ASYMPTOTIC BEHAVIOUR IN QUANTUM FIELD THEORY

R.J. Crewther

CERN - Geneva

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INTRODUCTION

The organizers have asked me to talk about the renormalization group and its applications. I shall concentrate on explaining the rules for short-distance behaviour in field theory and current algebra. Nothing will be said about applications to form factors, on-shell scattering amplitudes, charmonium, etc.¹⁾.

The renormalization group originated in papers by Stueckelberg and Petermann²⁾ (SP) and Gell-Mann and Low³⁾ (GML) in 1953/54 and was revived, after a long quiescent period, by Wilson⁴⁾ (operator-product expansions and broken scale invariance) and Callan and Symanzik⁵⁾ (CS) (1968/70). Many people prefer the CS approach because there is no need to introduce unfamiliar renormalization prescriptions, but I think that it is unwise to completely ignore the original ideas. For example, a comparison of the GML and CS methods led to the use of mass-independent renormalization prescriptions [such as⁶⁾ dimensional renormalization] which generate "improved" CS equations⁷⁾. Consequently, all of these methods will be reviewed. Applications will be restricted to the results of assuming asymptotic freedom⁸⁾ or broken scale invariance. I shall also report progress on an awkward technical problem, the properties of operator-product expansions for non-Abelian gauge theories.

1. RENORMALIZATION GROUP: CALLAN-SYMANZIK METHOD

Any renormalized amplitude A possesses a corresponding Callan-Symanzik (CS) equation⁵⁾. The main ingredient in the derivation is knowing the mode of renormalization of A : is it multiplicative, subtractive, or something more complicated? So we begin with a mini-review of the renormalization procedure itself⁹⁾.

1.1 Renormalization

Let Λ represent a cut-off mass which regulates ultraviolet divergences in Feynman diagrams. Typically, these divergences appear as powers and logarithms of Λ which blow up as Λ tends to infinity. Divergences are caused by loop integrations, so they can be isolated as divergences in one-particle-irreducible (1PI) (or "proper") subdiagrams^{*}). Therefore, we consider the Feynman integral for an ℓ -loop 1PI amplitude with L external legs:

$$\Gamma(q_1, \dots, q_{L-1}) = \int^\Lambda d^4 p_1 \dots d^4 p_\ell I(p_1 \dots p_\ell; q_1 \dots q_{L-1}) \quad (1.1)$$

The integrand I is a linear combination of products of vertices and internal propagators which depend on loop momenta $p_1, \dots, p_\ell$ and external momenta $q_1, \dots, q_{L-1}$.

^{*}) A 1PI diagram is a connected graph which cannot be separated into two disconnected pieces by cutting one of its internal lines. The corresponding 1PI amplitude is understood not to include propagators for external lines.

Convergence is tested by applying power-counting arguments^{10,11}). Imagine that a subset $S = \{p'_1, \dots, p'_m\}$ of the loop momenta $\{p_1, \dots, p_\ell\}$ is scaled to infinity [all $p'_j = O(\eta)$ with $\eta \rightarrow \infty$] with all other momenta held fixed^{*}). The integrand I develops a power-law behaviour in η for which the formula

$$|I(\eta)| \lesssim \eta^{c(S)} \cdot \{\text{CONSTANT}\}$$

provides an asymptotic bound. Here $c(S)$ is the sum of individual powers contributed by each vertex and propagator depending on $p'_1, \dots, p'_m$; (for example, there is a power -2 for each boson propagator involved). Thus each subset S can contribute cut-off dependence

$$\left| \int^\Lambda d^4 p'_1 \dots d^4 p'_m I \right| \lesssim \Lambda^{d(S)} \{\text{logs of } \Lambda\}$$

to Γ , where the power

$$d(S) = 4 m(S) + c(S) \quad (1.2)$$

is called the superficial degree of divergence of the subintegration over S . The integral (1.1) converges if $d(S)$ is negative for all S .

Consider the special case in which the only divergence in Γ is that caused by all of the loop momenta $p_1, \dots, p_\ell$ growing large together; in other words, Γ is superficially divergent [$d(\Gamma) \geq 0$] but internally convergent [e.g., $d(S) < 0$ for all $S \neq \{p_1, \dots, p_\ell\}$]. Suppose that Γ is differentiated with respect to one of the external momenta q_i . When $\partial/\partial q_i$ acts on the integrand I , the characteristic exponent $c(\Gamma)$ for its asymptotic behaviour as $p_1, \dots, p_\ell$ become large is reduced by at least 1, while no increase occurs in the other $c(S)$. Therefore, if $\partial/\partial q_i$ is applied a sufficient number of times to Γ , the result is completely convergent. Since Γ_{div} (the divergent part of Γ) is annihilated by these differentiations, it must be a polynomial of degree $\leq d(\Gamma)$ in $q_1, \dots, q_{L-1}$. In diagrammatic language (Fig. 1), this means that Γ_{div} is a local vertex produced by shrinking the blob Γ to a point.

An ℓ -loop divergence of this type is easily cancelled by including a counterterm $\Delta \mathcal{L}_\ell$ in the Lagrangian $\mathcal{L}$ with bare vertex equal to $-\Gamma_{\text{div}}$. However, many 1PI graphs do not possess this property of being "primitively divergent"^{**}). An arbitrary ℓ -loop 1PI graph may contain divergent ℓ' -loop 1PI subgraphs ($\ell' < \ell$) which must first be made convergent by including suitable counterterms $\Delta \mathcal{L}_{\ell'}$ in $\mathcal{L}$; i.e., it will be necessary to include internal counterterm vertices in the blob in Fig. 1. So we try the following prescription^{10,12}):

*) More precisely, write $p'_j = \eta r_j$ and let η tend to ∞ , keeping all r_j, p_i ($\neq$ any p'_j), and q_i fixed and not allowing any partial sum $\sum' r$ of the r_j to vanish. The latter condition ensures that a p' -dependent propagator (which carries momentum $\eta \sum' r + \sum_{p \neq p'} p + \sum' q$) has the expected asymptotic dependence on η . The special directions $\sum' r = 0$ are covered by looking at appropriate subsets S' of S .

**) A primitively divergent graph becomes convergent if any internal line is cut.

- i) Start with $\mathcal{L}_0$, a Lagrangian from which propagators and vertices can be constructed.
- ii) Construct counterterms $\Delta\mathcal{L}_1$ which remove all divergences in 1-loop LPI graphs generated by $\mathcal{L}_0$.
- iii) Use a new Lagrangian $\mathcal{L}_1 = \mathcal{L}_0 + \Delta\mathcal{L}_1$ to generate 2-loop graphs and construct $\Delta\mathcal{L}_2$ to get rid of the resulting LPI divergences, and so on, to any finite number of loops and vertices¹³⁾:

$$\mathcal{L}_\ell = \mathcal{L}_{\ell-1} + \Delta\mathcal{L}_\ell$$

Obviously, we obtain finite results for diagrams in which, for each pair of divergent LPI subgraphs, one subgraph is entirely contained within the other ("nested" divergences) or the two subgraphs are disjoint. The difficult step in the proof¹²⁾ of convergence is to disentangle overlapping divergences (sets of divergent LPI graphs which are neither disjoint nor nested). The result, valid for all polynomial interactions (renormalizable or otherwise), is that the above procedure renders all subintegrations convergent by power-counting^{*}). In other words, the Lagrangian

$$\mathcal{L} = \mathcal{L}_\infty = \mathcal{L}_0 + \Delta\mathcal{L} \quad (1.3)$$

with counterterm Lagrangian $\Delta\mathcal{L}$ given by

$$\Delta\mathcal{L} = \Delta\mathcal{L}_1 + \Delta\mathcal{L}_2 + \dots + \Delta\mathcal{L}_\ell + \dots$$

generates cut-off independent perturbative amplitudes.

The form of $\Delta\mathcal{L}$ is restricted by the fact that the degree of the polynomial $\Gamma_{\text{div}}(q_1, \dots, q_{L-1})$ in Fig. 1 cannot be greater than $d(\Gamma)$. The set of allowed vertices is easily determined by counting mass dimensionalities. Each term in the Lagrangian has dimensionality 4, in four space-time dimensions. Boson and fermion fields ϕ and ψ have dimensionalities 1 and 3/2 corresponding to $\overline{\phi\phi} \sim p^{-2}$ and $\overline{\psi\psi} \sim p^{-1}$ for the propagators at large momentum p . [Higher-spin fields obey this rule if special circumstances, such as gauge invariance, ensure that longitudinal terms like $i q_\mu q_\nu / q^2 m^2$ in the spin-1 propagator $\overline{B}_\mu B_\nu$ (m = mass) are absent ($m = 0$) or cancelled off.] These rules fix the effective dimensionality $\dim(g)$ of a coupling constant g ; for example, for the vertex $g\phi^6$, $\dim(\phi^6) = 6$ implies $\dim(g) = -2$. A typical term in $\Delta\mathcal{L}$ is

$$\Delta\mathcal{L} \propto \Lambda^j (\ln \Lambda)^p m^r g^N O_d(\phi, \psi, \partial) \quad (1.4)$$

*) For the general case, the involved combinations of Hepp's proof¹²⁾ seem to be unavoidable. Authors of textbooks often choose to discuss a specific theory such as quantum electrodynamics, where skeleton expansions and Ward identities permit some simplification.

for integer powers $j, p, r \geq 0$ and $N > 0$, where the dimensionality d of the operator O_d is given by

$$4 = j + r + N \dim(g) + d$$

i.e.,

$$d \leq 4 - N \dim(g) \quad (1.5)$$

If $\dim(g)$ is negative, d is unbounded as the order of perturbation N increases. The inevitable result is a non-renormalizable theory in which the number of distinct vertices (and hence new coupling constants) is unbounded. Apart from the lack of predictive power of these theories¹⁴⁾, the prescription (1.3) implies a quantization procedure for high-derivative counterterms which is not obviously unitary. So we shall ensure renormalizability ($d \leq 4$) by requiring

$$\dim(g) \geq 0 \quad (1.6)$$

but frequently permit external composite operators^{13,15)} Q to couple to $\mathcal{L}$ via sources χ (to a fixed order in χ):

$$\mathcal{L} \longrightarrow \mathcal{L}(\chi) = \mathcal{L} + \chi(Q + \Delta Q) + O(\chi^2) \quad (1.7)$$

Given the constraint (1.6), the rule for renormalization counterterms ΔQ generated by 1PI diagrams with a single Q -insertion is simply

$$\dim(\Delta Q) \leq \dim(Q) \quad (1.8)$$

For example, consider

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - g\phi^4 \quad (1.9)$$

as a trial Lagrangian. Only the two- and four-legged 1PI amplitudes $\Gamma^{(2)}, \Gamma^{(4)}$ are superficially divergent. According to (1.4), we have $d \leq 4$, so the only possibilities are counterterms proportional to $(\partial\phi)^2$ and ϕ^2 [from $\Gamma_{\text{div}}^{(2)}$] and ϕ^4 [from $\Gamma_{\text{div}}^{(4)}$]. Therefore, the most general expression for the complete Lagrangian is

$$\mathcal{L} = \frac{1}{2} Z_3 (\partial\phi)^2 - \frac{1}{2} (m^2 - \delta m^2) \phi^2 - g Z_1 \phi^4 \quad (1.10)$$

where Z_1, Z_3 , and δm^2 are Λ -dependent. Note that there can be vertices in $\mathcal{L}$ which do not appear in $\mathcal{L}_0$. For example, the Yukawa trial Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\not{\partial} - M)\psi + g'\bar{\psi}\gamma_5\psi\phi \quad (1.11)$$

produces an additional ϕ^4 vertex in $\mathcal{L}$ because of divergences in four-point 1PI boson amplitudes $\Gamma^{(4)}$. The result is a theory with two coupling constants, g and g' :

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} Z_3 (\partial\phi)^2 - \frac{1}{2} (m^2 - \delta m^2) \phi^2 - g Z_1 \phi^4 \\ & + Z_2 \bar{\psi} i \not{\partial} \psi - (M - \delta M) \bar{\psi} \psi + g' Z'_1 \bar{\psi} \gamma_5 \psi \phi \end{aligned} \quad (1.12)$$

The Lagrangian $\mathcal{L}$ [Eq. (1.10) or (1.12)] produces finite 1PI amplitudes $\Gamma^{(L)}$ and connected Green's functions $G^{(L)}$ (with L legs). We have no further use for $\mathcal{L}_0$; it is just a crutch which helps us to arrive at a suitable $\mathcal{L}$.

Sometimes it is convenient to introduce a new set of Feynman rules (labelled with a B for "bare" or "unrenormalized") for the same $\mathcal{L}$ in terms of new quantities $\phi_B, m_B^2, g_B, \dots$. The substitution

$$\begin{aligned} \phi_B &= Z_3^{1/2} \phi \\ m_B^2 &= Z_3^{-1} (m^2 - \delta m^2) \\ g_B &= Z_3^{-2} Z_1 g \end{aligned} \quad (1.13)$$

in Eq. (1.10) produces a simple set of B rules:

$$\mathcal{L} = \frac{1}{2} (\partial\phi_B)^2 - \frac{1}{2} m_B^2 \phi_B^2 - g_B \phi_B^4 \quad (1.14)$$

Let $G_B^{(L)}$ be the L -point Green's function computed using the B rules. Numerically, the only difference between $G_B^{(L)}$ and $G^{(L)}$ is the normalization of each external line -- internally, everything cancels when propagators and vertices are changed according to (1.13), because the Lagrangian $\mathcal{L}$ remains the same. So we get:

$$G^{(L)}(q_1, \dots, q_{L-1}; g, m) = Z_3^{-L/2} G_B^{(L)}(q_1, \dots, q_{L-1}; g_B, m_B, \Lambda) \quad (1.15)$$

Therefore, if some cut-off dependence is suitably absorbed in coupling constant and mass renormalization, the remaining divergence of an unrenormalized Green's function can be segregated as a multiplicative constant¹⁰⁾ $Z_3^{L/2}$. There is a wave-function renormalization $Z_3^{-1/2}$ for each external boson line, and a similar factor $Z_2^{-1/2}$ for each fermion [if Eq. (3.12) is being considered] with

$$g' = (Z'_1)^{-1} Z_2 Z_3^{1/2} g_B \quad (1.16)$$

1.2 Renormalization group

Prescriptions like (1.2), (1.10), and (1.12) still contain a Λ -independent ambiguity, because all we have done so far is to require that divergent parts of 1PI graphs be cancelled. The corresponding finite parts have to be fixed by

imposing a renormalization prescription (call it R), and quantities such as δm^2 , ϕ , ..., must be regarded as R-dependent: δm_R^2 , ϕ_R , g_R , $Z_i(R)$, $\Gamma_R^{(L)}$, $G_R^{(L)}$, and so on. In order to completely specify each distinct vertex appearing in $\Delta \mathcal{L}$, we must impose a normalization condition on the corresponding 1PI amplitude. The label R refers to this set of conditions. The points in momentum space to which the conditions refer are called subtraction points.

A popular choice for R is to normalize on-shell. For example, boson mass and wave-function normalizations are fixed by choosing

$$\Gamma_R^{(2)}(q) = 0, \quad \partial \Gamma_R^{(2)} / \partial q^2 = 0, \quad (q^2 = m_R^2)$$

for the self-energy $\Gamma_R^{(2)}$, so that the dressed boson propagator

$$G_R^{(2)}(q) = i / (q^2 - m_R^2 - i \Gamma_R^{(2)}(q)) \quad (1.17)$$

has a pole at $q^2 = m_R^2$ with conventional residue. Another example is the charge of the electron $\sqrt{4\pi\alpha}$ (with $\alpha^{-1} = 137.036$, ...) in quantum electrodynamics (QED), which refers to the amplitude for an on-shell electron to absorb a zero-frequency photon.

Alternatively, R can be specified by intermediate renormalization^{13,16)} in which the subtraction point is at the origin of momentum space; e.g., R for Eq. (1.10) is fixed by

$$\Gamma_R^{(2)}(q) = -i m_R^2, \quad \partial \Gamma_R^{(2)} / \partial q^2 = i, \quad (q = 0);$$

$$\Gamma_R^{(4)}(q_1 = q_2 = q_3 = 0) = -i g_R 4! \quad (1.18)$$

It does not matter that g_R and m_R^2 are not directly measurable; they parametrize the theory as effectively as on-shell variables do.

More generally, R can be a function of several fixed "reference momenta" $\lambda_1, \lambda_2, \dots$, which parametrize the subtraction points: consider the prescription $R = R(\lambda_1, \dots, \lambda_5)$ for Eq. (1.10):

$$\Gamma_R^{(2)}(\lambda_1) = -i m_R^2, \quad \partial \Gamma_R^{(2)} / \partial q^2 \big|_{q=\lambda_2} = i,$$

$$\Gamma_R^{(4)}(\lambda_3, \lambda_4, \lambda_5) = -i g_R 4! \quad (1.19)$$

Note

- i) To avoid trouble with unitarity, we want g_R and m_R^2 to be real. However, for arbitrary λ in Eq. (1.19), $\Gamma^{(2)}$ and $\Gamma^{(4)}$ may have absorptive parts. In that case, the normalization conditions (1.19) apply only to the real parts of the 1PI amplitudes at non-singular points λ . Additional conditions for

absorptive parts are not needed: a 1PI absorptive part converges superficially because it is obtained by cutting internal lines of the full 1PI amplitude. The λ -momenta should be zero or space-like if a continuation of Feynman amplitudes to Euclidean space is being attempted.

- ii) Suppose we use (1.19) to specify $G^{(L)}$ and Z_3 in Eq. (1.15); then they depend on λ as well as the variables shown explicitly. Of course, the B rules make no reference to λ , so $G_B^{(L)}$, when written as a function of m_B and g_B , does not depend on λ .
- iii) Do not confuse the momenta $\lambda_3, \lambda_4, \lambda_5$ at which g_R is fixed with the momenta entering and leaving a vertex inside a Feynman diagram!

Now suppose that two theorists are thinking about the same physical situation (i.e. they use the same $\mathcal{L}$) but use different renormalization procedures, R and R':

$$\mathcal{L} = \mathcal{L}_R(\text{R-quantities}) = \mathcal{L}_{R'}(\text{R'-quantities})$$

Transformation equations relating R and R' are easily found by referring back to the B rules, which do not depend on R or R'. The equations

$$\phi_R = Z_3(R)^{-1/2} \phi_B, \quad \phi_{R'} = Z_3(R')^{-1/2} \phi_B$$

etc., imply [e.g., for Eq. (1.12)]

$$\begin{aligned} \phi_{R'} &= \zeta_3(R', R)^{-1/2} \phi_R \\ \psi_{R'} &= \zeta_2(R', R)^{-1/2} \psi_R \\ g_{R'} &= \zeta_1(R', R)^{-1} \zeta_3(R', R)^2 g_R \\ g'_{R'} &= \zeta'_1(R', R)^{-1} \zeta_2(R', R) \zeta_3(R', R)^{1/2} g_{R'} \\ m_{R'}^2 &= \zeta_3(R', R) m_R^2 + \delta(R', R) \\ M_{R'} &= \zeta_2(R', R) M_R + \Delta(R', R) \end{aligned} \tag{1.20}$$

and

$$G_{R'}^{(b,f)} = \zeta_3(R', R)^{-b/2} \zeta_2(R', R)^{-f/2} G_R^{(b,f)}, \tag{1.21}$$

(b bosons, f fermions),

where the transformation coefficients ζ_i, δ, Δ , given by

$$\begin{aligned} \zeta_i(R', R) &= Z_i(R')/Z_i(R), \quad (\text{ditto for } \zeta'_1) \\ \delta(R', R) &= \delta m_{R'}^2 - \zeta_3(R', R) \delta m_R^2 \\ \Delta(R', R) &= \delta M_{R'} - \zeta_2(R', R) \delta M_R \end{aligned} \tag{1.22}$$

are all cut-off independent. The ambiguity mentioned at the beginning of this subsection is now completely characterized in terms of these coefficients.

The set $\mathcal{G}$ of all transformations $R \rightarrow R'$ is called²⁾ the "Renormalization Group". Measurable quantities $\mathcal{M}$ like decay rates and cross-sections are renormalization-group invariant: $\mathcal{M}(R') = \mathcal{M}(R)$. In other words, $\mathcal{M}$ should not be theorist-dependent.

The group property is partially realized in the existence of an identity transformation $R \rightarrow R$, an inverse $R' \rightarrow R$ to $R \rightarrow R'$, and a product $R \rightarrow R''$ equivalent to successive transformations $R \rightarrow R'$ and $R' \rightarrow R''$, with

$$\begin{aligned} \zeta_i(R, R) &= 1 \\ \zeta_i(R, R') &= \zeta_i(R', R)^{-1} \\ \zeta_i(R'', R) &= \zeta_i(R'', R') \zeta_i(R', R), \text{ etc.} \end{aligned}$$

However, we do not have a group at this stage¹⁷⁾, because there is no rule for multiplying two arbitrary transformations $R \rightarrow R'$ and $R'' \rightarrow R'''$. A satisfactory rule can be obtained for special subsets of prescriptions which can be explicitly parametrized: $R = R(\lambda)$ [e.g., $R = R(\lambda_1, \dots, \lambda_s)$ discussed above]. Then it is sufficient to consider transformations in λ -space which possess the group property -- inversions, translations, rotations, scale transformations, etc. So the group property is certainly a feature of many subsets of $\mathcal{G}$, but it is not clear whether $\mathcal{G}$ itself can be characterized as a group or not. (Henceforth, I shall forget about this fine distinction.)

The idea of SP²⁾ was to look at sets of prescriptions R depending on continuous parameters λ from which a Lie group with associated Lie algebras can be obtained, and find equations giving the result of performing an infinitesimal λ -transformation. The only subgroup of any practical importance seems to be the trivial case $U(1)$ for scale transformations of the subtraction points λ (with generator $\lambda \cdot \partial / \partial \lambda$). Non-Abelian cases are easy to construct, but the results appear to be uninteresting.

