation. Let us explain the general mechanism of that example. Let $G = SU(N) \times SU(M)$ and let the Higgs' field lie in the (N, M) representation. Then the little group is automatically $U(1) \times SU(N-1) \times SU(M-1)$ for the following reasons. Let $\lambda_i^{(N)}/2$ be $SU(N)$ generators so

$$\text{Tr}(\lambda^{i(N)} \lambda^{j(N)}) = 2 \delta^{ij} \quad (3.3)$$

$$\sum_{i=1}^{N^2-1} \lambda_{\alpha\beta}^{i(N)} \lambda_{\gamma\delta}^{j(N)} + \frac{2}{N} \delta_{\alpha\beta} \delta_{\gamma\delta} = 2 \delta_{\alpha\delta} \delta_{\gamma\beta} \quad (3.4)$$

(3.4) follows from (3.2) and the fact that any $N \times N$ matrix can be expressed as a linear combination of 1 and $\lambda^{i(N)}$. Hence any N spinor χ is an eigenfunction of

$$\sum_{i=1}^{N^2-1} \chi^\dagger \lambda^{i(N)} \chi \lambda^{i(N)}$$

with eigenvalue $2(N-1)\chi^\dagger \chi / N$.

Hence if $\underline{T} = (\lambda^{i(N)}/2, \lambda^{j(M)}/2)$ are the generators of $SU(N) \times SU(M)$ and we can define a vector $\underline{\phi}$ in terms of the (N,M) spinor $\underline{\psi}$

$$\underline{\phi} = \left(\frac{N}{N-1} \psi^\dagger \lambda^{i(N)} \psi, \frac{M}{M-1} \psi^\dagger \lambda^{j(M)} \psi \right)$$

then $\underline{Q} \cdot \underline{T} \psi = 0$. Hence, $\underline{Q} \cdot \underline{T}$ is a generator of the little group of $\underline{\psi}$ (and is proportional to Q the electric charge). Also the little group lies within the little group of $\underline{Q}$ which is automatically $U(1)_{EM}$ times something.

If $N=4, M=2$, as 't Hooft chose, $K = SU(3)$. Here G is not simple (i.e., has two coupling constants) but it is still semi-simple and hence our analysis valid.

A possible advantage of this second type of Higgs' mechanism over the first one is that it can more easily be achieved with renormalizable self-interactions.

4. - NON-TRIVIAL SOLUTIONS FOR ANY COMPACT SEMI-SIMPLE COLOUR GROUP

It could be that K has other factors beyond $SU(3)$ that we do not know about, but whatever it is it seems very likely that it will be semi-simple and compact (but not necessarily simple) - semi-simple otherwise we should see a second photon - compact to ensure positive energies.

It is interesting that these assumptions are sufficient to say quite a lot about the solutions to (3.1) ; the possible points of intersection of $U(1)$ and K are finite in number and must lie in some cyclic subgroup of the centre of the colour group.

The element k satisfying (3.1) must commute with all elements of the colour group since $\exp(4\pi i g Q)$ does (Q commutes with all the colour generators). Hence it lies in what is called the centre of K . For a compact semi-simple group this is a finite Abelian group⁹⁾. In general only some of these elements can satisfy (3.1), namely those belonging to some cyclic subgroup. Cyclic means that each element can be expressed as some power of a single element. For example, in the $K = SU(N)$ example each solution was a power of

$$\begin{pmatrix} e^{\frac{2\pi i}{N}} I_N & 0 \\ 0 & 1 \end{pmatrix}$$

Thus for general K the possible monopole structure is rather like the $SU(N)$ case.

The proof follows : by Dirac's condition (1.1) - a consequence of (3.1) - the possible values of magnetic charge g are all multiples of some unit g_0 , say, i.e.,

$$g = n g_0 \quad (4.1)$$

If g is permitted it must satisfy (3.1). This and (4.1) imply

$$k = \left[\exp(4\pi i g_0 Q) \right]^n$$

and so is an element of the finite cyclic group generated by $\exp(4\pi i g_0 Q)$. But the possible solutions to (3.1) form a group which must therefore be a subgroup of this cyclic group. By a mathematical theorem [p. 38 of Ref. 10] any subgroup of a cyclic group is itself cyclic and our result proved. Let p be the number of elements of this subgroup, i.e., the number of intersections of $U(1)$ with K . Then $k^p = 1$ for any k satisfying (3.1). Hence

$$e g = n / 2 p \quad n = 0, \pm 1, \pm 2, \dots \quad (4.2)$$

This is the generalization of the Dirac condition (1.1) which applies to coloured particles also.

Now let us illustrate some correlations between the possible electric charge of a particle and its colour transformation properties that would follow from the existence of monopoles. Since Q and the colour generators commute all particles in an irreducible colour multiplet have the same charge. If a particle state $|s\rangle$ lies in the self-adjoint representation or any representation found by reducing tensor products of the self-adjoint representation (this includes the colour singlet), then

$$K |s\rangle = |s\rangle$$

if k is in the centre. Hence by (3.1)

$$\exp(4\pi i g Q) |s\rangle = |s\rangle$$

and the charge e must satisfy Dirac's condition (1.1), i.e., fractional charges are excluded.