Eq. (1.21) means that $G^{(b,f)}$ transforms multiplicatively under the group. The 1PI amplitudes transform similarly (with $\zeta^{1/2}$ for each line instead of $\zeta^{-1/2}$)

$$\Gamma_{R'}^{(b,f)} = \zeta_3(R', R)^{b/2} \zeta_2(R', R)^{f/2} \Gamma_R^{(b,f)}, \quad [(b,f) \neq (2,0), (0,2)] \quad (1.23)$$

except for self-energy amplitudes, where there is an additional subtractive renormalization: Eqs. (1.17) (valid irrespective of R) and (1.21) imply

$$\Gamma_{R'}^{(2,0)} = \zeta_3(R', R) \Gamma_R^{(2,0)}(q) + i(\zeta_3(R', R) - 1) q^2 + i\delta(R', R) \quad (1.24)$$

for the boson self-energy $[\Gamma^{(2)} = \Gamma^{(2,0)}]$, and there is an analogous equation for fermions:

$$\Gamma_{R'}^{(0,2)} = \mathcal{Z}_2(R', R) \Gamma_R^{(0,2)}(q) + i(\mathcal{Z}_2(R', R) - 1) \not{A} + i \Delta(R', R) \quad (1.25)$$

1.3 Callan-Symanzik equation

The CS equation⁵⁾ was the product of a search for the Ward identity of scale transformations in perturbation theory¹⁸⁾. The generator of scale transformations¹⁹⁾, $D(x_0) = \int d^3x x^\mu \theta_{\mu 0}(x)$, has a time-variation given by the trace of the energy momentum tensor $\theta_{\mu\nu}$. In free-field theory, θ_μ^μ is the sum of mass terms

$$\theta_\mu^\mu = m^2 \phi^2 + M \bar{\Psi} \Psi \quad (1.26)$$

To obtain something which looks like a scaling Ward identity for an amplitude A , we need the corresponding amplitude $A(\Delta)$ with an extra operator insertion Δ where, for free-field theory, Δ reduces to the zero-momentum mass operator $\int d^4x \theta_\mu^\mu(x)$.

Consider the L-leg boson Green's function $G^{(L)}$ (connected), which is multiplicatively renormalized [Eqs. (1.15), (1.21)], and assume for convenience that there is only one mass m and dimensionless coupling constant g . If we use the B rules (1.14) to compute²⁰⁾ $G_B^{(L)}$, the only place in which m_B appears is in undressed propagators $i/(p^2 - m_B^2)$. Hence the identity

$$m_B \frac{\partial}{\partial m_B} \frac{i}{p^2 - m_B^2} = \frac{i}{p^2 - m_B^2} (-2i m_B^2) \frac{i}{p^2 - m_B^2}$$

implies that the operation $m_B \partial / \partial m_B$ (with external momenta and g_B, Λ fixed) is equivalent to the insertion of a new vertex $-i m_B^2 \phi_B^2$ at zero-momentum transfer (i.e. $-i\Delta$). So we define

$$G_B^{(L)}(\Delta) = i m_B \frac{\partial}{\partial m_B} G_B^{(L)}(q; g_B, m_B, \Lambda) \quad (1.27)$$

where q is shorthand for $(q_1, \dots, q_{L-1})$. Of course, $G_B^{(L)}(\Delta)$ is cut-off dependent, but its renormalization is very simple. If the operator Q in (1.7) is chosen to be ϕ^2 , the only counterterm operator ΔQ which can satisfy (1.8) is again ϕ^2 [ϕ does not appear because of $\mathcal{L}(\phi) = \mathcal{L}(-\phi)$]:

$$\mathcal{L}(\chi) = \mathcal{L}(0) + \chi Z_\Delta(\Lambda; R) \phi_R^2 + \chi^2 Z_{\Delta\Delta} \quad (1.7')$$

i.e. the composite operator $\phi^2(x)$ is multiplicatively renormalized. Hence there is a wave-function renormalization Z_Δ^{-1} for the external line represented by the source χ in addition to a factor $Z_3^{-1/2}$ for each boson line:

$$G_R^{(L)}(\Delta) = Z_\Delta^{-1} Z_3^{-L/2} G_B^{(L)}(\Delta) \quad (1.28)$$

It is convenient (but not essential) to use the intermediate renormalization prescription (1.18), because then we do not have to worry about additional dependence on reference momenta λ . In order to fix the finite ambiguity in Z_Δ , we have to choose a normalization condition for $\Gamma^{(2)}(\Delta)$, the 1PI amplitude with two boson legs and a Δ insertion:

$$\Gamma_R^{(2)}(\Delta; q=0) = 2 m_R^2 \quad (1.18')$$

Note the relation

$$G^{(2)}(\Delta; q) = \{G^{(2)}(q)\}^2 \Gamma_R^{(2)}(\Delta; q) \quad (1.29)$$

Now we construct two identities, each the result of applying $m_R d/dm_R$ (fixed q, g_B, Λ) to $G_R^{(L)}$ and changing variables:

$$\begin{aligned} m_R \frac{d}{dm_R} \Big|_{q, g_B, \Lambda} G_R^{(L)} &= \left[m_R \frac{\partial}{\partial m_R} + m_R \frac{dg_R}{dm_R} \Big|_{\Lambda, g_B} \frac{\partial}{\partial g_R} \right] G_R^{(L)}(q; g_R, m_R) \\ &= -\frac{1}{2} L Z_3^{-1} G_R^{(L)} \{m_R \partial / \partial m_R\} Z_3(\Lambda/m_R, g_B) \\ &\quad + Z_3^{-L/2} (m_R/m_B) (dm_B/dm_R)_{g_B, \Lambda} \{m_B \partial / \partial m_B\} G_B^{(L)}(q; g_B, m_B) \end{aligned}$$

The second identity depends on Eq. (1.15). The result of combining these identities with (1.28) is the CS equation^{5,21)}

$$\begin{aligned} \left[m_R \frac{\partial}{\partial m_R} + \beta_R \frac{\partial}{\partial g_R} + L \gamma_R \right] G_R^{(L)}(q; g_R, m_R) \\ = -i \delta_R G_R^{(L)}(\Delta; q, g_R, m_R) \end{aligned} \quad (1.30)$$

with

$$\begin{aligned} \beta_R &= m_R (dg_R/dm_R)_{g_B, \Lambda} \\ \gamma_R &= -m_R \partial / \partial m_R \ln \{Z_3(\Lambda/m_R, g_B)^{-1/2}\} \\ \delta_R &= Z_\Delta (m_R/m_B) (dm_B/dm_R)_{g_B, \Lambda} \end{aligned} \quad (1.31)$$

The functions γ, δ are cut-off independent, because we can apply the conditions

$$\begin{aligned} G_R^{(2)}(q=0) &= -i/m_R^2 \\ (\partial/\partial q^2) G_R^{(2)}(q=0) &= -i/m_R^4 \\ G_R^{(2)}(\Delta; q=0) &= -2/m_R^2 \end{aligned} \quad (1.32)$$

[implied by Eqs. (1.17), (1.18), (1.18'), (1.29)] to Eq. (1.30) with $L = 2$:

$$\begin{aligned}\delta_R &= 1 - \gamma_R \\ \gamma_R &= 2 + \frac{1}{2} m_R^4 \delta_R (\partial/\partial q^2) G_R^{(L)}(\Delta; q=0)\end{aligned}\quad (1.33)$$

It follows from (1.30) that β is also cut-off independent. Hence the dimensionless functions β, γ, δ can only depend on g_R , in this renormalization prescription. The formulas

$$\begin{aligned}\beta_R(g_R) &= -g_R \Lambda \partial/\partial \Lambda \ln Z_g(\Lambda/m_R, g_B) \quad , \quad (g_R = Z_g g_B) \\ \gamma_R(g_R) &= \Lambda \partial/\partial \Lambda \ln \{Z_3(\Lambda/m_R, g_B)\}^{-1/2} \quad , \quad (\Lambda \rightarrow \infty)\end{aligned}\quad (1.34)$$

[from (1.31)] show the connection between β and γ and infinities in coupling-constant and wave-function renormalization.

Let us introduce a parameter η which scales all the momenta together: $q \rightarrow \eta q$. Since $G_R^{(L)}$ has mass dimensionality $4 - 3L$, purely dimensional arguments imply the identity

$$[\eta \partial/\partial \eta + m_R \partial/\partial m_R + 3L - 4] G_R^{(L)}(\eta q; g_R, m_R) = 0 \quad (1.35)$$

so the CS equation can be cast in a form which shows the connection with scale transformations:

$$\begin{aligned}[\eta \partial/\partial \eta - \beta_R(g_R) \partial/\partial g_R + 3L - 4 - L \gamma_R(g_R)] G_R^{(L)}(\eta q; g_R, m_R) \\ = -i \delta_R(g_R) G_R^{(L)}(\Delta; \eta q; g_R, m_R)\end{aligned}\quad (1.36)$$

The term $\beta_R \partial/\partial g_R$, caused by coupling-constant renormalization, means that $\int d^4x \theta_\mu^\mu(x)$ effectively picks up a dimension 4 vertex^{5,22)} $-i \beta_R \int d^4x \phi^4(x)$.

If there are several coupling constants $g_R(i)$ ($i = 1, \dots, j$), the $\beta_R \partial/\partial g_R$ term becomes $\sum_{i=1}^j \beta_R^{(i)} \partial/\partial g_R(i)$; i.e. there is a β -function $\beta_R^{(i)}$ for each $g_R(i)$, and it depends on all of the $g_R(i)$'s:

$$\begin{aligned}\vec{g}_R &= (g_R^{(1)}, \dots, g_R^{(j)}) \\ \vec{\beta}_R &= (\beta_R^{(1)}, \dots, \beta_R^{(j)}) \\ \vec{\beta}_R &= \vec{\beta}_R(\vec{g}_R) \quad , \quad \gamma_R = \gamma_R(\vec{g}_R) \\ \vec{\beta}_R \cdot \partial/\partial \vec{g}_R &\equiv \sum_{i=1}^j \beta_R^{(i)} \partial/\partial g_R(i)\end{aligned}\quad (1.37)$$

If there are also several masses $m_R(k)$, each mass can generate its own CS equation, β -functions $\beta_R^{(i,k)}$, and γ -function $\gamma_R^{(k)}$. Occasionally, these CS equations are separately useful²³⁾, but in most cases, it is better to sum them because then $\vec{m}_R \cdot \partial/\partial \vec{m}_R$ can be converted to $\eta \partial/\partial \eta$ using the generalization of (1.35).

Subscripts R have been attached to the symbols β, γ, δ in Eq. (1.36) to indicate that their functional forms depend on the renormalization prescription. To see this, consider a prescription R' (such as on-shell normalization) in which

$g_{R'}$, $\tilde{Z}_3(R', R)$, and $m_{R'}/m_R$ depend on g_R alone; i.e. we avoid introducing dependence on a separate dimensionful quantity λ . Then a change of variables (g_R, m_R) to $(g_{R'}, m_{R'})$ in Eq. (1.30) produces a CS equation of the same form, with R' replacing R and

$$\begin{aligned}\beta_{R'}(g_{R'}) &= \beta_R \{dg_{R'}/dg_R\} / \{1 + \beta_R d/dg_R \ln(m_{R'}/m_R)\} \\ \gamma_{R'}(g_{R'}) &= \{\gamma_R + \tilde{Z}_3^{-1} \beta_R dg_R/dg_{R'}\} / \{1 + \beta_R d/dg_R \ln(m_{R'}/m_R)\}\end{aligned}\quad (1.38)$$

Observe that β, γ are R -independent in the 1-loop approximation.

The main complication for $R = R(\lambda)$ is that $m_R \partial / \partial m_R$ in Eq. (1.30) has to be replaced by $\{m_R \partial / \partial m_R + \lambda \partial / \partial \lambda\}$ so that we still get the infinitesimal scale transformation $\eta \partial / \partial \eta$ in (1.36). In general, the β and γ functions depend on dimensionless ratios such as m_R^2/λ^2 , λ_1^2/λ_2^2 , ..., as well as on g_R .

The generalization to amplitudes $G_{ABC}^{(b,f)}$ involving several composite operators $A, B, C, \dots$, can be readily carried through. The analogue of Eq. (1.7) is

$$\begin{aligned}\mathcal{L}(\chi) &= \mathcal{L}(0) + \chi_A(A + \Delta A) + \chi_B(B + \Delta B) + \dots \\ &+ \chi_A^2 \Delta(AA) + \chi_A \chi_B \Delta(AB) + \chi_A \chi_C \Delta(AC) + \dots + \chi_B^2 \Delta(BB) \\ &+ \dots + \chi_A^3 \Delta(AAA) + \dots \\ &+ \dots\end{aligned}\quad (1.39)$$

where $\Delta(ABC \dots)$ is a local vertex with cut-off-dependent normalization. Consider the terms linear in χ . For general A , there may be many counterterms ΔA with dimensionality less than or equal to that of A [see Eq. (1.8)]. In other words, the Z -factors have to be treated as matrices. However, by adding suitable linear combinations of the counterterm vertices $\Delta A, \Delta B, \dots$, we can always re-define $A, B, C, \dots$, such that they are multiplicatively renormalized:

$$A + \Delta A = Z_A(\lambda) A, \quad B + \Delta B = Z_B(\lambda) B, \dots (1.40)$$

Equation (1.39) also contains counterterms $O(\chi^p)$ ($p > 1$) produced by superficially divergent 1PI subdiagrams with p composite-operator insertions. However, these terms are very easy to isolate. For example, if $A(x)$ and $B(y)$ (x, y = coordinates) couple to the same 1PI diagram, the induced counterterm must be proportional to $P(\partial_x) \delta^4(x - y)$, where P is a polynomial; this is so because a 1PI diagram (made convergent internally) shrinks to a *point* when the divergent part is taken (Fig. 1). In momentum space, we have $P(-iq_{AB})$, where q_{AB} is the momentum transferred from A to B . So the mode of renormalization is given by

$$\begin{aligned}\{G_{ABC\dots}^{(b,f)}\}_R &= Z_A^{-1} Z_B^{-1} Z_C^{-1} \dots Z_3^{-b/2} Z_2^{-f/2} \{G_{ABC\dots}^{(b,f)}\}_{\text{Bare}} \\ &+ \sum_i P_i(q_{\text{subset } i}) A_i\end{aligned}\quad (1.41)$$

where P_i is a polynomial in momentum-transfer variables $q_{\text{subset } i}$ for the i^{th} subset of $\{A, B, C, \dots\}$ and t_i is a non-trivial amplitude involving the remaining operators plus the counterterm operator $\Delta(i^{\text{th}} \text{ subset})$. There is a factor Z_Q^{-1} for each insertion Q , in addition to the usual factors $Z_3^{-1/2}$, $Z_2^{-1/2}$ for each external boson and fermion line in a complete or connected Green's function. The subtractive renormalizations $\sum_i P_i t_i$ lack absorptive parts in channels corresponding to the polynomial dependence $P_i(q_{\text{subset } i})$.

By repeating the previous analysis, we end up with a CS equation for $G_{ABC\dots}^{(b,f)}$,

$$\begin{aligned} & \left[m_R \frac{\partial}{\partial m_R} + \beta_R(g_R) \frac{\partial}{\partial g_R} + b \gamma_R(g_R) + f \bar{\gamma}_R(g_R) + \gamma_{ABC\dots}^{(R)}(g_R) \right] \left\{ G_{ABC\dots}^{(b,f)}(q; g_R, m_R) \right\}_R \\ & + \sum_i P_i(q_{\text{subset } i}) \mathcal{G}_i(\text{other } q; g_R, m_R) \\ & = -i \delta_R(g_R) \left\{ G_{ABC\dots}^{(b,f)}(\Delta; q; g_R, m_R) \right\}_R \end{aligned} \quad (1.42)$$

where all factors are cut-off independent, $\mathcal{G}_i$ is a polynomial, and

$$\bar{\gamma}_R(g_R) = \Lambda \frac{\partial}{\partial \Lambda} \ln \left\{ Z_2(\Lambda/m_R, g_B) \right\}^{-1/2} \quad (1.43)$$

is the fermionic analogue of $\gamma_R(g_R)$. The important thing to notice is that there is an addition rule satisfied by

$$\gamma_{ABC\dots}^{(R)}(g_R) = -\Lambda \frac{\partial}{\partial \Lambda} \ln \left[Z_A Z_B Z_C \dots \right] \quad (1.44)$$

the γ -function describing the combined effects of the product $ABC \dots$ on wave-function renormalization. For each operator Q , there is a characteristic γ -function

$$\gamma_Q^{(R)}(g_R) = -\Lambda \frac{\partial}{\partial \Lambda} \ln Z_Q(\Lambda/m_R, g_B) \quad (1.45)$$

which satisfies the formula

$$\gamma_{ABC\dots}^{(R)}(g_R) = \gamma_A^{(R)}(g_R) + \gamma_B^{(R)}(g_R) + \gamma_C^{(R)}(g_R) + \dots \quad (1.46)$$

Conserved currents (e.g. $\theta_{\mu\nu}$ and the electromagnetic current J_μ) obey identities such as

$$i(p' - p)_\mu G_{J_\mu}^{(0,2)}(p', p) = G^{(0,2)}(p') - G^{(0,2)}(p) \quad (1.47)$$

irrespective of which rules (B or any R) are chosen. Hence there is no wave-function renormalization factor for a conserved current

$$\mathcal{Z}_{J_\mu}(R', R) = Z_{J_\mu} = 1 \quad (1.48)$$

and the corresponding γ -function (1.45) vanishes identically:

$$\gamma_{J_\mu} = 0 \quad (1.49)$$

Of course, there can still be subtractive renormalizations induced by time-ordering products of conserved currents. For example, the renormalization of $T\langle 0|J_\mu(x)J_\nu(0)|0\rangle$ involves a subtraction proportional to $(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\delta^4(x)$. Partially conserved currents such as the chiral $SU(3)\times SU(3)$ currents $(\tilde{T}_\mu^a, \tilde{T}_{5\mu}^a)$ of current algebra become conserved at short distances, so they also satisfy (1.48) and (1.49).

2. ASYMPTOTIC SOLUTIONS

This chapter concerns situations in which the mass-inserted amplitude $G(\Delta)$ can be neglected in the CS equation. In perturbation theory, an appropriate asymptotic limit is specified by Weinberg's theorem¹¹⁾.

2.1 Logarithms in perturbation theory

Weinberg's theorem extends the power-counting method to the problem of estimating the asymptotic behaviour of a Feynman amplitude as some of its external momenta become large. Actually, it is a theorem in real-variable analysis which describes the asymptotic behaviour of infinite multiple integrals, so its application to Feynman integrals is necessarily restricted to the Euclidean region. However, Pohlmeier²⁴⁾ has proven a corresponding theorem for Minkowski space.

Let $\mathcal{A}$ be a Feynman amplitude which depends on external momenta $q_1, \dots, q_K$ and $r_1, \dots, r_N$, where the q -momenta are large:

$$\begin{aligned} \mathcal{A} &= \mathcal{A}(q_1, \dots, q_K; r_1, \dots, r_N) \\ q_i &= \eta l_i + c_i, \quad (\eta \rightarrow \infty; l_i, c_i, r_j \text{ all fixed}) \\ i &= 1, \dots, K, \quad j = 1, \dots, N \end{aligned} \quad (2.1)$$

All momenta are understood to be incoming, so momentum conservation requires

$$\sum_{j=1}^N r_j + \sum_{i=1}^K q_i = 0 \quad (2.2a)$$

$$\sum_{i=1}^K l_i = 0 \quad (2.2b)$$

Imagine the large $O(\eta)$ momenta (indicated by heavy lines in Fig. 2) percolating through all vertices and propagators of a subgraph $\mathcal{G}'$ of a typical graph $\mathcal{G}$. Naturally, we arrange that the external lines of $\mathcal{G}'$ include *all* external lines of $\mathcal{G}$ which carry the $O(\eta)$ momenta $q_1, \dots, q_K$, and that it is kinematically possible for all internal lines of $\mathcal{G}'$ to carry $O(\eta)$ momenta. Each subgraph can

contribute to the asymptotic behaviour of $\mathcal{G}$. The idea is to count the powers produced by propagators, vertices and loop integrations in $\mathcal{G}'$, keeping the intermediate momenta $k_1, \dots, k_M$ fixed. The effect of loop integration is illustrated by the 1-loop boson self-energy amplitude (Fig. 3) for a scalar Yukawa theory (interaction $g\bar{\psi}\psi\phi$)*):

$$\begin{aligned} \Gamma_R^{(2,0)}(q^2)_{1\text{-loop}} &= -g_R^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left\{ (\not{p} - m)^{-1} (\not{p} + \not{q} - m)^{-1} \right. \\ &\quad \left. - \text{subtraction} \right\} \\ &\sim - (ig_R^2/8\pi^2) q^2 \ln q^2, \quad (q \rightarrow \infty). \end{aligned} \quad (2.3)$$

This example shows that it is necessary to count 4 powers of η for each loop integral $\int d^4p$ inside**) $\mathcal{G}'$. Consequently, each $\mathcal{G}'$ contributes a characteristic power $O(\eta^{D(\mathcal{G}')})$ (with logarithmic corrections), where the *dimensionality* $D(\mathcal{G}')$ of $\mathcal{G}'$ is given by

$$\begin{aligned} D(\mathcal{G}') &= \sum \dim \{ \text{propagators and vertices of } \mathcal{G}' \} \\ &\quad + 4 \{ \text{number of independent loops in } \mathcal{G}' \} \end{aligned} \quad (2.4)$$

So far, the only restriction placed on the vectors ℓ_i is the momentum-conservation equation (2.2b). Another obvious requirement is that none of the ℓ_i should vanish -- otherwise, some of the q -momenta would not be $O(\eta)$. However, the power-counting rules introduced above do not necessarily work unless additional conditions are imposed. For example, the 4 boson and 3 fermion propagators of the subgraph $\mathcal{G}'$ of Fig. 4 (Yukawa theory) normally contribute $O(\eta^{-11})$. However, at the "exceptional" momentum point²⁵⁾ $\ell_1 = -\ell_2$, the fermion propagator in the middle of $\mathcal{G}'$ no longer carries momentum $O(\eta)$:

$$A(\mathcal{G}'; \ell_i^2 \neq 0; \ell_1 = -\ell_2) = O(\eta^{-10}).$$

Additional exceptional points exist in Minkowski space:

$$A(\mathcal{G}'; \ell_i^2 \neq 0; \ell_1 \neq -\ell_2; (\ell_1 + \ell_2)^2 = 0) = O(\eta^{-9}) \text{ or } O(\eta^{-10})$$

[depending on whether $(\ell_1 + \ell_2) \cdot (c_1 + c_2 + k_1)$ vanishes or not]. The general conclusion is that tree graphs can be more singular than $O(\eta^{D(\mathcal{G}')})$ if some of the partial sums $\sum' \ell$ happen to be light-like.

Not surprisingly, loop integrals also misbehave at these exceptional points. Consider the 1-loop amplitude (Fig. 5)

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- *) Here, the relevant subgraph $\mathcal{G}'$ is the complete graph $\mathcal{G}$. All other subgraphs involve restricting $\int d^4p$ to a finite volume, so they contribute $O(q^{-1})$.
- **) The intermediate lines $k_1, \dots, k_M$ may be parts of closed loops in $\mathcal{G}$, but this is not relevant for the subgraph $\mathcal{G}'$. It becomes relevant only when a larger subgraph is considered.

$$I(q_1 \dots q_K) = i^K \int \frac{d^4 p}{(2\pi)^4} \prod_{i=1}^K [(p + Q_i)^2 - m^2]^{-1}, \quad (K \geq 3) \quad (2.5a)$$

$$(Q_i = \sum_{j=1}^i q_j, \quad q_i = \eta \ell_i + c_i, \quad \sum_{i=1}^K \ell_i = \sum_{i=1}^K c_i = 0)$$

which can be simply represented as a Feynman-parameter integral²⁶⁾:

$$I = \frac{i^{K+1} (K-3)!}{16\pi^2} \int_0^1 \prod_{i=1}^K d\alpha_i \delta(1 - \sum \alpha) / [\sum_{i < j} \alpha_i \alpha_j (Q_i - Q_j)^2 - m^2]^{K-2} \quad (2.5b)$$

The limit $\eta \rightarrow \infty$ is equivalent to setting m and c_i to zero in (2.5a) or (2.5b).

If the massless amplitude $I(m = 0 = c_i)$ exists, the result is

$$I(\eta) \sim \eta^{4-2K} I(m = 0 = c_i; \eta = 1), \quad (\eta \rightarrow \infty) \quad (2.6)$$

which agrees with naive power-counting:

$$D(\mathcal{G}') = 4 - 2K \quad (\text{with } \mathcal{G}' = \text{complete graph } \mathcal{G}).$$

However, $I(m = 0 = c_i)$ may not exist because of infrared singularities; in that case, $\eta^{2K-4} I(\eta)$ blows up as η tends to infinity. For example, if all partial sums $\sum' \ell$ are light-like but non-zero, there is a singularity at $p = 0$:

$$I(\eta) \sim i^K \eta^{4-2K} \int \frac{d^4 p}{(2\pi)^4} \prod_{i=1}^K [p^2 + 2p \cdot \sum_{j=1}^i \ell_j + O(\eta^{-1})]^{-1}$$

The true asymptotic behaviour can be found by substituting

$$(Q_i - Q_j)^2 = \eta r_{ij} + O(1)$$

into Eq. (2.5b), with

$$r_{ij} = 2 \left(\sum_{k=i}^j \ell_k \right) \cdot \left(\sum_{k=i}^j c_k \right)$$

thus, if r_{ij} all have the same sign, the answer is

$$I(\eta) \sim \eta^{2-K} \left(\frac{i^{K+1} (K-3)!}{16\pi^2} \right) \int_0^1 \prod_{i=1}^K d\alpha_i \delta(1 - \sum \alpha) / [\sum_{i < j} \alpha_i \alpha_j r_{ij}]^{K-2}, \quad (2.6')$$

$$((\sum' \ell)^2 = 0, \quad \sum' \ell \neq 0, \quad \eta \rightarrow \infty; \quad K \geq 3)$$

because the resulting α -integral is manifestly convergent.

In general, loop integrals become infrared singular at thresholds for the production of intermediate zero-mass particles. These thresholds occur when at least one of the momenta $\sum' \ell$ becomes light-like. This is precisely the condition for exceptional asymptotic behaviour of tree graph amplitudes, so we conclude that the general requirement for momenta to be non-exceptional²⁵⁾ is

$$(\sum' \ell)^2 \neq 0 \quad (2.7)$$

for all partial sums $\sum' \ell$ associated with connected amplitudes.

Note

- i) In coordinate space, the non-exceptional $\eta \rightarrow \infty$ limit corresponds to the short-distance limit

$$x_1 \dots x_K \longrightarrow y \quad (\text{non-exceptional}),$$

where $x_1, \dots, x_K$ are coordinates conjugate to the momenta $q_1, \dots, q_K$. If K' partial sums $\sum' l$ become light-like ($0 < K' < K$), the x -coordinates are free to tend to $K' + 1$ distinct y -coordinates:

$$x_1 \dots x_K \longrightarrow y_1 \dots y_{K'+1} \quad (\text{exceptional}).$$

- ii) I have glossed over a complication caused by renormalization of ultraviolet divergences. Counterterms which remove logarithmic divergences develop logarithmic infrared singularities

$$\int^\Lambda d^{4l} p (\ln p)^{l'}/p^{4l}, \quad (l' < l)$$

as m, c_i , and all subtraction points λ are effectively scaled to zero by the $\eta \rightarrow \infty$ limit. Consequently, there are logarithmic corrections to naive power-counting at non-exceptional momenta [as in Eq. (2.3)]:

$$\mathcal{G}' \text{ contribution} = O[\eta^{D(\mathcal{G}')} \{\text{Polynomial in } \ln \eta\}] \quad (2.8)$$

Additional powers of η are not generated: the candidate

$$\text{counterterm } (m = \lambda = 0) = \int^\Lambda d^{4l} p / p^{4l - d(\Gamma)}, \quad (d(\Gamma) < 0)$$

is not permitted because it is ultraviolet convergent^{*)}.

The main features and limitations of the theorem have now been exposed. It states that the amplitude $\mathcal{A}(\mathcal{G})$ for a graph $\mathcal{G}$ satisfies the asymptotic bound

$$\mathcal{A}(\mathcal{G}) = O[\eta^{\text{Max } D(\mathcal{G}')} \ln^{\beta(\mathcal{G})} \eta] \quad (2.9)$$

for the non-exceptional limit $\eta \rightarrow \infty$ specified by Eqs. (2.1), (2.2), and (2.7).

The power $\text{Max } D(\mathcal{G}')$ denotes the maximum dimensionality attained in the set of all subgraphs $\mathcal{G}'$ of $\mathcal{G}$; (see Fig. 2). The logarithmic power $\beta(\mathcal{G})$ (integer ≥ 0) is determined by counting powers of $\ln \Lambda$ in counterterms which remove the ultraviolet divergences of subgraphs with maximal dimensionality²⁷⁾.

If propagators for external lines of $\mathcal{G}'$ are excluded, theories with dimensionless coupling constants obey the rule

*) In other words, we forbid "oversubtraction": once a 1PI amplitude Γ becomes superficially convergent [$d(\Gamma) < 0$], counterterms which further reduce $d(\Gamma)$ are not introduced in Eq. (1.3).

$$\mathcal{D}(\mathcal{G}') = 4 - \{\text{number of bosons external to } \mathcal{G}'\} - \frac{3}{2} \{\text{number of fermions external to } \mathcal{G}'\} \quad (2.10)$$

Hence the maximum dimensionality is found by minimizing the number M of intermediate lines in Fig. 2. If composite operators (with sources χ) are present, the generalization of (2.10) is

$$\mathcal{D}(\mathcal{G}') = 4 - \sum \dim \{\text{external lines of } \mathcal{G}'\} \quad (2.10')$$

where each source χ is represented by a line (external or intermediate in Fig. 2) and given an appropriate dimensionality. For example, the source χ_{ϕ^2} for the composite operator $\phi^2(x)$ carries dimensionality 2, so the complete graph shown in Fig. 5 has dimensionality $4 - 2K$; [compare Eq. (2.6)].

According to Eqs. (2.10) and (2.10'), the value of $\mathcal{D}(\mathcal{G}')$ in renormalizable theories is completely specified by the number and nature of the lines carrying momenta $k_1, \dots, k_M$ and $q_1, \dots, q_K$ in Fig. 2. This means that Fig. 2 can be applied to a collection of Feynman diagrams for the amplitude $\mathcal{A}(q_1, \dots, q_K; r_1, \dots, r_N)$ if all sets of intermediate lines $\{k_1, \dots, k_M\}$ permitted by selection rules are considered.

Now we examine the limit $\eta \rightarrow \infty$ for the CS equation [Eqs. (1.36), (1.42)]

$$\begin{aligned} \partial_G G(\eta q) + \{\text{subtraction terms}\} &= -i\delta G(\Delta; \eta q) \\ \partial_G &= \eta \partial / \partial \eta - \beta \partial / \partial g - \gamma_G \end{aligned} \quad (2.11)$$

for connected amplitudes $G(\eta q)$, $G(\Delta; \eta q)$ at non-exceptional momenta q . Graphs contributing to $G(\eta q)$ do not have external legs with fixed momenta (i.e. $N = 0$ in Fig. 2), so for each graph $\mathcal{G}$, the dominant subgraph is $\mathcal{G}$ itself:

$$G(\eta q) = O[\eta^{\mathcal{D}(\mathcal{G})} \ln^p \eta], \quad [\mathcal{D}(G) = \mathcal{D}(\mathcal{G})] \quad (2.12)$$

The dominant subgraphs^{*} for $G(\Delta)$ are shown in Fig. 6:

$$\begin{aligned} \mathcal{D}(\mathcal{G}'; \text{Fig. 6a}) &= \begin{cases} \mathcal{D}(G) - 2 & (k_1, k_2 = \text{bosons}), \\ \mathcal{D}(G) - 3 & (k_1, k_2 = \text{fermions}); \end{cases} \quad (2.13) \\ \mathcal{D}(\mathcal{G}'; \text{Fig. 6b}) &= \mathcal{D}(G) - \dim \chi_\Delta \end{aligned}$$

^{*}) Graphs with one intermediate boson line and $\mathcal{D}(\mathcal{G}') = \mathcal{D}(G) - 2$ are possible if a ϕ^3 interaction is added to the Lagrangian (1.10) or (1.12).

Here χ_Δ is the source for $\bar{\psi}\psi$ or ϕ^2 at zero momentum:

$$\dim \chi_\Delta = 1 \text{ or } 2 \quad (2.14)$$

Equations (2.13) and (2.14) imply

$$G(\Delta; \eta q) = O[\eta^{D(G)-1} \ln^{\beta'} \eta] \quad (\text{fermions present}), \quad (2.15a)$$

$$G(\Delta; \eta q) = O[\eta^{D(G)-2} \ln^{\beta''} \eta] \quad (\text{no fermions}), \quad (2.15b)$$

so the leading power $G^{\text{as.}}$ of G , defined by

$$G(\eta q) = G^{\text{as.}}(\eta q) + O[\eta^{D(G)-1} \ln^{\beta'''} \eta] \quad (2.16)$$

satisfies the equation^{*)}

$$\partial_G G^{\text{as.}}(\eta q) + \{\text{sub. terms}\}^{\text{as.}} = 0 \quad (2.17)$$

with

$$\begin{aligned} \{\text{subtraction terms}\} &= \sum_i \mathcal{P}_i \mathcal{G}_i \quad [\text{Eq. (1.42)}] \\ &= \{\text{sub. terms}\}^{\text{as.}} + O[\eta^{D(G)-1} \ln^{\beta'''} \eta] \end{aligned} \quad (2.18)$$

In other words, $G(\Delta)$ is asymptotically negligible⁵⁾.

The restriction to non-exceptional momenta²⁵⁾ is essential. For example, apart from a few special cases¹⁾, $G(\Delta)$ is *not* asymptotically negligible if G is an on-shell amplitude. Also, additional mass insertions usually do not reduce the asymptotic behaviour further: in most cases, the amplitudes $G(n \Delta\text{'s})$ and $G(n+1 \Delta\text{'s})$ in the CS equation^{**)}

$$\begin{aligned} \partial_{G(n \Delta\text{'s})} G(n \Delta\text{'s}) + \{\text{sub. terms}\} &= -i\delta G(n+1 \Delta\text{'s}) \\ \partial_{G(n \Delta\text{'s})} &= \partial_G - n\gamma_\Delta \\ \gamma_\Delta &= -\Lambda \partial/\partial \Lambda \ln Z_\Delta(\Lambda/m_R, g_B) \end{aligned} \quad (2.19)$$

have the same asymptotic power $D(G) - 2$. The situation for $G(\Delta\Delta)$ is illustrated in Fig. 7:

*) Unless a selection rule forces it to vanish, $G^{\text{as.}}$ almost invariably receives contributions from the simplest Feynman graphs for G . If desired, the subtraction terms can be removed by considering unordered products instead of the T-product $G_{ABC}^{(b,f)} \dots$ in Eq. (1.42).

**) See Eqs. (1.28) and (1.45), and the addition rule (1.46).

$$\mathcal{D}(\mathcal{G}'; \text{Fig. 7a}) = \begin{cases} \mathcal{D}(G) - 2 & (k_1, k_2 = \text{bosons}), \\ \mathcal{D}(G) - 3 & (k_1, k_2 = \text{fermions}); \end{cases} \quad (2.20)$$

$$\mathcal{D}(\mathcal{G}'; \text{Fig. 7b}) = \mathcal{D}(G) - 2 \dim \chi_\Delta.$$

If Δ is proportional to $\int d^4x \bar{\psi}\psi(x)$, the asymptotic power is reduced from $\mathcal{D}(G) - 1$ for $G(\Delta)$ to $\mathcal{D}(G) - 2$ for $G(\Delta\Delta)$. Otherwise, there is no reduction; clearly, any number of Δ insertions can be attached to the lower blob in Fig. 7a without changing the asymptotic power.

In perturbation theory, the general form of $G^{\text{as.}}$ consistent with Eq. (2.17) is

$$G^{\text{as.}}(\eta) = \eta^{\mathcal{D}(G)} \sum_N g^N \sum_{p=0}^{\ell(N)} \ln^p \eta G_{N,p} \quad (2.21)$$

where $\ell(N)$ is the number of loops in N^{th} order diagrams for the amplitude G . The presence of $\partial/\partial g$ in the CS operator ∂_G means that (2.17) relates the N^{th} order coefficient $G_{N,p}$ to lower-order coefficients. The consequences of this are best illustrated with a simple example, the photon propagator in QED²⁸⁾:

$$D'_{F\mu\nu}(q) = - \frac{i}{q^2} (g_{\mu\nu} - q_\mu q_\nu / q^2) d(q^2/m^2, \alpha) + \{q_\mu q_\nu / q^4 \text{ term}\}, \quad (2.22)$$

$$[\alpha = (\text{charge})^2 / 4\pi].$$

Charge renormalization is simpler than in Eq. (1.16) because gauge invariance implies⁹⁾ $Z_1 = Z_2$:

$$\alpha_R = Z_3 \alpha_B \quad (2.23)$$

It follows that the combination $\alpha d(q^2/m^2, \alpha)$ has no wave-function renormalization, so its CS equation has no γ -function or subtraction terms and the leading asymptotic power $\alpha d^{\text{as.}}(\eta^2 q^2/m^2, \alpha)$ satisfies the equation^{*})

$$[\eta \partial/\partial \eta - \alpha \beta(\alpha) \partial/\partial \alpha] (\alpha d^{\text{as.}})^{-1} = 0 \quad (2.24)$$

with

$$\beta(\alpha) = - \Lambda \partial/\partial \Lambda \ln Z_3(\Lambda/m_R, \alpha_B) \quad (2.25)$$

If the perturbative expansions

$$\begin{aligned} [d^{\text{as.}}(\eta, \alpha)]^{-1} &= \sum_{n=0}^{\infty} \alpha^n a_n(\eta), \quad (a_0(\eta) \equiv 1) \\ \beta(\alpha) &= \sum_{n=1}^{\infty} \alpha^n b_n \end{aligned} \quad (2.26)$$

^{*}) Note that $(1 - d^{-1})$ is the photon self-energy amplitude (1PI).

are substituted in Eq. (2.24), the result is a set of equations

$$\eta \partial / \partial \eta \alpha_{n+1}(\eta) = \sum_{r=0}^n \alpha_r(\eta) (r-1) b_{n-r+1}, \quad (n = 0, 1, 2, \dots) \quad (2.27)$$

which imply that a two-loop calculation²⁹⁾ of $\beta(\alpha)$ ($b_1 = 2/3\pi$, $b_2 = 1/2\pi^2$) is sufficient to determine the leading logarithm in *any* order:

$$\begin{aligned} \alpha_1(\eta) &= -b_1 \ln \eta + \alpha_1(\eta) \\ \alpha_n(\eta) &= -b_2 b_1^{n-2} (\ln \eta)^{n-1} / (n-1) + O[(\ln \eta)^{n-2}], \quad (n \geq 2) \end{aligned} \quad (2.28)$$

Examination of the cut-off dependence of photon self-energy diagrams shows that the n -loop graphs of Fig. 8 (plus suitable counterterm graphs) are responsible for the $(\ln \eta)^{n-1}$ dependence of $\alpha_n(\eta)$ for $n \geq 2$ *).

The next problem is to decide how to sum the logarithms in Eq. (2.21). The leading-logarithm approximation is very easy to analyse, but it usually fails. For example, Eq. (2.28) yields

$$(\alpha d^{as})^{-1} = \{\text{leading logs}\} + \{1^{\text{st}} \text{ non-leading}\} + \dots \quad (2.29)$$

with

$$\{\text{leading logs}\} = \alpha^{-1} \{1 - (2\alpha/3\pi) \ln \eta\} \quad (2.30)$$

$$\{1^{\text{st}} \text{ non-leading}\} = \alpha_1(1) + (3/4\pi) \ln \{1 - (2\alpha/3\pi) \ln \eta\} \quad (2.31)$$

Equation (2.30) is unsatisfactory as an asymptotic approximation for $(\alpha d^{as})^{-1}$ because it implies that αd^{as} has a "ghost" pole³⁰⁾ (i.e. a pole at a space-like momentum point). This is not a genuine difficulty, because the "non-leading" term (2.31) dominates (2.30) in the relevant region ($\ln \eta \sim 3\pi/2\alpha$). In fact, the leading-logarithm approximation works only if η is much smaller:

$$1 - (2\alpha/3\pi) \ln \eta \gg \alpha \quad (2.32)$$

As η increases, more and more non-leading logarithms [i.e. higher orders in the expansion of $\beta(\alpha)$] become important. When η becomes sufficiently large, the perturbation expansion for $\beta(\alpha)$ cannot be truncated at finite order, and all of the logarithms in (2.21) must be summed.

*) The sum of the two-loop graphs goes as $-b_2 \alpha_B^2 \ln \Lambda$, and there is an additional power of $\ln \Lambda$ for each of the $n - 2$ fermion-loop insertions in the internal photon line: Z_3^{-1} (Fig. 8) = $O[(\ln \Lambda)^{n-1}]$. All other subsets of n -loop graphs (with $Z_1 = Z_2$ for each subset) are $O[(\ln \Lambda)^{n-2}]$ so, according to the discussion of Eqs. (2.8) and (2.9), they cannot influence the leading logarithmic behaviour of $\alpha_n(\eta)$.

Further progress depends on the following assumptions:

- I) The theory exists at finite values of the coupling constant g , where $\beta(g)$, $\gamma_G(g)$, $G(g)$, $G^{as.}(g)$, ..., are differentiable functions of g . Perturbation theory is generated by asymptotically expanding these functions about $g = 0$.
- II) At finite g , the leading power of G is $G^{as.}$, as in perturbation theory.

Very little is known about the validity of (I) in non-trivial theories; for example:

- i) Does the radius of convergence of the perturbation series vanish³¹⁾?
- ii) If so, does unitarity (or something equally respectable) uniquely specify the continuation from $g = 0$ to finite g ?
- iii) Are inequivalent theories generated by different orders of summation³²⁾ [e.g. of diagrams, or of terms in (2.21)]?

In general, the status of assumption (II) is equally uncertain. However, for asymptotically free theories⁸⁾, it is not difficult to show⁷⁾ that (II) works, provided that the validity of (I) is assumed.

2.2 Characteristics of solutions

Consider Eq. (2.17) when subtraction terms are absent and there is only one coupling constant g :

$$\left[\eta \partial / \partial \eta - \beta(g) \partial / \partial g - \gamma_G(g) \right] G^{as.}(\eta q; g, m) = 0 \quad (2.33)$$

The general solution of (2.33) is most conveniently written¹⁰⁾

$$G^{as.}(\eta q; g, m) = G^{as.}(q; \bar{g}, m) \exp \int_g^{\bar{g}} dx \gamma_G(x) / \beta(x) \quad (2.34)$$

where the auxiliary function $\bar{g} = \bar{g}(g, \eta)$ (known as the "effective coupling constant") is defined by

$$\int_g^{\bar{g}} dx / \beta(x) = \ln \eta \quad ; \quad \bar{g}(g, 1) = g \quad (2.35)$$

The η -dependence of $G^{as.}(\eta q)$ is entirely contained in the $\bar{g}$ -dependence of the right-hand side of Eq. (2.34). For consistency, we must suppose that the integral $\int dx / \beta(x)$ diverges at least once to $+\infty$ and once to $-\infty$. Otherwise, Eq. (2.35) does not permit the variable η to run freely from 0 (the infrared limit) to ∞ (the ultraviolet limit).