We now make some general deductions from our condition (3.1). If there exist an electric charge e and a magnetic charge g violating Dirac's condition, i.e.,

$$e g \neq n/2 \quad \text{for any integer } n \quad (4.3)$$

then

- a) e must be coloured (and hence may be confined) since if it is not Dirac's condition (1.1) is valid and contradicts (4.3);
- b) e must be fractional, i.e., a multiple of $1/(2pg)$ but not $1/(2g)$ by (4.2) and (4.3);

- c) g must emanate a coloured magnetic flux [since in (3.1) $k \neq 1$] ;
- d) the $U(1)_{EM}$ and K_{colour} groups must intersect at points distinct from 1 [to get (4.2) rather than (1.1)] ;
- e) magnetic charges which are multiples of pg must emanate no coloured magnetic flux [since $\exp(4\pi pgQ) = 1$].

APPENDIX

If $\underline{A}$ and $\underline{B}$ are two vectors - i.e., transform by the self-adjoint representation - and f_{ijk} the structure constants of G , we define the cross-product $(\underline{A} \wedge \underline{B})_i = f_{ijk} A_j B_k$. Then

$$[\underline{A} \cdot \underline{I}, \underline{B} \cdot \underline{I}] = i(\underline{A} \wedge \underline{B}) \cdot \underline{I} \quad (\text{A.1})$$

Suppose $g_{\alpha\beta\gamma}$ are the structure constants of the colour group. Then by (A.1)

$$\underline{K}_\alpha \wedge \underline{K}_\beta = g_{\alpha\beta\gamma} \underline{K}_\gamma \quad (\text{A.2})$$

$$\underline{K}_\alpha \wedge \underline{\phi} = 0 \quad (\text{A.3})$$

If (2.3) is not valid we can satisfy it by redefining

$$\underline{K}'_\alpha = \underline{K}_\alpha - \underline{\phi} \underline{K}_\alpha \cdot \underline{\phi}$$

which preserves the algebraic structure (A.2) and (A.3). To check (A.2) we note

$$\underline{K}'_\alpha \wedge \underline{K}'_\beta = g_{\alpha\beta\gamma} \underline{K}_\gamma$$

but by (A.3)

$$g_{\alpha\beta\gamma} \underline{K}_\gamma \cdot \underline{\phi} = (\underline{K}'_\alpha \wedge \underline{K}'_\beta) \cdot \underline{\phi} = \underline{K}'_\alpha \cdot (\underline{K}'_\beta \wedge \underline{\phi}) = 0$$

Hence

$$\underline{K}'_\alpha \wedge \underline{K}'_\beta = g_{\alpha\beta\gamma} \underline{K}'_\gamma$$

So without loss of generality (2.3) can be taken to be satisfied. Now we turn to proving (2.7). Since

$$[\mathcal{D}^\mu, \mathcal{D}^\nu] \psi = ie \mathcal{D}(\zeta^{\mu\nu} \cdot \underline{I}) \psi$$

(2.5) and (2.4) imply that $\underline{G}^{\mu\nu}$ is a linear combination of $\underline{\phi}$ and the $\underline{K}_\alpha$. This is also true of $\underline{G}_0^{\mu\nu}$ and hence of their difference :

$$\underline{G}^{\mu\nu} = \underline{G}_0^{\mu\nu} + A^{\mu\nu} \underline{\phi} + B_\alpha^{\mu\nu} \underline{K}_\alpha \quad (\text{A.4})$$

It remains to show that $A^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ where A^μ is defined by (2.6). We find explicitly from (2.6), using (2.5), that

$$\underline{G}^{\mu\nu} = \underline{G}_0^{\mu\nu} + (\partial^\mu A^\nu - \partial^\nu A^\mu) \underline{\phi} + (\mathfrak{D}_0^\mu \underline{B}^\nu - \mathfrak{D}_0^\nu \underline{B}^\mu - \underline{B}_\mu \wedge \underline{B}_\nu) \quad (\text{A.5})$$

where $\underline{B}_\mu = B_{\alpha-\alpha}^\mu K_\alpha$. Taking the scalar product of (A.4) and (A.5) with $\underline{\phi}$ will give the desired result. For

$$\underline{\phi} \cdot \mathfrak{D}_0^\mu \underline{B} = \partial^\mu (\underline{\phi} \cdot \underline{B}) - (\mathfrak{D}_0^\mu \underline{\phi}) \cdot \underline{B}$$

which vanishes by (2.3) and (2.5). Also, by (A.3)

$$\underline{\phi} \cdot (\underline{B}_\mu \wedge \underline{B}_\nu) = (\underline{\phi} \wedge \underline{B}_\mu) \cdot \underline{B}_\nu = 0$$

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