If the divergence in $\int dx / \beta(x)$ is caused by the range of integration becoming infinite,

$$\left| \int_g^{\pm \infty} dx / \beta(x) \right| = \infty \quad (2.36)$$

then Eq. (2.35) implies that $\bar{g}$ diverges as $\ln \eta$ becomes infinite. There is no good argument for supposing that this does not happen in real life. Nevertheless, people usually do not bother with this case because of its lack of predictive power: additional assumptions for the $g \rightarrow \infty$ behaviour of the functions $G^{\text{as.}}(q; g, m)$ must be introduced in order to determine their asymptotic behaviour from Eq. (2.34).

Alternatively, $\beta(x)$ may possess "eigenvalues" or "fixed points" g_∞ ,

$$\beta(g_\infty) = 0 \quad (2.37)$$

which make $\int dx/\beta(x)$ diverge; (e.g. the origin $x = 0$ is a fixed point). According to Eq. (2.35), $\bar{g}$ approaches the nearest fixed point g_∞ as $\ln \eta$ tends to $+\infty$ (ultraviolet-stable fixed point) or to $-\infty$ (infrared-stable fixed point)³³; see Fig. 9. A series of fixed points $g'_\infty, g''_\infty, \dots$ (as in Fig. 10), produces independent regions I, II, III, ..., in coupling-constant space. The effective coupling constant is restricted to the region in which the "physical" coupling constant g lies*). For example, if g lies within region III in Fig. 10, the relevant IR- and UV-stable fixed points are g''_∞ and g'''_∞ :

$$g''_\infty = \lim_{\eta \rightarrow 0} \bar{g} < \bar{g}(g, \eta) < g'''_\infty = \lim_{\eta \rightarrow \infty} \bar{g} \quad (2.38)$$

Generally, IR-stable fixed points are less interesting because the correction $(G - G^{\text{as.}})$ from mass insertions is expected to dominate $G^{\text{as.}}$ when η becomes too small.

It is convenient to discuss the asymptotic properties of Eq. (2.34) for the case $\bar{g} \rightarrow g_\infty$ in terms of the function^{34, 35})

$$\mathcal{E}(\eta) = (\ln \eta)^{-1} \int_g^{\bar{g}} dx \left\{ \gamma_G(x) - \gamma_G(g_\infty) \right\} / \beta(x) \quad (2.39)$$

which is closely related to the exponential factor in (2.34):

$$\exp \int_g^{\bar{g}} dx \gamma_G(x) / \beta(x) = \eta^{\gamma_G(g_\infty)} + \mathcal{E}(\eta) \quad (2.40)$$

A change of integration variable $x \rightarrow v$ in Eq. (2.39), with

$$x = \bar{g}(g, \eta^v), \quad dx/\beta(x) = \ln \eta \, dv$$

yields the formula

$$\mathcal{E}(\eta) = \int_0^1 dv \left\{ \gamma_G[\bar{g}(g, \eta^v)] - \gamma_G(g_\infty) \right\} \quad (2.41)$$

*) If g is exactly equal to g_∞ , Eq. (2.33) can be solved directly:

$$G^{\text{as.}}(\eta q; g_\infty, m) = \eta^{\gamma_G(g_\infty)} G^{\text{as.}}(q; g_\infty, m)$$

From Eq. (2.41), we conclude³⁴⁾

$$\varepsilon(\eta) \longrightarrow 0 \quad , \quad (|\ln \eta| \longrightarrow \infty) \quad (2.42)$$

because $\bar{g}(g, \eta^v)$ tends to g_∞ for $v \neq 0$, and $\gamma_G(x)$ is bounded and continuous [according to assumption (I)].

The asymptotic behaviour of $G^{as.}(\eta q)$ is obtained by expanding $\bar{g}$ about g_∞ in Eq. (2.34). The leading term

$$G^{as.}(\eta q; g, m) \sim \eta^{\gamma_G(g_\infty) + \varepsilon(\eta)} G^{as.}(q; g_\infty, m) \quad (2.43)$$

is almost entirely determined by properties of the theory at the fixed point^{3-5, 33)}. All dependence on the region $g \leq x < g_\infty$ is contained in the factor $\eta^{\varepsilon(\eta)}$, which controls the over-all normalization of the leading term. If the region $x \approx g_\infty$ generates a singularity of the integral in Eq. (2.39), this factor also modifies the leading power $\eta^{\gamma_G(g_\infty)}$ in Eq. (2.43): e.g.,

$$\eta^{\varepsilon(\eta)} \sim \eta^{c/(\ln \eta)^p} \quad \text{or} \quad \eta^{c/(\ln |\ln \eta|)^p} \quad (2.44)$$

The precise form of the modification depends on³⁴⁾:

- i) the strength of the singularity in $\{\gamma_G(x) - \gamma_G(g_\infty)\}/\beta(x)$ as x tends to g_∞ ;
- ii) the rate at which $\bar{g}$ approaches g_∞ . This is controlled by the strength of the zero in $\beta(x)$ at $x = g_\infty$; [see Eq. (2.35)]. For example³²⁾, if g_∞ is an infinite-order zero of $\beta(x)$, $\bar{g}$ approaches g_∞ very slowly, e.g.

$$\bar{g} - g_\infty = O[(\ln |\ln \eta|)^{-p}] \quad , \quad (p > 0)$$

compared with the rate at which $\bar{g}$ tends to a simple zero:

$$\bar{g} - g_\infty = O[\eta^{\beta'(g_\infty)}] \quad , \quad (\beta'(g_\infty) \neq 0) \quad (2.45)$$

Note

- a) If there are several coupling constants $\vec{g}$ as in Eq. (1.37), the effective coupling constant $\vec{g}$ is a vector in coupling-constant space defined by the equations

$$\eta \partial / \partial \eta \vec{g}(\vec{g}, \eta) = \vec{\beta}(\vec{g}) \quad ; \quad \vec{g}(\vec{g}, 1) = \vec{g} \quad (2.35')$$

The exponent in Eq. (2.34) becomes

$$(\ln \eta)^{-1} \int_0^1 dv \gamma_G[\vec{g}(\vec{g}, \eta^v)]$$

and the condition for a fixed point $\vec{g}_\infty$ is

$$\vec{\beta}(\vec{g}_\infty) = 0 \quad (2.37')$$

A fixed point is UV-stable if it attracts $\vec{g}$ from *all* directions in $\vec{g}$ -space as η tends to infinity.

- b) The simplest example of a subtractively renormalized amplitude is the two-point function

$$\int d^4x e^{iq \cdot x} T \langle 0 | J_\mu(x) J_\nu(0) | 0 \rangle^{1PI} = (q_\mu q_\nu - g_{\mu\nu} q^2) \pi(q^2/m^2, g) \quad (2.46)$$

whose absorptive part is measured in the inclusive process $e^+e^- \rightarrow \text{hadrons}$. According to Eq. (1.49), there is no g -dependent γ -function, so the analogue of (2.33) is

$$[\eta \partial / \partial \eta - \beta(g) \partial / \partial g] \pi^{as.}(q^2/m^2, g) + K(g) = 0 \quad (2.47)$$

with solution

$$\pi^{as.}(q^2/m^2, g) = \pi^{as.}(q^2/m^2, \bar{g}) - \int_{\bar{g}}^g dx K(x)/\beta(x) \quad (2.48)$$

2.3 Broken scale invariance

Now we consider the possibility that Green's functions are asymptotically scale-invariant⁴⁾. This means that the leading term (2.43) should be proportional to a pure power $\eta^{\gamma_G(g_\infty)}$ in the UV limit $\eta \rightarrow \infty$.

To ensure that corrections of the type (2.44) are absent, we have to assume that the function $\varepsilon(\eta)$ tends to zero sufficiently rapidly:

$$\varepsilon(\eta) = O[(\ln \eta)^{-1}] \quad (2.49)$$

Referring to Eq. (2.39), we see that the condition for asymptotic scale invariance is^{3,4)}

$$I_G = \int_{\bar{g}}^{g_\infty} dx \{ \gamma_G(x) - \gamma_G(g_\infty) \} / \beta(x) = \text{convergent} \quad (2.50)$$

Equation (2.50) is supposed to be valid for all Green's functions G , so this looks like a very strong assumption. However, if g_∞ is a simple zero of $\beta(x)$,

$$\beta(g_\infty) = 0, \quad \beta'(g_\infty) < 0, \quad (g_\infty = \text{UV-stable}) \quad (2.51)$$

(2.50) is automatically satisfied because assumption (I) says that $\gamma_G(x)$ is differentiable^{*}).

^{*}) Conversely, if g_∞ is not a simple zero of $\beta(x)$, it is unlikely that asymptotic scale invariance can be valid: for every Green's function G , $\{\gamma_G(x) - \gamma_G(g_\infty)\}$ would have to display a sufficiently strong zero at $x = 0$ to compensate for the zero of $\beta(x)$ in Eq. (2.50).

The asymptotic expansion of Eq. (2.34) can now be specified more precisely than in Eqs. (2.42) and (2.43)^{5,18}):

$$G^{as}(\eta q; g, m) = \eta^{\gamma_G(g_\infty)} \exp I_G G^{as}(q; g_\infty, m) + O[\eta^{\{\gamma_G(g_\infty) - |\beta'(g_\infty)|\}}], \quad (\eta \rightarrow \infty) \quad (2.52)$$

Equation (2.52) indicates that non-leading terms decrease as a power $O[\eta^{-|\beta'(g_\infty)|}]$ relative to the leading term. This result is a consequence of Eq. (2.45) and the fact that the most important non-leading contributions arise from $O(\bar{g} - g_\infty)$ terms in the expansions of $G^{as}(q; \bar{g}, m)$ and the exponential factor in Eq. (2.34).

According to assumption (II), contributions to the asymptotic behaviour of $G(\eta q)$ due to mass insertions also decrease as a power $O[\eta^{-p(\Delta)}]$ ($p(\Delta) > 0$) relative to the leading term in the expansion of $G^{as}(\eta q)$. Equation (2.52) and assumption (II) imply the result

$$G(\eta q; g, m) = \eta^{\gamma_G(g_\infty)} \{1 + O(\eta^{-P})\} \exp I_G G^{as}(q; g_\infty, m), \quad (2.53)$$

$$P = \text{Min}\{p(\Delta), |\beta'(g_\infty)|\}$$

Now suppose that G is the complete Green's function $G_{ABC} \dots$, where $A, B, C, \dots$, are renormalized field operators (simple or composite). Equation (2.53) is not directly applicable because of subtractive renormalizations (1.41) induced by the time-ordering operation in

$$G_{ABC} \dots \delta^4(q_1 + q_2 + \dots) = \int d^4x_1 d^4x_2 \dots \exp i(q_1 \cdot x_1 + q_2 \cdot x_2 + \dots) T \langle 0 | A(x_1) B(x_2) C(x_3) \dots | 0 \rangle \quad (2.54)$$

However, this problem can be trivially circumvented by considering the unordered product instead. According to the addition rule (1.46) and assumption (I), the characteristic power $\gamma_G(g_\infty)$ in Eq. (2.53) is given by the rule

$$\gamma_G(g_\infty) - \gamma_G(0) = \gamma_A(g_\infty) + \gamma_B(g_\infty) + \dots \quad (2.55)$$

Thus each operator Q contributes $\eta^{\gamma_Q(g_\infty)}$ to the asymptotic behaviour of G . In coordinate space, the result is

$$\langle 0 | A(\rho x_1) B(\rho x_2) C(\rho x_3) \dots | 0 \rangle \sim \rho^{-(d_A + d_B + d_C + \dots)} f(x_1, x_2, x_3, \dots), \quad (\rho = \eta^{-1} \rightarrow 0). \quad (2.56)$$

where

$$d_Q = \{\text{canonical dimension of } Q\} + \gamma_Q(g_\infty) \quad (2.57)$$

is the *dynamical dimension*^{4,33)} of the operator Q . Equation (2.56) says that dynamical dimension is an additively conserved quantity at short distances. This rule, abstracted from the preceding field-theoretic machinery, is the basis for Wilson's theory of broken scale invariance⁴⁾.

If Q is a conserved operator, the "anomalous" term $\gamma_Q(g_\infty)$ vanishes because of Eq. (1.49): for example,

$$d_{J_\mu} = 3, \quad d_{\theta_{\mu\nu}} = 4 \quad (2.58)$$

In particular, if J_μ is the electromagnetic current for hadrons, the two-point function $\langle 0 | J_\mu J_\nu | 0 \rangle$ obeys the familiar condition

$$\langle 0 | J_\mu(x) J_\nu(0) | 0 \rangle \sim (R/12\pi^4) (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) (x^2 - i\epsilon x_0)^{-2}, \quad (2.59)$$

$$(x_\alpha \rightarrow 0),$$

so the total cross-section for $e^+e^- \rightarrow$ hadrons is predicted to be asymptotically scale-invariant³⁶⁾:

$$\sigma(e^+e^- \rightarrow \text{hadrons}) / \sigma^{\text{as.}}(e^+e^- \rightarrow \mu^+\mu^-) \rightarrow R, \quad (q^2 \rightarrow \infty). \quad (2.60)$$

Here q is the sum of the e^+ and e^- momenta, and

$$\sigma^{\text{as.}}(e^+e^- \rightarrow \mu^+\mu^-) = 4\pi\alpha^2/3q^2$$

is a convenient normalizing factor. Asymptotic corrections to Eq. (2.60) are $O[(q^2)^{-P/2}]$, where P is the power defined in Eq. (2.53). [In fact, Eqs. (2.59) and (2.60) are valid for any UV-stable fixed point g_∞ ; if g_∞ is not a simple zero of $\beta(x)$, the corrections to (2.60) decrease logarithmically.]

A practical difficulty of this theory is that we do not know how to compute quantities like R , because they are characteristic of the non-trivial interacting theory at $g = g_\infty$.

2.4 Asymptotic freedom

A theory is asymptotically free^{8,37)} if

- i) the fixed point at the origin is UV-stable: for each coupling constant g_i , the condition

$$g_i \beta_i(g_1, g_2, \dots) < 0, \quad (i = 1, 2, \dots) \quad (2.61)$$

is satisfied in the neighbourhood of the origin $\vec{g} = 0$.

- ii) the value of $\vec{g}$ is chosen such that, as η increases from 1 to ∞ , $\vec{g}$ describes a path linking $\vec{g}$ with the origin:

$$\vec{g} \rightarrow 0, \quad (\eta \rightarrow \infty). \quad (2.62)$$

Equation (2.62) is not an automatic consequence of (2.61) because it assumes the absence of barriers such as non-trivial fixed points which can prevent $\vec{g}$ from reaching points close to the origin. The important feature of these theories is that most of the terms in Eq. (2.43) can be computed explicitly. In particular, $G^{as.}(q; g_\infty, m)$ and $\gamma_G(g_\infty)$ are trivially given by free-field theory because the relevant fixed point g_∞ vanishes.

Checking Eq. (2.61) is just a matter of computing one-loop^{*)} contributions to $\beta(g)$ in perturbation theory. Most theories do not obey Eq. (2.61): in QED, the coefficient b_1 of the one-loop term in (2.26) is positive ($b_1 = + 2/3\pi$), and more generally³⁸⁾, a four-dimensional renormalizable field theory cannot be asymptotically free unless non-Abelian gauge mesons are present. Furthermore, Eq. (2.61) is satisfied only by special classes of gauge theories. The simplest and most interesting case^{8, 39)} involves massless gauge fields A_μ^a interacting with themselves and with fermions ψ^i belonging to a representation R of the gauge group G . For applications to strong interactions^{1, 8, 39)} the fermions are supposed to be current quarks⁴⁰⁾ distinguished by properties of "colour" and "flavour":

$$\{\psi^i\} = \begin{pmatrix} u_1 & d_1 & s_1 & c_1 & \dots \\ u_2 & d_2 & \dots & \dots & \\ \vdots & & & & \end{pmatrix} \quad (2.63)$$

The flavours $u, d, s, c, \dots$ (= up, down, strange, charmed, ... quarks) are gauge-invariant and transform under observable symmetry groups G_{obs} such as chiral $SU(3) \times SU(3)$. Each flavour carries a colour index K which is transformed by G but not by G_{obs} (e.g. $u \rightarrow u_K$, $K = 1, 2, \dots$). For example⁴⁰⁾, if $K = 1, 2, 3$ refers to the fundamental representation $\underline{3}$ of $G = SU(3)$, the complete fermionic representation R is a direct sum:

$$R = \underline{3}_u + \underline{3}_d + \underline{3}_s + \dots \quad (2.64)$$

The Feynman rules of the theory involve a coupling constant g , the structure constants c^{abc} of the gauge group G , and the matrix generators τ^a of G for the representation R . If necessary, the fermions can be provided with a gauge-invariant mass matrix $\mathcal{M}$ in flavour space to ensure that symmetries like chiral $SU(3) \times SU(3)$ are softly broken. Propagators and vertices are generated by the Lagrangian^{**)}

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\Psi}(i\not{D}_f - \mathcal{M})\Psi + \mathcal{L}_{g.f.} + \mathcal{L}_{ghost} \quad (2.65)$$

*) Two-loop, if the one-loop terms accidentally vanish.

**) For background, consult review articles⁴¹⁻⁴³⁾ on the quantization and renormalization of gauge theories.

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g c^{abc} A_\mu^b A_\nu^c \quad (2.66)$$

is the gauge-covariant field-strength tensor, and

$$(D_f)_\mu = \partial_\mu - ig A_\mu^a \tau^a \quad (2.67)$$

is the gauge-covariant derivative for the fermionic representation R. The terms F^2 , $\bar{\psi} \not{D}_f \psi$ and $\bar{\psi} M \psi$ are invariant under infinitesimal gauge transformations

$$\begin{aligned} \delta A_\mu^a &= -D_\mu^{ab} \delta \omega^b + O(\delta \omega^2) \\ \delta \psi &= -ig \delta \omega^a \tau^a \psi + O(\delta \omega^2) \end{aligned} \quad (2.68)$$

where

$$D_\mu^{ab} = \partial_\mu \delta^{ab} - g c^{abc} A_\mu^c \quad (2.69)$$

is the covariant derivative for the adjoint representation of G, and $\delta \omega^a(x)$ are arbitrary non-singular functions of the coordinate x. The term $\mathcal{L}_{g.f}$ in Eq. (2.65) specifies a gauge for which the propagator of A_μ^a is well defined; e.g. a Fermi-type gauge-fixing term

$$\mathcal{L}_{g.f.} = -\frac{1}{2\xi} (\partial^\mu A_\mu^a)^2 \quad (2.70)$$

generates the propagator

$$\overline{A_\mu^a} A_\nu^b = -\frac{i\delta^{ab}}{k^2} \left\{ g_{\mu\nu} - (1-\xi) k_\mu k_\nu / k^2 \right\} \quad (2.71)$$

Unphysical contributions of $\mathcal{L}_{g.f.}$ to loop integrals must be cancelled by including a ghost Lagrangian^{44, 45)} $\mathcal{L}_{ghost}$ in Eq. (2.65). The choice of $\mathcal{L}_{g.f}$ in Eq. (2.70) produces a ghost Lagrangian

$$\mathcal{L}_{ghost} = (\partial^\mu \phi^a)^* D_\mu^{ab}(A) \phi^b \quad (2.72)$$

where the ghost field ϕ^a is a Lorentz scalar with Fermi-Dirac statistics. Equations (2.65) to (2.72) refer to the unrenormalized (B) representation.

One way of computing coupling-constant renormalization is to compare $O(A^2)$ and $O(A^3)$ terms in $\mathcal{L}$. Wave-function renormalization

$$(A_\mu^a)_R = Z_3^{-1/2} (A_\mu^a)_B \quad (2.73)$$

and a rescaling of the gauge parameter^{*)}

$$\xi_R = Z_3^{-1} \xi_B \quad (2.74)$$

*) The scaling factor in (2.74) is Z_3^{-1} because Ward identities forbid $(\partial \cdot A)^2$ counterterms⁴⁵⁾: i.e. $\xi_B^{-1} (\partial \cdot A)_B^2 = \xi_R^{-1} (\partial \cdot A)_R^2$. Because of (2.74), the CS differential operator ∂_G for ξ -dependent Green's functions G contains an additional term $-2\gamma(g, \xi) \xi \partial / \partial \xi$.

account for the $O(A^2)$ terms. Three-meson 1PI divergences produce a Z_1 -factor [similar to that in Eq. (1.10)]

$$-\frac{1}{2} g_B c^{abc} (\partial^\mu A^\nu - \partial^\nu A^\mu)_B^a (A_\mu^b A_\nu^c)_B = -\frac{1}{2} g_R Z_1 c^{abc} (\partial^\mu A^\nu - \partial^\nu A^\mu)_R^a (A_\mu^b A_\nu^c)_R \quad (2.75)$$

so coupling-constant renormalization is given by

$$g_R = Z_3^{3/2} Z_1^{-1} g_B \quad (2.76)$$

Note

Instead of (2.75), we could have considered one of the other g -dependent vertices in $\mathcal{L}$: $O(A^4)$, $O(\bar{\psi}A\psi)$, or $O(\phi^*A\phi)$. For example, if $\tilde{Z}_1, \tilde{Z}_3$ are the renormalization factors for ghost amplitudes,

$$\phi_R = \tilde{Z}_3^{-1/2} \phi_B$$

$$g_B c^{abc} (\partial^\mu \phi^a)_B^* (A_\mu^b \phi^c)_B = g_R \tilde{Z}_1 c^{abc} (\partial^\mu \phi^a)_R^* (A_\mu^b \phi^c)_R \quad (2.77)$$

the result is

$$g_R = \tilde{Z}_1^{-1} Z_3^{1/2} \tilde{Z}_3 g_B \quad (2.78)$$

As long as the regularization procedure^{46,47)} is chosen such that gauge Ward identities^{43,48-50)} are satisfied, the consistency of equations such as (2.76) and (2.78) is assured⁴⁸⁾:

$$Z_1/\tilde{Z}_1 = Z_3/\tilde{Z}_3 \quad (2.79)$$

A computation of the one-loop contributions to g_R/g_B yields (after a bit of algebra) the basic result^{8,39)}

$$\beta(g) = -bg^3 + O(g^5)$$

$$b = \left\{ \frac{11}{3} C_2(G) - \frac{4}{3} T(R) \right\} / 16\pi^2 \quad (2.80)$$

with

$$C_2(G) \delta^{ab} = c^{acd} c^{bcd}, \quad (C_2(G) > 0) \quad (2.81)$$

$$T(R) \delta^{ab} = \text{Tr}\{\tau^a \tau^b\}, \quad (T(R) \gg 0) \quad (2.82)$$

Hence Eq. (2.61) ($b > 0$) is satisfied if there are not too many fermions in the theory. The rate at which $\bar{g}$ approaches the origin is a direct consequence of Eqs. (2.35) and (2.80):

$$\bar{g}^2 = (2b \ln \eta)^{-1} + O[\ln \ln \eta / (\ln \eta)^2], \quad (\eta \rightarrow \infty). \quad (2.83)$$

Most Green's functions $G(\eta q)$ are not asymptotically scale-invariant because the one-loop term $c_G g^2$ in the perturbative expansion

$$\gamma_G(g) - \gamma_G(0) = c_G g^2 + O(g^4) \quad (2.84)$$

causes the integral

$$\int dx (\gamma_G(x) - \gamma_G(0)) / \beta(x)$$

in Eq. (2.50) to diverge logarithmically at $x = 0$. The rate at which the additional power $\varepsilon(\eta)$ [Eq. (2.39)] approaches zero is governed by the equation

$$\begin{aligned} \mathcal{J}_G &\equiv \int_g^0 dx [(\gamma_G(x) - \gamma_G(0)) / \beta(x) + c_G / bx] \\ &= \lim_{\eta \rightarrow \infty} [\varepsilon(\eta) \ln \eta + (c_G / b) \ln(\bar{g} / g)] \end{aligned} \quad (2.85)$$

i.e. $\varepsilon(\eta)$ decreases more slowly than $(\ln \eta)^{-1}$:

$$\varepsilon(\eta) \sim (c_G / 2b) \ln \ln \eta / \ln \eta$$

This means that the leading singularity of $G^{as.}(\eta q)$ differs from the singularity $\eta^{\gamma_G(0)}$ observed in free-field theory; there exist logarithmic modifications of the form^{8,39)} $(\ln \eta)^{c_G/2b}$:

$$G^{as.}(\eta q; g, m) = \eta^{\gamma_G(0)} (2b g^2 \ln \eta)^{c_G/2b} G^{as.}(q; 0, m) \exp \mathcal{J}_G \left\{ 1 + O\left(\frac{\ln \ln \eta}{\ln \eta}\right) \right\} \quad (2.86)$$

This is a non-perturbative result. It depends on the validity of assumption (I), because the factor $\exp \mathcal{J}_G$ depends on the values of $\beta(x)$ and $\gamma_G(x)$ in the region $0 < x < g$. Also, a $(\ln \eta)^{c_G/2b}$ singularity cannot be generated by finite-order perturbation theory for $G(\eta q)$, because in general, the power $c_G/2b$ is not an integer.

Of course, if there is no wave-function renormalization for G [i.e. $\gamma_G(x) \equiv \gamma_G(0)$], the leading singularity is the same as in free-field theory (parton model). In particular, the ratio

$$\begin{aligned} R(q^2/m^2; g) &= \sigma(e^+e^- \rightarrow \text{hadrons}) / \sigma^{as.}(e^+e^- \rightarrow \mu^+\mu^-) \\ &= R^{as.}(q^2/m^2; g) + \{\text{non-leading powers}\} \end{aligned} \quad (2.87)$$

tends to the asymptotic value^{8,39)}

$$R = \sum_{\substack{\text{flavours} \\ \text{colours}}} Q^2, \quad (Q = \text{quark charge}) \quad (2.88)$$

It is not difficult to find the leading asymptotic correction to this result. The two-loop contribution $R^{as.}(q^2/m^2; g)$ is $3g^2 T(R) / 16\pi^2$ relative to the one-loop contribution $\sum Q^2$; [the factor $3g^2 / 16\pi^2$ can be extracted from the Jost-Luttinger calculation²⁹⁾ for QED]. So, if the right-hand side of the equation

$$R^{as.}(\eta^2 q^2/m^2; g) = R^{as.}(q^2/m^2; \bar{g}) \quad (2.89)$$

is expanded in $\bar{g}$ about $\bar{g} = 0$, the result is⁵¹⁾

$$R^{as.}(\eta^2 q^2/m^2; g) = \sum Q^2 \{1 + 3T(R)\bar{g}^2/16\pi^2 + O(\bar{g}^4)\}, (\eta \rightarrow \infty). \quad (2.90)$$

Thus the first non-leading term is positive and decays logarithmically. This result illustrates the difference between asymptotic freedom and asymptotic free-field behaviour; in free-field theory, the leading correction is negative and approaches zero very rapidly: $O(\eta^{-4})$.

Asymptotic freedom allows us to make very precise statements about amplitudes which involve a *short-distance limit*. Unfortunately the reverse is true elsewhere, especially for the problem of deciding whether the theory possesses a respectable S-matrix or not. Most Yang-Mills theories [e.g. Eq. (2.65)] do not possess a perturbative S-matrix because of infrared singularities at thresholds for the production of massless gauge mesons, and in any case, we do not want massless states to appear in the hadronic spectrum. The perturbative solution to this problem is to break the gauge symmetry spontaneously, i.e. to introduce enough Higgs-Kibble⁵²⁾ scalar fields Φ to make all of the gauge mesons massive. The presence of Φ^4 couplings means that Eq. (2.61) is much more difficult to satisfy^{39, 53)}. Nevertheless, asymptotically free theories with scalar mesons do exist⁵⁴⁾ and in particular, there are models⁵⁵⁾ in which all perturbative states are massive. An immediate reason for not pursuing this line further is that it involves an unrealistic perturbative constraint. The spectrum of a summed-up theory is unlikely to bear much resemblance to the perturbative spectrum^{*}), and hadronic states are definitely not perturbative.

Another approach is to assume that there is a dynamical mechanism which breaks gauge invariance spontaneously⁵⁸⁾. There are no Φ fields; instead, the infrared singularities of perturbation theory are supposed to sum to scalar meson poles which simulate the Higgs-Kibble mechanism in a non-perturbative manner. Dynamical generation of mass from Lagrangians which contain no dimensionful parameters has been exhibited⁵⁹⁾ for four-dimensional scalar QED and some two-dimensional models.

However, the most likely possibility is that the strong interaction gauge symmetry (colour) is exact^{39, 60)}. One would break gauge invariance spontaneously only if it were desirable to produce a hadronic spectrum with lots of non-degenerate coloured states, including states with quark-like quantum numbers. Note that

*) A variety of classes of graphs can be summed to produce non-perturbative states. The traditional bound-state picture⁵⁶⁾ typically involves the ladder approximation for a scattering amplitude. The semi-classical approaches⁵⁷⁾ which are currently fashionable treat observable states as coherent superpositions of infinitely many perturbative states.

isospin and ordinary SU(3) have nothing to do with breaking a strong gauge group, because they are approximate degeneracy symmetries. Similarly, observable hadronic currents j_μ (electromagnetic, chiral, etc.) are colour-invariant; otherwise, instead of current conservation or partial conservation, we would have

$$D^{ab} \cdot j^b = \text{zero or soft operator} \quad (2.91)$$

where D^{ab} is the appropriate covariant derivative (2.67) with strong coupling constant g .

So we return to the colour-flavour model mentioned at the beginning of this section. All flavours, observable operators and observable states are colour singlets. Perturbative amplitudes become infrared singular either as some of the external momenta are taken on-shell, or as a partial sum of the external momenta becomes light-like. We are better off *with* the singularities because then we can at least contemplate the possibility⁶⁰⁾ that these infrared effects, summed to all orders in perturbation theory, confine quarks, gluons and coloured "bound states" (constituent quarks, etc.). The precise criterion for this is poorly understood, but there are some hopeful signs:

- i) The Bloch-Nordsieck solution⁶¹⁾ of the infrared problem in QED, which results in the existence of observable photon and fermion states, fails when applied to Yang-Mills theories^{62, 63)}.
- ii) The lattice approximation⁶⁴⁾ for the colour-flavour model indicates quark confinement.

It is surprising that perturbative infrared singularities in *gauge-invariant* channels receive so little attention, because there we *do* have a precise criterion: observable amplitudes must not be infrared singular. In other words, when summed to all orders, the singularities should conspire to shift all gauge-invariant thresholds away from zero mass. Gauge-invariant singularities occur even if the fermions in Eq. (2.65) are all massive ($\mathcal{M} \neq 0$). For example, consider the stress-energy tensor

$$\begin{aligned} \theta_{\mu\nu} = & - F_{\mu\alpha} F_{\nu}^{\alpha} + g_{\mu\nu} F^2/4 + \frac{i}{4} \bar{\Psi} (\gamma_{\mu} \overleftrightarrow{D}_{\nu} + \gamma_{\nu} \overleftrightarrow{D}_{\mu}) \Psi \\ & - g_{\mu\nu} \bar{\Psi} (i \not{D}_f - \mathcal{M}) \Psi + \{\text{renormalization counterterms}\} \end{aligned} \quad (2.92)$$

The diagram shown in Fig. 11a is responsible for the singularity

$$\begin{aligned} i \int d^4x e^{iq \cdot x} T \langle 0 | \theta_{\alpha\beta}(x) \theta_{\gamma\delta}(0) | 0 \rangle \sim & - (d(G)/60\pi^2) q_{\alpha} q_{\beta} q_{\gamma} q_{\delta} \ln q^2 \\ & + O(q^2), \quad (q^2 \rightarrow 0) \end{aligned} \quad (2.93)$$

where $d(G)$ is the number of generators of the gauge group G . More generally, amplitudes $T \langle 0 | 0_1 0_2 0_3 \dots | 0 \rangle$ involving gauge-invariant operators 0_i can be

infrared singular whenever the diagrams of Fig. 11b (intermediate gluon or ghost-antighost pair) are permitted by selection rules. Of course, additional singularities appear if some of the fermions are massless.

Note that observable amplitudes must be regular (not merely non-singular) at light-like momentum transfers. For example, if Q is the gauge-invariant operator

$$Q(x) = \varepsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta} + \{\text{renormalization counterterms}\} \quad (2.94)$$

there is a non-singular branch cut at $q^2 = 0$ in lowest-order perturbation theory for the two-point function

$$i \int d^4x e^{iq \cdot x} T \langle 0 | Q(x) Q(0) | 0 \rangle \sim - (2d(G)/\pi^2) q^4 \ln q^2 + O(q^2), \quad (q^2 \rightarrow 0) \quad (2.95)$$

The theory makes sense only if branch cuts like $q^4 \ln q^2$ disappear when the sum to all orders in perturbation theory is performed. This means that a complete analysis of the infrared problem involves non-singular graphs with more than two intermediate gluons or ghosts (Fig. 11c), as well as the diagrams of Fig. 11b.

The non-perturbative nature of the problem can be readily appreciated by considering the extreme case $\mathcal{M} = 0$ in (2.65), as advocated by Gross and Neveu⁵⁹⁾ [but without breaking the colour symmetry^{*})]. The only dimensional parameter in the theory is the renormalization subtraction point λ . Amplitudes obey equations such as (2.33) and (2.48) exactly (with λ replacing m), so the infrared behaviour for the limit in which all external momenta approach zero can be discussed in terms of a non-trivial infrared-stable fixed point g_∞ (possibly at infinity). For example, consider Eq. (2.48), (which is also valid for form factors of $T(\theta_{\alpha\beta}^{\theta\gamma\delta})$). The criterion is that

$$\lim_{\eta \rightarrow 0} \pi(\eta^2 q^2 / \lambda^2; g)_{\mathcal{M}=0} = \pi(q^2 / \lambda^2; g_\infty)_{\mathcal{M}=0} - \int_g^{g_\infty} dx \frac{K(x)}{\beta(x)} \quad (2.96)$$

should be finite with zero absorptive part, so $K(x)$ must approach zero sufficiently strongly as $x \rightarrow g_\infty$ to make $\int^{g_\infty} dx K(x)/\beta(x)$ converge.

The case $\mathcal{M} \neq 0$ is more complicated. Apart from the lattice approximation, the main lines of analysis will probably involve:

- i) theorems^{63, 65)} that infrared singularities of amplitudes are given by the theory with massive fields omitted (except as external lines), up to finite renormalizations;
- ii) the improved CS equation⁷⁾, to be considered in the next chapter.

^{*}) In order to generate ordinary SU(3) breaking for the $\mathcal{M} = 0$ case, it must be supposed that there exist inequivalent methods of summing the theory, and that one of these methods is SU(3)-asymmetric.

3. OTHER METHODS

3.1 Gell-Mann -- Low analysis³⁾

Let us return to the subject of coupling-constant renormalization in quantum electrodynamics.

The coupling constant α used in practical calculations ($\alpha^{-1} = 137.036, \dots$) refers to the amplitude for an electron to emit a zero-frequency photon. According to Eq. (2.23), charge renormalization is controlled by the wave-function renormalization of the photon propagator

$$D'_{F\mu\nu}(q) = -\left(\frac{i}{q^2}\right) (g_{\mu\nu} - q_\mu q_\nu / q^2) d(q^2/m^2, \alpha) - i\xi q_\mu q_\nu / q^4 \quad (3.1)$$

so the choice of α as coupling constant corresponds to the normalization condition

$$d(0, \alpha) = 1 \quad (3.2)$$

for $D'_{F\mu\nu}(q)$ at $q^2 = 0$.

Consider another renormalization procedure in which the coupling constant α_λ is given by the amplitude for the emission of a virtual photon with space-like momentum λ . Now the photon propagator $(D'_{F\mu\nu})_\lambda$ depends on q , λ , m , and α_λ ,

$$\alpha_\lambda (D'_{F\mu\nu})_\lambda = -\left(\frac{i}{q^2}\right) (g_{\mu\nu} - q_\mu q_\nu / q^2) D(q^2/\lambda^2, m^2/\lambda^2, \alpha_\lambda) - i\alpha_\lambda \xi_\lambda q_\mu q_\nu / q^4 \quad (3.3)$$

and is normalized at $q^2 = \lambda^2$:

$$D(1, m^2/\lambda^2, \alpha_\lambda) = \alpha_\lambda \quad (3.4)$$

When the renormalization prescription is altered by changing λ , the normalization of $(D'_{F\mu\nu})_\lambda$ changes by a factor λ_2^{-1} [as in Eq. (1.21)], but the combination $\alpha_\lambda (D'_{F\mu\nu})_\lambda$ is invariant:

$$D(q^2/\lambda_1^2, m^2/\lambda_1^2, \alpha_{\lambda_1}) = D(q^2/\lambda_2^2, m^2/\lambda_2^2, \alpha_{\lambda_2}) \quad (3.5)$$

$$= \alpha d(q^2/m^2, \alpha) \quad (3.6)$$

For $q^2 = \lambda_2^2$, Eqs. (3.5) and (3.6) become

$$\alpha_{\lambda_2} = D(\rho, m^2/\lambda^2, \alpha_\lambda) \quad (3.7)$$

$$= \alpha d(\rho\lambda^2/m^2, \alpha) \quad (3.8)$$

where ρ is a convenient scale parameter:

$$\lambda_2 = \lambda_1 \sqrt{\rho}, \quad \lambda_1 = \lambda \quad (3.9)$$

The result of substituting (3.7) into (3.5) is a functional equation for the renormalization-group transformation^{*}) $\lambda \rightarrow \lambda\sqrt{\rho}$:

$$D(q^2/\lambda^2, m^2/\lambda^2, \alpha_\lambda) = D(q^2/\rho\lambda^2, m^2/\rho\lambda^2, D(\rho, m^2/\lambda^2, \alpha_\lambda)) \quad (3.10)$$

Application of $\partial/\partial\rho$ to Eq. (3.10) at $\rho = 1$ produces the renormalization-group equation

$$[X\partial/\partial X + M\partial/\partial M - \psi(\alpha_\lambda, M)\partial/\partial\alpha_\lambda] D(X, M, \alpha_\lambda) = 0 \quad (3.11)$$

with $X = q^2/\lambda^2$, $M = m^2/\lambda^2$, and

$$\psi(\alpha_\lambda, M) = \left[\partial D(\rho, M, \alpha_\lambda) / \partial \rho \right]_{\rho=1} \quad (3.12)$$

In general, Eq. (3.11) cannot be integrated because ψ depends on two variables. However, GML³⁾ observed that all renormalized Feynman integrals remain convergent when M is set equal to zero:

$$m = 0, \quad q^2 \neq 0, \quad \lambda^2 \neq 0. \quad (3.13)$$

For example, the subtracted integral

$$I(q, \lambda, m) = \int d^4p (p^2 - m^2)^{-1} \{ [(p+q)^2 - m^2]^{-1} - [(p+\lambda)^2 - m^2]^{-1} \} \quad (3.14)$$

remains both infrared and ultraviolet convergent when the condition (3.13) is imposed. For the general case, ultraviolet power counting is not affected by setting m to zero, but infrared convergence has to be checked, either by inspection³⁾ or by appealing to Kinoshita's theorem^{1, 67, 68)}. The result is an integrable equation of the same form as Eq. (2.33),

$$[X\partial/\partial X - \psi(\alpha_\lambda)\partial/\partial\alpha_\lambda] D(X, 0, \alpha_\lambda) = 0 \quad (3.15)$$

where

$$\psi(z) = \psi(z, 0) \quad (3.16)$$

is known as the Gell-Mann/Low function.

Instead of analysing $D(X, 0, \alpha_\lambda)$ further, let us make use of the connection (3.6)-(3.8) between $D(X, M, \alpha_\lambda)$ and the conventionally renormalized theory. Thus Eq. (3.12) can be rewritten

$$\begin{aligned} \psi(\alpha d(\lambda^2/m^2, \alpha), m^2/\lambda^2) &= \left[(\partial/\partial\rho) \alpha d(\rho\lambda^2/m^2, \alpha) \right]_{\rho=1} \\ &= \partial/\partial \ln(-\lambda^2) [\alpha d(\lambda^2/m^2, \alpha)] \end{aligned} \quad (3.17)$$

Order-by-order in perturbation theory, the leading power in m^2 at $m = 0$ can be isolated by substituting the expansions

^{*}) The connection between the GML analysis and the work of Stueckelberg and Petermann²⁾ was noted by Bogoliubov and Shirkov^{28, 66)}.

$$\psi(z, m^2/\lambda^2) = \psi(z) + O[m^2 \{\log s \text{ of } m\}] \quad (3.18)$$

$$\alpha d(\lambda^2/m^2, \alpha) = \alpha d^{as.}(\lambda^2/m^2, \alpha) + O[m^2 \{\log s \text{ of } m\}] \quad (3.19)$$

Equation (3.18) depends on the existence of the $M \rightarrow 0$ limit (3.13), while (3.19) is just a special case of (2.16); [i.e. $\alpha d^{as.}$ is the amplitude which appears in Eqs. (2.24) and (2.26)]. The result is an integrable equation

$$\psi(\alpha d^{as.}(\lambda^2/m^2, \alpha)) = \partial/\partial \ln(-\lambda^2) [\alpha d^{as.}(\lambda^2/m^2, \alpha)] \quad (3.20)$$

with solution

$$\ln(-\lambda^2/m^2) = \int_{\alpha d^{as.}(-1, \alpha)}^{\alpha d^{as.}(\lambda^2/m^2, \alpha)} \frac{dz}{\psi(z)} \quad (3.21)$$

The analysis of the GML equation (3.21) is very similar to that of Eq. (2.35). The sign of the first term in the perturbative expansion^{3, 69)}

$$\psi(z) = z^2/3\pi + z^3/4\pi^2 + z^4(\zeta(3) - \frac{101}{96})/3\pi^3 + O(z^5) \quad (3.22)$$

implies that $z = 0$ is an infrared-stable fixed point. If there is an ultraviolet-stable fixed point α_0 ,

$$\psi(\alpha_0) = 0, \quad (\alpha < \alpha_0) \quad (3.23)$$

it follows from (3.21) that α_0 is the asymptotic charge of the photon propagator at short distances:

$$\alpha d^{as.}(\lambda^2/m^2, \alpha) \longrightarrow \alpha_0, \quad (-\lambda^2 \rightarrow \infty) \quad (3.24)$$

If $\psi(z)$ does not vanish for $z > 0$, we must suppose that $\int dz/\psi(z)$ diverges at $z = \infty$:

$$\alpha d^{as.}(\lambda^2/m^2, \alpha) \longrightarrow \infty, \quad (-\lambda^2 \rightarrow \infty) \quad (3.25)$$

Otherwise the theory is inconsistent because of the presence of a tachyonic singularity in $\alpha d^{as.}(\lambda^2/m^2, \alpha)$ at

$$(\lambda^2)_{tach.} = -m^2 \exp \int_{q(\alpha)}^{\infty} dz/\psi(z) \quad (3.26)$$

[where $q(\alpha)$ is shorthand for $\alpha d^{as.}(-1, \alpha)$]. It is now obvious why the "ghost" (tachyon) problem³⁰⁾ discussed previously [(Eq. (2.30)] is not a genuine difficulty: it is not reasonable to assume that the small- z behaviour $z^2/3\pi$ is a good approximation for $\psi(z)$ at large z .

The connection between $\psi(z)$ and the β -function (2.25) is found by applying the CS differential operator

$$[2 \partial/\partial \ln(-\lambda^2) - \alpha \beta(\alpha) \partial/\partial \alpha]$$

to Eq. (3.21)³²⁾:

$$\begin{aligned} \psi(q(\alpha)) &= \frac{1}{2} \alpha \beta(\alpha) dq(\alpha)/d\alpha \\ q(\alpha) &= \alpha d^{as.}(-1, \alpha) \end{aligned} \quad (3.27)$$

For conventional on-shell renormalization, the perturbative expansions of $q(\alpha)$ and $\beta(\alpha)$ are^{3, 29)}

$$\begin{aligned} q(\alpha) &= \alpha - 5\alpha^2/9\pi + \left(\zeta(3) + \frac{65}{648}\right) \alpha^3/\pi^2 + O(\alpha^4) \\ \beta(\alpha) &= 2\alpha/3\pi + \alpha^2/2\pi^2 - 121\alpha^3/144\pi^3 + O(\alpha^4) \end{aligned} \quad (3.28)$$

A feature of the GML formulation is that $\psi(z)$ is a universal function. Unlike $\beta(z)$, it does not depend on the renormalization prescription λ . Adler's proposal³²⁾ for computing the fine-structure constant

$$\beta(\alpha_\infty) = 0, \quad (\alpha_\infty^{-1} = 137.036 \dots ??) \quad (3.29)$$

illustrates this point. Equation (3.29) involves the β -function for on-shell renormalization. This makes sense because the fine-structure constant is the on-shell coupling constant. A different renormalization procedure would produce a different β -function [$O(\alpha^3)$ terms and higher] with an eigenvalue numerically different from α_∞ . On the other hand, the corresponding asymptotic charge

$$\alpha_0 = q(\alpha_\infty) \quad (3.30)$$

is renormalization-group invariant. It does not refer to a particular point in λ -space, so apart from the constraint $\alpha < \alpha_0$, its value has nothing to do with the value of the fine-structure constant.

To obtain renormalization-group equations for multiplicatively renormalized amplitudes G , one can use the fact that the ratio

$$G(p^2/\lambda^2, m^2/\lambda^2, \alpha_\lambda) / G(q^2/\lambda^2, m^2/\lambda^2, \alpha_\lambda)$$

is λ -independent for arbitrary sets of external momenta p and q . For example, the analogue of Eq. (3.11) is

$$[X \partial/\partial X + M \partial/\partial M - \psi(\alpha_\lambda, M) \partial/\partial \alpha_\lambda - \chi(\alpha_\lambda, M)] G(X, M, \alpha_\lambda) = 0 \quad (3.31)$$

$$\chi(\alpha_\lambda, M) = - \left[(\partial/\partial \rho) \ln G(\rho^{-1}, M/\rho, D(\rho, M, \alpha_\lambda)) \right]_{\rho=1} \quad (3.32)$$

The analysis of the $M = 0$ limit is the same as that of Eq. (2.33), so $\chi(\alpha_0, 0)$ is the power of X which characterizes the large- X behaviour of G :

$$\text{anomalous dimension} = \gamma_G(g_\infty) - \gamma_G(0) = \frac{1}{2} [\chi(\alpha_0, 0) - \chi(0, 0)] \quad (3.33)$$

Of course, we expect anomalous dimension to be an invariant of the renormalization group. This can be checked from Eq. (1.38) and by considering transformations on Eq. (3.32).

More complicated cases of coupling-constant renormalization can be handled by finding the appropriate generalization of the invariant function D in Eq. (3.3). For example, consider the invariant

$$\bar{D} = [G^{(2)}(p_1) G^{(2)}(p_2) G^{(2)}(p_3) G^{(2)}(p_4)]^{1/2} \Gamma^{(4)}(p_1, p_2, p_3, p_4) \quad (3.34)$$

with $G^{(2)}$ = boson propagator, $\Gamma^{(4)}$ = 4-leg 1PI vertex, p_i = external momenta, $\sum p = 0$. In analogy with (3.4), the coupling constant g_λ for the ϕ^4 interaction is given by $\bar{D}$ at the subtraction point $p_i = \lambda_i$. The $M \rightarrow 0$ limit can be taken if p_i and λ_i are not exceptional. The resulting $\psi(g)$ function is universal with respect to transformations which scale all of the λ_i together. In gauge theories, one vertex (e.g. 3-gluon) can be used to define $\bar{D}$ and g_λ , and then gauge Ward identities fix the normalization of the other vertices (4-gluon, ghost-gluon, etc.).

The GML approach involves the same assumptions [(I) and (II) in the discussion after (2.32)] as the CS method, so the choice of CS or GML method (or variation thereof) is mostly a matter of convenience; [e.g. the GML method is ideal for two-dimensional QED⁷⁰⁾].

3.2 Improved CS equation

According to assumption (II), non-leading powers in the asymptotic expansion of a perturbative amplitude do not compete with the leading power when summed to all orders. These non-leading powers reflect the presence of masses (or other dimensionful coupling constants) in the Lagrangian $\mathcal{L}$. The rule^{4,7,33,71)} for estimating these terms involves the maximal dynamical dimension d_Δ of the mass operators of $\mathcal{L}$; e.g.

$$d_\Delta = \begin{cases} 3 + \gamma_\Delta(g_\infty) & , \quad (\Delta = \bar{\psi}\psi) \\ 2 + \gamma_\Delta(g_\infty) & , \quad (\Delta = \phi^2/2) \end{cases} \quad (3.35)$$

where γ_Δ is the γ -function (1.46) for the composite operator Δ . It states that the N^{th} set of non-leading powers of a perturbative amplitude $G(\eta q)$ sums to an amplitude which is^{*)}

$$O\left[\eta^{-N(4-d_\Delta)} \{\text{logs of } \eta\}\right] \quad , \quad (N = 1, 2, \dots \text{ not too large}) \quad (3.36)$$

relative to the summed-up leading power $G^{\text{as.}}(\eta q)$. Thus we must assume⁴⁾

$$d_\Delta < 4 \quad (3.37)$$

in order to ensure the validity of assumption (II). Equation (3.37) is automatically satisfied⁷⁾ if the theory is asymptotically free because the relevant anomalous dimension $\gamma_\Delta(0)$ vanishes.

Wilson introduced Eq. (3.36) as one of the rules of his theory of broken scale invariance⁴⁾ and later justified it^{**)} within the context of the renormalization group by applying a variant of the GML analysis due to Eriksson⁷²⁾. The

*) So $p(\Delta)$ can be replaced by $4-d_\Delta$ in Eq. (2.53).

**) See Section IV of Ref. 33. Similar conclusions were obtained by Symanzik⁷¹⁾ for $\lambda\phi^4$ theory in the CS formalism.

conventional GML procedure is to renormalize the mass on-shell; the renormalized mass is taken to be the position of the pole in the propagator and is regarded as being independent of the subtraction point λ . [This accounts for the absence of a non-trivial coefficient for the operator $M\partial/\partial M$ in Eqs. (3.11) and (3.31).] Eriksson's method involves a mass parameter m_λ defined in terms of the propagator at the subtraction point λ . In bare outline, what happens is that a change of λ induces a mass renormalization

$$m_{\lambda'} = \mathcal{Z}_m(\lambda', \lambda) m_\lambda \quad (3.38)$$

as well as coupling-constant and wave-function renormalization. Because of the factor $\mathcal{Z}_m$, there is a function $(m_\lambda^2/\lambda^2)\chi_m(\alpha_\lambda, m_\lambda^2/\lambda^2)$ associated with $\lambda\partial/\partial\lambda(m_\lambda^2/\lambda^2)$ in the same way that $\psi(\alpha_\lambda, m_\lambda^2/\lambda^2)$ is related to $\lambda\partial/\partial\lambda(\alpha_\lambda)$ [as in Eqs. (3.7) and (3.12), except that the change in renormalization prescription produces a different two-variable ψ -function]. The variable m_λ^2/λ^2 acts like a coupling constant in the renormalization-group equations. Thus the anomalous dimension of Δ , $\chi_m(\alpha_0, 0)/2$ [or $\chi_m(\alpha_0, 0)$ for scalars], appears when the theory is expanded about the fixed point $(\alpha_0, 0)$ in $(\alpha_\lambda, m_\lambda^2/\lambda^2)$ space. Equation (3.37) is the condition for this fixed point to be UV-stable in the m_λ^2/λ^2 direction.

The result (3.36) was rediscovered by Weinberg⁷⁾ as a direct consequence of his improved CS equation

$$[\mu\partial/\partial\mu + \beta(g_R)\partial/\partial g_R + \gamma_m(g_R)m_R\partial/\partial m_R + \gamma(g_R)]G_R(\eta q; g_R, m_R, \mu) = 0 \quad (3.39)$$

Equation (3.39) is characteristic of "mass-independent" renormalization prescriptions in which there is a renormalization parameter μ but the β and γ functions do not depend on m_R/μ . Instead of a mass-insertion term, there is an ordinary derivative $m_R\partial/\partial m_R$ multiplied by a γ function $\gamma_m(g_R)$ associated with mass renormalization. This means that Eq. (3.39) can be directly integrated without taking a zero mass limit.

The main step in the derivation of (3.39) is to find a suitable renormalization prescription. One possibility⁷⁾ is to multiply g_B , m_B , and G_B by Z-factors given by setting $m_B = 0$ in the corresponding factors $Z(g_B, \Lambda/\mu, m_B/\mu)$ which relate bare and renormalized quantities in the GML prescription. Here, μ is the GML reference momentum λ and $Z_m = m_R/m_B$ is the Z-factor for the composite operator $\bar{\psi}\psi$. Assuming that cut-off independent renormalized amplitudes are produced^{7, 73)}, this prescription is obviously mass-independent:

$$\begin{aligned} \beta(g_R) &= \lim_{\Lambda \rightarrow \infty} (\mu\partial/\partial\mu) g_R(g_B, \Lambda/\mu) \\ \gamma_m(g_R) &= \lim_{\Lambda \rightarrow \infty} (\mu\partial/\partial\mu) \ln Z_m(g_B, \Lambda/\mu) \\ \gamma(g_R) &= \lim_{\Lambda \rightarrow \infty} (\mu\partial/\partial\mu) \ln Z_G(g_B, \Lambda/\mu) \end{aligned} \quad (3.40)$$

Equation (3.39) is an immediate consequence of combining the formulas

$$\begin{aligned} G_R &= Z_G G_B \\ (\mu \partial / \partial \mu) G_B(\text{fixed } q, g_B, m_B) &= 0 \end{aligned} \quad (3.41)$$

Weinberg also observed that all of the properties needed in the derivation of Eq. (3.39) are contained in 't Hooft's analysis⁷⁴⁾ of the renormalization group for dimensionally renormalized^{46,75-77)} amplitudes. This line was subsequently studied in detail by many authors⁷⁸⁾. Briefly, the analysis depends on the following theorem^{74,79)}: in the dimensional renormalization scheme, counterterm vertices are polynomials in the renormalized mass parameter m_R as well as in the external momenta q *). Thus Z-factors depend only on the unity of mass⁷⁴⁾ μ (which accompanies each loop integral $\mu^{4-n} \int d^n p$), the complex parameter n , and g_B , which has dimensionality $(4-n)\kappa$ (with $\kappa = 1/2$ for gauge theories, $\kappa = 1$ for ϕ^4 coupling):

$$Z = Z(g_B \mu^{(n-4)\kappa}, n) \quad (3.42)$$

Once again, Eq. (3.39) follows directly from (3.41) with⁷⁸⁾

$$\begin{aligned} \beta(g_R) &= \lim_{n \rightarrow 4} (\mu \partial / \partial \mu) g_R(g_B \mu^{(n-4)\kappa}, n) \\ \gamma_m(g_R) &= \lim_{n \rightarrow 4} (\mu \partial / \partial \mu) \ln Z_m(g_B \mu^{(n-4)\kappa}, n) \\ \gamma(g_R) &= \lim_{n \rightarrow 4} (\mu \partial / \partial \mu) \ln Z_G(g_B \mu^{(n-4)\kappa}, n) \end{aligned} \quad (3.43)$$

The analogue of Eq. (1.35) is

$$[\eta \partial / \partial \eta + m_R \partial / \partial m_R + \mu \partial / \partial \mu - d_G] G_R(\eta q; g_R, m_R, \mu) = 0 \quad (3.44)$$

where the integer d_G is the mass dimensionality of G , so if d_G is absorbed into the definition of the γ function

$$\gamma_G(g_R) = d_G + \gamma(g_R), \quad (3.45)$$

Eq. (3.39) can be written

$$[\eta \partial / \partial \eta - \beta(g_R) \partial / \partial g_R + \{1 - \gamma_m(g_R)\} m_R \partial / \partial m_R - \gamma_G(g_R)] G_R(\eta q; g_R, m_R, \mu) = 0 \quad (3.46)$$

The general solution of Eq. (3.46) is⁷⁾

$$G_R(\eta q; g_R, m_R, \mu) = G_R(q; \bar{g}, \bar{m}, \mu) \exp \int_{g_R}^{\bar{g}} dx \gamma_G(x) / \beta(x) \quad (3.47)$$

where, in addition to the effective coupling constant $\bar{g} = \bar{g}(g_R, \eta)$, there is an "effective mass"

$$\bar{m} = m_R \eta^{-1} \exp \int_{g_R}^{\bar{g}} dx \gamma_m(x) / \beta(x) \quad (3.48)$$

*) One proves (e.g. by induction in ℓ) that the ℓ -loop amplitude $(\partial / \partial m_R)^p \Gamma$ converges for sufficiently large integers p ; here, Γ is a 1PI amplitude (Fig. 1) with ℓ' -loop counterterms ($\ell' < \ell$) included in the Lagrangian. Observe that the result is not true for conventionally renormalized theories: $Z = Z(\Lambda / m_R, g_B)$ is polynomial in g_B and $\ln(\Lambda / m_R)$.

The $\eta \rightarrow \infty$ limit of Eq. (3.47) is controlled by the asymptotic behaviour of both $\bar{g}$ and $\bar{m}$ *):

$$\begin{aligned} \bar{m} &= m_R \eta^{\gamma_m(g_\infty) - 1} \exp \int_{g_R}^{\bar{g}} dx \{ \gamma_m(x) - \gamma_m(g_\infty) \} / \beta(x) \\ &\sim m_R \eta^{\gamma_m(g_\infty) - 1} \{ \log s \text{ of } \eta \} , \quad (\bar{g} \rightarrow g_\infty) \end{aligned} \quad (3.49)$$

Now, apart from finite renormalizations, the massless amplitude $G_R(q; g_R, 0, \mu)$ may be identified with the leading-power amplitudes $G^{as.}(q; g, m)$ and $G(X, 0, g_\lambda)$ of the CS and GML analyses. Therefore the condition⁷⁾

$$\bar{m} \rightarrow 0 , \quad \gamma_m(g_\infty) < 1 \quad (3.50)$$

is essential if assumption (II) is to be satisfied. Making the identification

$$d_\Delta = \begin{cases} 3 + \gamma_m(g_\infty) & , \quad (\Delta = \bar{\psi}\psi) \\ 2 + 2\gamma_m(g_\infty) & , \quad (\Delta = \phi^2/2) \end{cases} \quad (3.51)$$

we see that Eq. (3.50) is equivalent to Eq. (3.37).

Evidently, the terms $N = 1, 2, \dots$, of Eq. (3.36) correspond to increasing powers of $\bar{m}$ in the Taylor expansion of Eq. (3.47) about $\bar{m} = 0$:

$$G_R(\eta q; g_R, m_R, \mu) \sim \left\{ \exp \int_{g_R}^{\bar{g}} dx \gamma_G(x) / \beta(x) \right\} \sum_{r=0}^{\infty} \frac{\bar{m}^r}{r!} \left(\frac{\partial}{\partial \bar{m}} \right)^r G_R(q; g_R, m, \mu)_{m=0} \quad (3.52)$$

However, it is necessary to insert the warning "not too large" in (3.36) because in general, the limit

$$\lim_{m \rightarrow 0} \left(\frac{\partial}{\partial \bar{m}} \right)^r G_R(q; g_R, m, \mu) , \quad (\text{non-exceptional } q) \quad (3.53)$$

does not exist if r is too large⁷⁾. If m_R is a fermionic mass parameter, trouble first appears when $(\partial/\partial m)^3$ acts on a single internal propagator, producing a logarithmic infrared singularity in the limit $m \rightarrow 0$:

$$\bar{m}^3 \left(\frac{\partial}{\partial \bar{m}} \right)^3 \int^\Lambda d^4 p / (p - \bar{m}) = O(\bar{m}^3 \ln \bar{m}) \quad (3.54)$$

Higher derivatives ($r > 3$) do not reduce the power further:

$$\text{e.g.} \quad \bar{m}^5 (\partial/\partial \bar{m})^5 \int^\Lambda d^4 p / (p - \bar{m}) = O(\bar{m}^3) \quad (3.55)$$

For scalar masses m_R , the $r = 2$ term (3.53) is infrared singular.

The restriction of (3.36) to small values of N means that the non-leading terms are not completely under control. In particular, the logarithmic powers β in $O(\bar{m}^3 \ln^\beta \bar{m})$ (for fermions) should be further investigated; for example, are there models in which β is bounded as the order of perturbation grows? I think

*) Note the analogy between the η -dependence of $(\bar{g}, \bar{m}/m_R)$ and the λ -dependence of $(\alpha_\lambda, m_\lambda^2/\lambda^2)$ in Wilson's analysis³³⁾. The mass-independence property simplifies the analysis: $\eta \partial/\partial \eta (\bar{g}, \ln \bar{m}/m_R)$ is given by the integrable expression $(\beta(\bar{g}), \gamma_m(\bar{g}) - 1)$, whereas the equations for $\lambda \partial/\partial \lambda (\alpha_\lambda, m_\lambda^2/\lambda^2)$ are coupled.

that β is larger than 1 in general, because the coefficient of the term (3.54) in the expansion of $\bar{m}^3(\partial/\partial\bar{m})^3 G_R$ is a zero-mass amplitude evaluated at exceptional momenta:

$$\text{coefficient} = \lim_{m \rightarrow 0} G_R(p=0, -p=0; q; \bar{q}, m, \mu) \quad (3.56)$$

Here $(p, -p)$ are external momenta for the fermion-antifermion pair obtained by cutting the internal propagator $(p-m)^{-1}$ in (3.54). In particular, the zero-mass limit can produce logarithms in the fermionic self-energy which modify the denominator

$$(p - \bar{m})^{-1} \longrightarrow (p \ln^\alpha(p^2/\mu^2) - \bar{m})^{-1} \quad (3.57)$$

and generate extra logarithms of $\bar{m}$ in Eq. (3.54).

Equation (3.46) is valid for all momenta q , so there is no need to restrict its application to ultraviolet limits. Thus the discussion of infrared singularities at the end of Chapter II can be extended in an obvious way to include the case $\mathcal{M} \neq 0$. For example, the analogue of Eq. (2.96) is

$$\lim_{\eta \rightarrow 0} \pi_R(\eta^2 q^2; g_R, \mathcal{M}_R, \mu) = \pi_R(q^2; g_\infty, \mathcal{M}_\infty, \mu) - \int_{g_R}^{g_\infty} dx \frac{K(x)}{\beta(x)} \quad (3.58)$$

Unfortunately, the asymptotic value $\mathcal{M}_\infty$ of $\mathcal{M}(\eta \approx 0)$ is not known because the IR-stable fixed point g_∞ is not at the origin.

4. OPERATOR-PRODUCT EXPANSIONS

In order to apply the renormalization group to topics such as current algebra and deep inelastic electroproduction, it is necessary to consider ultraviolet limits for subsets of the external lines of amplitudes. The appropriate tool is Wilson's expansion for operator products at short distances⁴⁾.

4.1 Short-distance limit

The short-distance expansion^{4,15,80-83)}

$$\prod_{i=1}^N A_i(x + \rho \xi_i) \sim \sum_n \mathcal{C}_n(\rho \xi_j - \rho \xi_k) O_n(x), \quad (\rho \rightarrow 0, \text{fixed } x, \xi_i) \quad (4.1)$$

is an asymptotic expansion of the operator product ΠA_i into local operators $O_n(x)$ (including the identity operator I), with c-number coefficient functions $\mathcal{C}_n$ depending on the various coordinate differences $\rho(\xi_j - \xi_k)$. The product ΠA_i may be unordered (e.g. multiple commutators), or ordered (T-product, anti-T-product, ...), or some combination thereof. The terms $\mathcal{C}_n O_n$ in the sum $\sum_n$ are ordered according to the strength of the singularity of $\mathcal{C}_n$ in the scaling variable ρ *):

$$\lim_{\rho \rightarrow 0} \left[\left\{ \Pi A_i - \sum_{n=0}^N \mathcal{C}_n(\rho) O_n(0) \right\} / \mathcal{C}_N(\rho) \right] = 0, \quad (N = 0, 1, 2, \dots) \quad (4.2)$$

*) Unlike its cousin, the light-cone expansion^{84,85)}, the short-distance expansion does not involve performing an infinite summation $\sum_{n=0}^{\infty}$.

The effect of expanding about $x' = x - \rho\xi$ instead of x is to change the contributions of derivative operators:

$$O_n(x' + \rho\xi) \sim \sum_{r=0}^{\infty} \frac{\rho^r}{r!} \left(\xi \cdot \frac{\partial}{\partial x} \right)^r O_n(x') \quad (4.3)$$

In free-field theory, the expression (4.1) is trivially obtained by Taylor-expanding the Wick expansion of ΠA_i . For example, consider the Wick expansion

$$\begin{aligned} T\{\mathcal{F}_\mu^i(\xi) \mathcal{F}_\nu^j(0)\} &= \left\{ \overline{\psi}(\xi) \gamma_\mu(\lambda^i/2) \overline{\psi}(\xi) \overline{\psi}(0) \gamma_\nu(\lambda^j/2) \psi(0) \right\} I \\ &+ : \overline{\psi}(\xi) \gamma_\mu(\lambda^i/2) \overline{\psi}(\xi) \overline{\psi}(0) \gamma_\nu(\lambda^j/2) \psi(0) : + : \overline{\psi}(0) \gamma_\nu(\lambda^j/2) \overline{\psi}(0) \overline{\psi}(\xi) \gamma_\mu(\lambda^i/2) \psi(\xi) : \\ &+ : \overline{\psi}(\xi) \gamma_\mu(\lambda^i/2) \psi(\xi) \overline{\psi}(0) \gamma_\nu(\lambda^j/2) \psi(0) : \end{aligned} \quad (4.4)$$

for SU(3) currents

$$\mathcal{F}_\mu^i(x) = : \overline{\psi}(x) \gamma_\mu(\lambda^i/2) \psi(x) : , \quad (\lambda^i = \text{matrices for ordinary SU(3)}) \quad (4.5)$$

with

$$\begin{aligned} \overline{\psi}(0) \overline{\psi}(\xi) &= i \int \frac{d^4 q}{(2\pi)^4} e^{iq \cdot \xi} (\not{q} - m + i\varepsilon)^{-1} \\ &\sim - \frac{i}{2\pi^2} \not{\xi} / (\xi^2 - i\varepsilon)^2 \end{aligned} \quad (4.6)$$

When expanded about $\xi = 0$, each term on the right-hand side of (4.4) contributes to the Wilson expansion:

$$\begin{aligned} O_n &= I , \quad (\text{first term of (4.4)}) ; \\ O_n &= \text{two-fermion operators} = : \overline{\psi}(x) \{ \lambda, \gamma \text{ matrices} \} \{ \overleftrightarrow{\partial} \dots \partial \} \psi(x) : , \\ &\quad (\text{second and third terms of (4.4)}) ; \\ O_n &= \text{four-fermion operators} \\ &= : \overline{\psi}(x) \{ \lambda, \gamma \} \{ \overleftrightarrow{\partial} \dots \partial \} \psi(x) \overline{\psi}(x) \{ \lambda, \gamma \} \{ \overleftrightarrow{\partial} \dots \partial \} \psi(x) : , \\ &\quad (\text{last term of (4.4)}) . \end{aligned} \quad (4.7)$$

The expansion of $\overline{\psi}(0) \overline{\psi}(\xi)$ produces terms of the form ξ^p or $\xi^p \ln(\xi^2 m^2)$, so we have ($p = \text{integer}$)

$$\mathcal{O}_n(\xi) = \xi_\alpha \xi_\beta \dots \xi_\omega (\xi^2)^p \{ \text{constant, or constant} \cdot \ln(m^2 \xi^2) \} \quad (4.8)$$

The logarithm is at least $O(m^3 \ln m)$ in Eq. (4.6) [compare (3.54)], so it contributes to unimportant non-leading terms in the Wilson expansion.

The expansion (4.1) remains valid in renormalized perturbation theory^{71,81,83,86}. The operators A_i, O_n are provided with counterterms so that they yield finite matrix elements. In general, there are more operators O_n than in free-field theory. For example, the expansion of $T\{\mathcal{F}_\mu \mathcal{F}_\nu\}$ involves six-fermion, eight-fermion, ..., operators as well as renormalized versions of the operators in

(4.7). The operators O_n can also depend on other fields appearing in the Lagrangian (e.g. gluon and ghost fields in gauge theories). The properties^{71,87)} of the coefficient functions C_n are very similar to those of the asymptotic parts G^{as} of renormalized Green's functions; they become coupling-constant dependent and exhibit extra logarithmic singularities compared with Eq. (4.8),

$$C_n(\xi) \sim \xi_\alpha \xi_\beta \dots \xi_\omega (\xi^2)^p \text{Polynomial}(\ln \xi^2, g_R) \quad (4.9)$$

or more generally [in analogy with Eq. (2.21)]

$$C_n(\rho \xi_j - \rho \xi_k) \sim \rho^p \text{Polynomial}(\ln \rho, g_R), \quad (p = \text{integer}) \quad (4.10)$$

Since the most interesting applications of Eq. (4.1) are non-perturbative, a proof based on the general principles of axiomatic field theory would be very desirable. The main difficulty to be overcome is that there is no general argument which forbids the appearance of infinite oscillations in matrix elements of ΠA_i as ρ tends to zero⁸²⁾. A rather complete analysis is possible⁸²⁾ if oscillations are assumed to be absent.

Most practical applications^{4,88)} involve products ΠA_i of observable hadronic currents: the electromagnetic current J_μ , the energy-momentum tensor $\theta_{\mu\nu}$, the currents $F_\mu^i, F_{5\mu}^i$ associated with chiral $SU(3) \times SU(3)$, and so on. It is a consequence of (4.1) that the corresponding operators O_n are also observable. Naturally⁴⁾, the expansion respects the conservation and covariance properties of ΠA_i for exact symmetries (Poincaré invariance, isospin, charge conjugation, etc.), so it is often convenient to isolate a sector (J, I, ...) of the expansion containing operators O_n with spin J, isospin I, etc. As a rule⁴⁾, the leading power of ρ in a given sector also conserves symmetries which are only approximate [e.g. ordinary $SU(3)$, chiral $SU(2) \times SU(2)$]. For example, the leading contribution proportional to $\theta_{\mu\nu}$ in the expansion

$$F_{5\mu}^i(x+\xi) F_{5\nu}^j(x-\xi) \sim \dots + C_{\mu\nu\alpha\beta}^{ij}(\xi) \theta^{\alpha\beta}(x) + \dots, (\xi \rightarrow 0) \quad (4.11)$$

satisfies the constraints

$$\begin{aligned} C_{\mu\nu\alpha\beta}^{ij} &= \delta^{ij} C_{\mu\nu\alpha\beta} \\ \partial^\mu C_{\mu\nu\alpha\beta}(\xi) &= 0 = \partial^\nu C_{\mu\nu\alpha\beta}(\xi) \end{aligned} \quad (4.12)$$

As in the discussion of the asymptotic properties of Green's functions, it is desirable to classify contributions to a given sector by powers of the scaling variable ρ , not by logarithms^{*)}:

$$[C_{n+1}(\rho)/C_n(\rho)]_{(J,I,\dots)} = O[\rho^{-p(n)}], \quad (p(n) > 0) \quad (4.13)$$

*) The consistency conditions used to analyse $\pi^0 \rightarrow 2\gamma$ decay⁸⁹⁾ involve coefficient functions classified according to (4.13).

The most important point is that the expansion (4.1) should not depend on the matrix element to which it is applied. The general condition for this to be true is

$$\langle F | \prod_{i,j} \{ A_i(x + \rho \xi_i) B_j(y_j) \} | I \rangle \sim \sum_n \mathcal{C}_n(\rho) \langle F | \prod_j \{ O_n(x) B_j(y_j) \} | I \rangle, \quad (4.14)$$

where $|I\rangle, |F\rangle$ are vacuum or on-shell states, the matrix element is complete (i.e. includes disconnected pieces), $\Pi_{i,j}$ and Π_j refer to unordered or ordered products*), and the limit $\rho \rightarrow 0$ is taken with

- i) all momenta and other parameters of $|I\rangle, |F\rangle$ fixed,
- ii) the coordinates x, ξ, y_j fixed ($y_j \neq x$).

Note

- a) When ordered products (such as T-products) are involved, there can be ambiguities proportional to products of $\delta(\xi_i - \xi_j)$ and $\delta(y_i - y_k)$ and derivatives thereof. For example, the renormalization of the left-hand side of Eq. (4.14) may involve counterterms of the form

$$\chi_{A_1} \chi_{A_2} \dots \chi_{B_1} \chi_{B_2} \dots \Delta(A_1 A_2 \dots B_1 B_2 \dots)$$

in Eq. (1.39). The $\rho \rightarrow 0$ limit of these δ -function terms need not match the δ -functions associated with counterterms

$$\chi_{O_n} \chi_{B_1} \chi_{B_2} \dots \Delta(O_n B_1 B_2 \dots)$$

on the right-hand side.

- b) Even in free-field theory, the state vector

$$|\varphi(\rho)\rangle = \int d^4x \prod_i d^4\xi_i f(x, \xi_1, \xi_2, \dots) \left[\{ \pi A_i - \sum_{n=0}^N \mathcal{C}_n O_n \} / \mathcal{C}_N \right] | I \rangle \quad (4.15)$$

(f = suitable smearing function)

does not vanish:

$$\begin{aligned} \langle \varphi(\rho) | \varphi(\rho) \rangle &\not\rightarrow 0, \\ \langle F | \varphi(\rho) \rangle &\longrightarrow 0, \quad (\rho \rightarrow 0) \end{aligned} \quad (4.16)$$

i.e. the limit $\rho \rightarrow 0$ is "weak".

For most operators $B_j(y_j)$, it is absolutely mandatory that the *coordinates* y_j (and *not* the corresponding momenta) be held fixed as ρ tends to zero. This point will be illustrated by considering the connected amplitude of Fig. 12a in the tree approximation, with

$$B(y) = : \bar{\Psi}(y) (\partial^2)^{25} \Psi(y) : \quad (4.17)$$

*) More precisely, to products which can be constructed from discontinuities of the completely time-ordered product.

The most singular q-number contribution generated by the Wick expansion (4.4) is

$$T\{\mathcal{F}_\mu^i(\xi) \mathcal{F}_\nu^j(0)\} \sim f^{ijk} \mathcal{C}_{\mu\nu}^{\lambda}(\xi) \mathcal{F}_\lambda^k(0) \quad , \quad (\xi \rightarrow 0) \quad (4.18)$$

where $\mathcal{C}_{\mu\nu\lambda}(\xi)$ scales as ξ^{-3} and f^{ijk} are the structure constants for ordinary SU(3). [This singularity generates the equal-time commutator

$$[\mathcal{F}_0^i, \mathcal{F}_0^j]_{E.T.} = i f^{ijk} \delta^3(\vec{\xi}) \mathcal{F}_0^k$$

of current algebra.] Thus the momentum-space singularity is

$$\int d^4\xi e^{iq \cdot \xi} \mathcal{C}_{\mu\nu\lambda}(\xi) = O(q) \quad , \quad (q \rightarrow \infty). \quad (4.19)$$

The graphs responsible for this contribution are shown in Fig. 12b.

We must also consider the graphs of Fig. 12c. If the coordinates y, z_1, z_2 are held fixed as $\xi \rightarrow 0$, the leading term contributed by these graphs is independent of ξ [i.e. $O(\xi^+)$ relative to (4.18)] and is given by the graph of Fig. 12d, with

$$Q_{\mu\nu}(0) = : \bar{\Psi}(0) \gamma_\mu(\lambda^i/2) \Psi(0) \bar{\Psi}(0) \gamma_\nu(\lambda^j/2) \Psi(0) : \quad (4.20)$$

This agrees with the result of applying the limit $\xi \rightarrow 0$ to the last term of Eq. (4.4). Thus the contribution $Q_{\mu\nu}$ is always present in the expansion of $T\{\mathcal{F}_\mu \mathcal{F}_\nu\}$ and does not depend on whether $B(y)$ is present or not.

On the other hand, if we transform to momentum space and let the momenta of the currents $\mathcal{F}_\mu, \mathcal{F}_\nu$ become large ($q = \eta l + c$; $\eta \rightarrow \infty$, $l^2 \neq 0$) with the momenta of $\psi, \bar{\psi}$, and B held fixed, the result

$$\{\text{Fig. 12(c) graphs}\} = O(q^{48}) \quad (4.21)$$

dominates the "expected" term (4.19). Thus naive substitution of the operator-product expansion for $T\{\mathcal{F}_\mu \mathcal{F}_\nu\}$ gives the wrong answer.

However, it is very easy to explain the result (4.21) in the language of operator-product expansions. For simplicity, let us set the momentum of the B operator equal to zero and consider the limit $\xi \rightarrow 0$ of the amplitude

$$\mathcal{A}(z_1, z_2, \xi) = \int d^4y T \langle 0 | \bar{\Psi}(z_1) \Psi(z_2) B(y) \mathcal{F}_\alpha^i(\xi) \mathcal{F}_\beta^j(0) | 0 \rangle \quad (4.22)$$

The point is that, in addition to the expansion of $T\{\mathcal{F}_\mu \mathcal{F}_\nu\}$, it is necessary to consider the expansion

$$T\{B(y) \mathcal{F}_\alpha^i(\xi) \mathcal{F}_\beta^j(0)\} \sim \mathcal{C}_{\alpha\beta}^{ij}(\xi, y) : \bar{\Psi} \Psi : + \dots, \quad (\xi, y \rightarrow 0) \quad (4.23)$$

$$\mathcal{C}_{\alpha\beta}^{ij}(\rho\xi, \rho y) = O(\rho^{-56}) \quad (4.24)$$

Substituting into (4.22), we find

$$\mathcal{A}(z_1, z_2, \xi) \sim \int d^4 y \mathcal{C}_{\alpha\beta}^{ij}(\xi, y) T \langle 0 | \bar{\Psi}(z_1) \Psi(z_2) : \bar{\Psi}(0) \Psi(0) : | 0 \rangle, (\xi \rightarrow 0) \quad (4.25)$$

Counting powers in coordinate space, we have four powers of ρ for the integral $\int d^4 y$ to be added to the power -56 in (4.24):

$$\int d^4 y \mathcal{C}_{\alpha\beta}^{ij}(\xi, y) = O(\xi^{-52}), \quad (\xi \rightarrow 0) \quad (4.26)$$

Fourier transforming (4.26) yields the result

$$\int d^4 \xi e^{iq \cdot \xi} \int d^4 y \mathcal{C}_{\alpha\beta}^{ij}(\xi, y) = O(q^{48}), \quad (q \rightarrow \infty) \quad (4.27)$$

in agreement with Eq. (4.21). In other words, it is not sufficient to substitute (4.18) into (4.22) because, for any non-zero value of ξ (no matter how small), the integral $\int d^4 y$ always contains the region $y = 0(\xi)$ in which the contribution (4.23) dominates.

It can be seen from the literature^{15,71,81,86,90} that a complete derivation of the Wilson expansion in perturbation theory is necessarily non-trivial. Indeed, explicit derivations of (4.14) have been given only for the following special cases

$$\prod_{i=1}^L A_i(x + \rho \xi_i) = \prod_{i=1}^L \left\{ \phi(x + \rho \xi_i) \text{ or } \partial_\alpha \partial_\rho \dots \partial_\omega \phi(x + \rho \xi_i) \right\} \quad (4.28)$$

$$\prod_{j=1}^N B_j(y_j) = \prod_{j=1}^N \phi(y_j)$$

in ϕ^4 theory; (for L arbitrary, see Ref. 90). However, the main idea of the derivation is the same for all cases (i.e. simple or composite operators A_i, B_j in any renormalizable theory) and can be summarized fairly simply.

Consider the complete time-ordered amplitude (Fig. 13)

$$\mathcal{A} = T \langle 0 | \prod_{i=1}^L \prod_{j=1}^N A(\rho \xi_i) B_j(y_j) | 0 \rangle \quad (4.29)$$

which corresponds to the choice $x = 0$, $|I\rangle = |F\rangle = |0\rangle$ in (4.14). In Fig. 13, wavy lines correspond to sources χ for the operators A_i, B_j [as in Eq. (1.7)] and non-wavy lines indicate propagators of fields (gluon, ghost, fermion, etc.) from which the Lagrangian and A_i, B_j are constructed. Some of the source functions χ can be chosen to generate the field operators themselves; e.g. see the lines labelled A_1 and B_2 in Fig. 13.

The amplitude $\mathcal{A}$ includes disconnected contributions, so the first step is to extract all factors of the form

$$G_S = T \langle 0 | \prod_{i \in S} A_i(\rho \xi_i) | 0 \rangle \quad (4.30)$$

where S is any non-empty subset of $\{1, \dots, L\}$. We already know how to deal with the $\rho \rightarrow 0$ limit of G_S , because in momentum space, it becomes the $\eta \rightarrow \infty$ limit of

$G_S(\eta q + c)$ at non-exceptional q . Summing over all possible sets S (including the complete set $\{1, \dots, L\}$), we have^{15, 81)}

$$T \prod_{i=1}^L A_i(\rho \xi_i) = : \prod_{i=1}^L A_i(\rho \xi_i) : + \sum_S T \langle 0 | \prod_{i \in S} A_i(\rho \xi_i) | 0 \rangle : \prod_{i \notin S} A_i(\rho \xi_i) : \quad (4.31)$$

where the generalized Wick products^{*)}

$$: \prod_i' A_i(\rho \xi_i) : \quad , \quad (\pi' = \prod_{i \notin S} \text{ or } \prod_{i=1}^L)$$

correspond to time-ordered amplitudes

$$\mathcal{A}' = T \langle 0 | \prod_{j=1}^N : \prod_i' A_i(\rho \xi_i) : B_j(y_j) | 0 \rangle \quad (4.32)$$

constructed from graphs in which every A_i operator in the product $\prod' A$ is connected to a B_j operator. The problem is now reduced to that of obtaining expansions for the $\mathcal{A}'$ amplitudes. The expansion for the original amplitude $\mathcal{A}$ can then be reconstructed from Eq. (4.31), with the amplitudes G_S appearing as factors in various coefficient functions. [Observe that we chose to extract only those factors (4.30) which do not depend on the B_j operators.]

Let us denote the Fourier transform of (4.32) by

$$F.T. [\mathcal{A}'] = \tilde{\mathcal{A}}'(q_1, \dots, q_K; r_1, \dots, r_N) \delta(\sum_{i=1}^K q_i + \sum_{j=1}^N r_j) \quad (4.33)$$

where K is the number of A_i operators in (4.32) and the momenta $\{q_1, \dots, q_K\}$ and $\{r_1, \dots, r_N\}$ are conjugate to the coordinates $\{\rho \xi_i\}$ and $\{y_1, \dots, y_N\}$, respectively. The notation is chosen such that Eqs. (2.1) and (2.2) and Fig. 2 can be applied directly to the amplitude $\tilde{\mathcal{A}}'$. Figure 2 involves decomposing $\tilde{\mathcal{A}}'$ into an "upper blob" $\mathcal{U}$ ($\mathcal{G}'$ in Fig. 2) and a "lower blob" $\mathcal{L}$:

$$\tilde{\mathcal{A}}' = \int \prod_{i=1}^M d^4 k_i \mathcal{U}(q_1, \dots, q_K; k_1, \dots, k_M) \mathcal{L}(k_1, \dots, k_M; r_1, \dots, r_N; \sum r = -\sum q) \quad (4.34)$$

The $\mathcal{U}$ amplitude is defined in terms of graphs $\mathcal{G}'$ which become 1PI when the coordinates ξ_i are set equal: i.e. $\xi_i = \xi$, for all i in $\prod_i'$ (as shown in Fig. 14). The rationale for this definition is that it isolates all graphs $\mathcal{G}'$ which are able to carry $O(\eta)$ momenta simultaneously along all internal lines; i.e. the decomposition (4.34) is designed to facilitate power-counting for the limit $q_1, \dots, q_K \rightarrow \infty$.

Note

- i) One of the consequences of the definition is that propagators for the intermediate lines $k_1, \dots, k_M$ (including self-energy corrections) are incorporated in the amplitude $\mathcal{L}$.

*) The name is appropriate because, in free-field theory, generalized Wick products are the same as ordinary Wick products.

ii) The expression for $\mathcal{U}$ includes a momentum-conservation δ -function

$$\delta(\sum q + \sum k) \quad (4.35a)$$

so $\mathcal{U}$ can be considered as a function of $(K + M)$ independent variables $(q_1, \dots, q_K, k_1, \dots, k_M)$. Also, some of the graphs $\mathcal{G}'$ may be disconnected; each connected component gives rise to a factor

$$\delta(\sum' q + \sum'' k) \quad (4.35b)$$

in the expression for $\mathcal{U}$, where $\sum'$ and $\sum''$ denote non-empty partial sums.

iii) In Fig. 2 (as applied to $\tilde{\mathcal{U}}'$), all external lines should be understood to be wavy source lines. Previously (Figs. 6b and 7b), we allowed some of the source lines $r_1, \dots, r_N$ to be intermediate as well. However, for reasons which will shortly become evident, we now require that *none* of the intermediate lines $k_1, \dots, k_M$ be wavy (as in Fig. 14).

The amplitude $\mathcal{U}$ can be expanded in $k = (k_1, \dots, k_M)$ about the point^{*)}

$$k = (0, 0, \dots, 0) \quad (4.36)$$

with all momenta $q = (q_1, \dots, q_K)$ held fixed:

$$\mathcal{U}(q, k) = \sum_{n=0}^{p-1} \frac{1}{n!} (k \cdot \frac{\partial}{\partial \lambda})^n \mathcal{U}(q, \lambda) \Big|_{\lambda=0} + \mathcal{R}_p(q, k), \quad (4.37)$$

($p \geq 1$; notation: $k \cdot \frac{\partial}{\partial \lambda} = \sum_{i=1}^M k_i^\mu \frac{\partial}{\partial \lambda_i^\mu}$)

Consider the dimensionality of the amplitudes appearing in (4.37). The definition of dimensionality given by Eq. (2.4) does not take account of the extra δ -functions (4.35) which we have chosen to incorporate in the definition of $\mathcal{U}$. Since each δ -function contributes a power -4 , we have

$$\mathcal{D}(\mathcal{U}) = \mathcal{D}(\mathcal{G}') - 4 \{ \text{number of connected components of } \mathcal{G}' \} \quad (4.38)$$

Each derivative $\partial/\partial \lambda$ acting on $\mathcal{U}$ lowers its dimensionality by 1:

$$\begin{aligned} \mathcal{D}\{(\partial/\partial \lambda)^n \mathcal{U}(q, \lambda)\} &= \mathcal{D}(\mathcal{U}) - n \\ \mathcal{D}\{\mathcal{R}_p\} &= \mathcal{D}(\mathcal{U}) - p \end{aligned} \quad (4.39)$$

We see that Eq. (4.37), when substituted into (4.34), looks like a Wilson expansion with coefficient functions $(\partial/\partial \lambda)^n \mathcal{U}(q, \lambda = 0)$ multiplying operators $\mathcal{O}_n$ given by momentum-space vertices

$$(k)^n = (k_{i_1})_{\mu_1} (k_{i_2})_{\mu_2} \dots (k_{i_n})_{\mu_n} \quad (4.40)$$

*) If zero-mass propagators are present, the expansion must be performed about a non-exceptional point $\lambda = (\lambda_1, \dots, \lambda_M)$ in k -space. The subsequent analysis is unchanged, apart from notational complications due to the λ -dependence of various amplitudes.

However, the prescription (4.37) is not generally satisfactory: the remainder amplitude $\mathcal{R}_p$ need not be $O(\eta^{-p} \ln^{\beta} \eta)$ relative to $\mathcal{U}$ if $\mathcal{G}'$ contains subgraphs with sufficiently large dimensionality. It is necessary to find a prescription for constructing a remainder amplitude in which the dimensionalities of all subgraphs are simultaneously lowered.

The answer is obvious if the complete graph $\mathcal{G}$ for the amplitude $\tilde{\mathcal{A}}'$ can be uniquely decomposed into a nested set of subgraphs (Fig. 15)

$$\mathcal{G}'_1 \subset \mathcal{G}'_2 \subset \dots \subset \mathcal{G}'_h \quad (4.41)$$

Let $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_h$ be the corresponding "upper blob" amplitudes, with

$$\begin{aligned} \mathcal{U}_2(\{k\}_2, q) &= \int \{dk\}_1 \mathcal{U}_2(\{k\}_2, \{k\}_1) \mathcal{U}_1(\{k\}_1, q) \quad , \quad \dots \\ \mathcal{U}_{(m+1)} &= \int \{dk\}_m \mathcal{U}_{(m+1)m} \mathcal{U}_m \quad , \quad \text{etc.} \end{aligned} \quad (4.42)$$

so that the complete amplitude can be written

$$\begin{aligned} \tilde{\mathcal{A}}' &= \int \{dk\}_h \mathcal{L}_h(r, \{k\}_h) \mathcal{U}_h(\{k\}_h, q) \\ &= \int \prod_{m=1}^h \{dk\}_m \mathcal{L}_h \mathcal{U}_{h(h-1)} \mathcal{U}_{(h-1)(h-2)} \dots \mathcal{U}_{21} \mathcal{U}_1 \end{aligned} \quad (4.43)$$

We begin by subtracting the smallest blob $\mathcal{U}_1$ as in (4.37),

$$\mathcal{U}_1 \rightarrow (1 - t_1) \mathcal{U}_1 \quad (4.44)$$

where t_m is shorthand for the Taylor operator

$$t_m = \sum_{r=0}^{p(m)-1} \frac{1}{r!} \left(k \cdot \frac{\partial}{\partial \lambda} \right)_m^r \Big|_{\lambda=0} \quad , \quad (m = 1, 2, 3, \dots, h) \quad (4.45)$$

For the next graph $\mathcal{G}'_2$, we have

$$\mathcal{U}_2 \rightarrow (1 - t_2) \int \{dk\}_1 \mathcal{U}_{21} (1 - t_1) \mathcal{U}_1 \quad (4.46)$$

and so on, until we arrive at a completely subtracted amplitude

$$\tilde{\mathcal{I}}'_p = \int \prod_{m=1}^h \{dk\}_m \mathcal{L}_h (1 - t_h) \mathcal{U}_{h(h-1)} (1 - t_{h-1}) \dots (1 - t_2) \mathcal{U}_{21} (1 - t_1) \mathcal{U}_1 \quad (4.47)$$

in which the dimensionalities of the subgraphs $\mathcal{G}'_1, \dots, \mathcal{G}'_h$ are reduced by $p(1), \dots, p(h)$, respectively. The difference between $\tilde{\mathcal{A}}'$ and $\tilde{\mathcal{S}}'_p$ is a finite series in the form of an operator-product expansion:

$$\tilde{\mathcal{A}}' = \sum_{n=0}^{n(p)} \tilde{\mathcal{C}}'_n(q) \mathcal{A}_n(r) + \tilde{\mathcal{I}}'_p \quad (4.48)$$

The prescription for constructing $\tilde{\mathcal{S}}'_p$ must be generalized to include cases in which there are overlapping subgraphs. Recall (Fig. 14) that the subgraphs $\mathcal{G}'$ are defined such that, when the coordinates ξ_i are identified, the result is a 1PI graph Γ' associated with the class of operators

$$Q(\xi) = : \pi'_i A_i(\xi) : , \dots , : \pi' \{ \partial \dots \partial A_i(\xi) \} : , \dots \quad (4.49)$$

The superficial degree of divergence of Γ' is trivially related to the dimensionality of the amplitude $\mathcal{U}$ associated with $\mathcal{G}'$:

$$\mathcal{D}(\mathcal{U}) = d(\Gamma') - 4K \quad (4.50)$$

Furthermore, the subtractions $t_1, t_2, \dots, t_h$ in (4.47) become renormalization counterterms ΔQ for the operators Q when the coordinates ξ_i are set equal to ξ . So for the general case, the prescription is that subtractions used to construct $\tilde{S}'_p$ from $\tilde{\mathcal{A}}'(\xi_i \neq \xi)$ should correspond to renormalization counterterms ΔQ (possibly oversubtracted) for the amplitude $\tilde{\mathcal{A}}'(\xi_i = \xi)$. This ensures that all subdimensionalities of $\tilde{S}'_p$ are reduced relative to those of $\tilde{\mathcal{A}}'(\xi_i \neq \xi)$, because the renormalization theorem¹²⁾ says that all of the $d(\Gamma')$ can be lowered to any desired value by including sufficient counterterms ΔQ in Eq. (1.7). Thus Eq. (4.48) remains valid for the general case.

An example of overlapping subgraphs is given in Fig. 16. There are two nested sets

$$\mathcal{G}'_1 \subset \mathcal{G}'_{2a} \subset \mathcal{G}'_3, \quad \mathcal{G}'_1 \subset \mathcal{G}'_{2b} \subset \mathcal{G}'_3$$

with $\mathcal{G}'_{2a}$ and $\mathcal{G}'_{2b}$ overlapping. The remainder amplitude is given by the formula

$$\begin{aligned} \tilde{\mathcal{P}}'_p = & \iint \{dk\}_3 \{dk\}_1 \mathcal{L}_3(1-t_3) \cdot \\ & \cdot \left[\int \{dk\}_{2a} \mathcal{U}_{3(2a)}(1-t_{2a}) \mathcal{U}_{(2a)1} - \int \{dk\}_{2b} \mathcal{U}_{3(2b)} t_{2b} \mathcal{U}_{(2b)1} \right] (1-t_1) \mathcal{U}_1 \end{aligned} \quad (4.51)$$

No account has been taken of the need for renormalization counterterms for the operator O_n (for example), so this discussion applies only to unrenormalized (B) amplitudes^{*}). In coordinate space, Eqs. (4.48) and (4.31) imply

$$\begin{aligned} \mathcal{A}_B &= \sum_{n=0}^{n(p)} (\mathcal{C}_n)_B (\mathcal{A}_n)_B + \mathcal{I}_p(\xi_i, y_j)_B, \\ (\mathcal{A}_n)_B &= T \langle 0 | \prod_{j=1}^N O_n(0) B_j(y_j) | 0 \rangle_B \end{aligned} \quad (4.52)$$

where $\mathcal{S}_p$ is constructed from Eq. (4.31) and the Fourier transforms of $\tilde{S}'_p$ amplitudes.

According to the rule (1.41), $\mathcal{A}$ and $\mathcal{A}_n$ are renormalized as follows:

$$\begin{aligned} \mathcal{A}_R &= \prod_{ij} Z_{A_i}^{-1} Z_{B_j}^{-1} \mathcal{A}_B + \delta[\xi_i - \xi_j, \rho \xi_i - y_j, y_i - y_j] \\ (\mathcal{A}_n)_R &= Z_n^{-1} \prod_j Z_{B_j}^{-1} (\mathcal{A}_n)_B + \delta[y_i, y_i - y_j] \end{aligned} \quad (4.53)$$

Some of the Z -factors (e.g. Z_n) may be matrices. The terms $\delta[x_i, y_j, \dots]$, which involve δ -functions of the coordinate differences $x_i, y_j, \dots$, are subtractions

*) Unlike Refs. 15, 71, 81, 86, 90, which deal directly with renormalized amplitudes. The present approach is less satisfactory but avoids some complicated algebra.

generated by counterterms $\Delta(A_1 A_2 \dots B_1 B_2 \dots)$ and $\Delta(O_n B_1 B_2 \dots)$ in Eq. (1.39); [see note (a) following Eq. (4.14)]. In general, the terms $\sum_n C_n \mathcal{A}_n$ and $\mathcal{S}_p$ mix under renormalization. However, in renormalizable theories, renormalization does not increase the subdimensionalities of an amplitude. Therefore, the term $O(\mathcal{S}_p)$ in the equation

$$(\mathcal{L}_n)_R = Z_n \prod_i Z_{A_i}^{-1} (\mathcal{L}_n)_B + \delta[\xi_i - \xi_j] + O(\mathcal{S}_p) \quad (4.54)$$

can be absorbed into the definition of the renormalized amplitude $(\mathcal{S}_p)_R$ without changing its maximum subdimensionality. So the result is

$$\mathcal{A}_R = \sum_{n=0}^{n(p)} (\mathcal{L}_n)_R (\mathcal{A}_n)_R + \{\delta\text{-functions}\} + (\mathcal{S}_p)_R \quad (4.55)$$

where the remainder amplitude obeys the bound

$$(\mathcal{S}_p/\mathcal{A})_R = O(\rho^p \ln^p \rho), \quad \left(\begin{array}{l} p = \text{positive integer;} \\ \rho \rightarrow 0 \text{ with } \xi_i, y_j \text{ fixed} \end{array} \right) \quad (4.56)$$

Note

- i) The terms $C_n O_n$ do not depend on the operators B_j because the sets of intermediate lines $\{k_1, \dots, k_{M(m)}\}_m$ do not involve sources. Because of this restriction on intermediate lines, the coordinates ξ_i, y_j must be held fixed as ρ tends to zero in (4.56).
- ii) Strictly speaking, the term $\{\delta\text{-functions}\}$ [absent for the special case (4.28)] violates Eq. (4.1) because it need not be negligible relative to C_n and may depend on the B_j operators. However, the violation is of the same form as the δ -function ambiguities of ordered products, so it is of little practical importance.
- iii) It is permissible to fix the momenta of simple (i.e. non-composite) operators B_j in Eqs. (4.55) and (4.56). For example, consider the operator B_2 in Fig. 13; even if its momentum is fixed, its source line cannot be intermediate because the corresponding subgraph would not be 1PI for $\xi_i = \xi$.
- iv) The on-shell states $|I\rangle, |F\rangle$ of Eq. (4.14) can be recovered by choosing some of the B_j in (4.55) to be suitable interpolating operators. If simple operators are used, remark (iii) is relevant. More generally, the on-shell limit isolates the set of graphs for which the interpolating operator (simple or composite) communicates with other external lines via a single internal line. For this subset, the source of the interpolating field cannot be an intermediate line, so the time-ordered case of (4.14) is obtained.

The set of allowed discontinuities of $T\langle F | \Pi_{ij} A_i B_j | I \rangle$ is restricted by the locality of the operator O_n on the right-hand side of Eq. (4.14). If the ordering of the A_i operators with respect to each other (but not to the B_j operators) is

partially or completely removed, $T\langle F | \Pi_j O_n B_j | I \rangle$ remains completely time-ordered, while the $i\epsilon$ -prescription for coefficient functions is adjusted accordingly; e.g.

$$\begin{aligned} T\{A_1(\xi) A_2(0)\} &\sim (\xi^2 - i\epsilon)^\alpha O_1(0) , \\ A_1(\xi) A_2(0) &\sim (\xi^2 - i\epsilon\xi_0)^\alpha O_1(0) , \quad (\xi \rightarrow 0) . \end{aligned} \quad (4.57)$$

It is also possible to unorder part of $\Pi_j B_j$ on each side of (4.14), retaining the time-ordering between $A_i(\rho\xi_i)$ (or O_n) and $B_j(y_i)$ for *all* i, j . Special precautions are necessary when unordering a $A_i B_j$ pair. In general, the relevant B_j operator should not appear in the middle of the partially or completely unordered product $\Pi_i A_i$: e.g.

$$\begin{aligned} \langle 0 | A_1(\xi) A_2(0) B(y) | 0 \rangle &\sim (\xi^2 - i\epsilon\xi_0)^\alpha \langle 0 | O_1(0) B(y) | 0 \rangle \\ \langle 0 | A_1(\xi) B(y) A_2(0) | 0 \rangle &\sim ?? , \quad (\xi \rightarrow 0 , y \text{ fixed}) \end{aligned} \quad (4.58)$$

These remarks are important⁹¹⁾ for the process

$$e^+ + e^- \longrightarrow h + X \quad (4.59)$$

where h is an observed hadron (momentum p_h) and X denotes other final-state hadrons with combined momentum p_X . The one-particle inclusive cross-section is proportional to

$$\bar{W}_{\mu\nu} = \sum_X \langle 0 | J_\mu(0) | Xh \rangle \langle Xh | J_\nu(0) | 0 \rangle (2\pi)^4 \delta^4(p_X + p_h - q) \quad (4.60)$$

where q is the momentum of the virtual photon. If Φ_h is an interpolating operator for h , the corresponding off-shell amplitude in coordinate space is

$$\mathcal{W}_{\mu\nu} = \langle 0 | T\{J_\mu(\xi) \Phi_h(y_1)\} \bar{T}\{J_\nu(0) \Phi_h(y_2)\} | 0 \rangle \quad (4.61)$$

where the coordinate ξ is conjugate to q and $\bar{T}$ denotes the anti-T-product. It is not possible to substitute an operator-product expansion for $J_\mu J_\nu$ because local operators O_n cannot be produced by the combination $T(A_1 B_1) \bar{T}(A_2 B_2)$ [in the notation of (4.14)]. Mueller⁹¹⁾ has suggested an expansion of $\mathcal{W}_{\mu\nu}$ in which there are coefficient functions (different from the Wilson coefficient functions) which do not multiply local operators O_n . Derivations have been given for some non-gauge theories^{91, 92)}.

4.2 Renormalization-group properties

A particular sector $(J, I, \dots)$ of the operator-product expansion (4.1) is obtained by projecting out intermediate sets $\{k_1, \dots, k_M\}$ with total quantum numbers $(J, I, \dots)$, or equivalently, by a judicious choice of operators B_j in Eq. (4.29):

$$\mathcal{A} \longrightarrow \mathcal{A}_{(J, I, \dots)}$$

The corresponding operators $\{O_n\}$ have quantum numbers $(J, I, \dots)$. For simplicity, let us assume that there is a single operator O_n (matrix element $\mathcal{A}_n$) which contributes the first p powers of the scaling parameter ρ to the asymptotic expansion of $\mathcal{A}$, order-by-order in perturbation theory:

$$\begin{aligned} \mathcal{A}_{(J,I,\dots)}/\mathcal{A}_n &= F_n(\rho \xi_i, y_j; g, m, \mu) \\ &= \mathcal{C}_n(\rho \xi_i; g, m, \mu) \{1 + O(\rho^p \ln^p \rho)\} , \quad (4.62) \\ &(\rho \rightarrow 0, \quad p = \text{positive integer}) \end{aligned}$$

As indicated by the notation, we choose to carry out the analysis using a mass-independent renormalization prescription*).

The improved CS equation for the ratio F_n is (neglecting δ -function subtractions)

$$\begin{aligned} [\mu \partial/\partial \mu + \beta(g) \partial/\partial g + \gamma_m(g) m \partial/\partial m + \gamma(g) - \gamma_n(g)] \cdot F_n(\rho \xi_i, y_j; g, m, \mu) &= 0 \quad (4.63) \end{aligned}$$

where $\gamma(g)$ and $\gamma_n(g)$ are the γ -functions for the amplitudes $\mathcal{A}$ and $\mathcal{A}_n$ [with $\gamma(0) = 0 = \gamma_n(0)$]. The addition rule (1.46) implies

$$\begin{aligned} \gamma(g) &= \sum_i \gamma_{A_i}(g) + \sum_j \gamma_{B_j}(g) \\ \gamma_n(g) &= \gamma_{O_n}(g) + \sum_j \gamma_{B_j}(g) \end{aligned}$$

so all dependence on γ -functions for the B_j operators drops out in the difference

$$\gamma(g) - \gamma_n(g) = \sum_i \gamma_{A_i}(g) - \gamma_{O_n}(g) \quad (4.64)$$

Equations (4.62)-(4.64) imply

$$\begin{aligned} [\mu \partial/\partial \mu + \beta \partial/\partial g + \gamma_m m \partial/\partial m + \sum_i \gamma_{A_i} - \gamma_{O_n}] \mathcal{C}_n &= 0 [\rho^p \ln^p \rho \mathcal{C}_n(\rho)] \quad (4.65) \end{aligned}$$

Since $\mathcal{C}_n$ does not depend on y_j , $\mu \partial/\partial \mu$ can be eliminated in terms of $m \partial/\partial m$ and either $\rho \partial/\partial \rho$ or $\eta \partial/\partial \eta$ ($\eta = \rho^{-1}$) using a homogeneity equation of the form (3.44): e.g.

$$[\mu \partial/\partial \mu + m \partial/\partial m + \eta \partial/\partial \eta - d_{\mathcal{C}}] \tilde{\mathcal{C}}_n(\eta q_i; g, m, \mu) = 0 \quad (4.66)$$

where $\tilde{\mathcal{C}}_n(q)$ is the Fourier transform of $\mathcal{C}_n(\xi)$ and $d_{\mathcal{C}}$ is the mass dimensionality of $\tilde{\mathcal{C}}_n$. Thus the analogue of Eq. (3.46) is

$$\begin{aligned} [\eta \partial/\partial \eta - \beta \partial/\partial g + (1 - \gamma_m) m \partial/\partial m - (d_{\mathcal{C}} + \sum_i \gamma_{A_i} - \gamma_{O_n})] \tilde{\mathcal{C}}_n &= 0 [\eta^{-p} \ln^p \eta \tilde{\mathcal{C}}_n] \quad (4.67) \end{aligned}$$

*) See Ref. 7. The corresponding analysis for the ordinary CS equation^{18,71,86,87) involves expanding $T(\Delta \Pi_{ij}; A_i B_j)$, where Δ is the zero-momentum mass-insertion operator [e.g. $\Delta = \int d^4x : \bar{\psi}(x) \psi(x) :$]. Straightforward substitution of (4.1) works for the leading power in a given sector, but for non-leading powers, the expansion of $:\bar{\psi} \psi: \Pi_{ij} A_i$ also contributes. Compare Eqs. (4.17)-(4.27).}

If m is a fermionic mass, the amplitudes

$$\tilde{\mathcal{C}}_n^{(r)}(\eta q_i; g, \mu) = \left(\frac{\partial}{\partial m}\right)^r \tilde{\mathcal{C}}_n(\eta q_i; g, m, \mu) \Big|_{m=0} \quad (4.68)$$

exist in perturbation theory for $r = 0, 1, 2$, but not for $r = 3, 4, \dots$, so

Eq. (4.67) becomes

$$\left[\eta \partial / \partial \eta - \beta \partial / \partial g - \left\{ d_{\mathcal{C}} + \sum_i \gamma_{A_i} - \gamma_{O_n} - r(1 - \gamma_m) \right\} \right] \tilde{\mathcal{C}}_n^{(r)} = 0, \quad (4.69)$$

$$0 \leq r \leq \text{Min}(2, p-1).$$

The consequences of (4.69) are analogous to those of Eq. (2.33):

i) Broken scale invariance:

$$\mathcal{C}_n^{(r)} \sim \text{const. } \rho^{-P(r,n)}, \quad (4.70)$$

$$P(r,n) = \sum_i d_{A_i} - d_{O_n} - r(4 - d_{\Delta})$$

For $r = 0$, Eq. (4.70) extends to coefficient functions⁴⁾ the rule (2.56) that dynamical dimensions are conserved at short distances. Non-leading terms $r = 1, 2, \dots$, are damped by a power $r(4 - d_{\Delta})$ of ρ , as in Eq. (3.36), if the condition (3.37) is satisfied. For the special case

$$A_i = (\text{partially}) \text{ conserved currents}, \quad O_n = \text{mass operator} \quad (4.71)$$

there is a cancellation^{4,7)} of anomalous dimensions for the leading term ($r = 1$)

$$P(r=1, n=\Delta) = \sum_i d_{A_i} - 4 = \text{integer} \quad (4.72)$$

which is important for the short-distance analysis of $\eta \rightarrow 3\pi$ decay and other weak and electromagnetic corrections to strong interactions.

ii) Asymptotic freedom^{7,93)}:

$$\mathcal{C}_n^{(r)} \sim \text{const. } \rho^{-p(r,n)} (\ln \rho)^{\lambda(r,n)} \{ 1 + O(\ln |\ln \rho| / \ln \rho) \} \quad (4.73)$$

$$p(r,n) = \sum_i \dim A_i - \dim O_n - r$$

$$\lambda(r,n) = \{ \sum_i c_{A_i} - c_{O_n} + r c_m \} / 2b$$

Here, $\dim Q$ denotes the canonical dimension of Q , b is defined in Eq. (2.80), and c_Q is the one-loop coefficient in the expansion

$$\gamma_Q(g) = c_Q g^2 + O(g^4) \quad (4.74)$$

The result (4.73) generalizes Eq. (2.86). If A_i and O_n are conserved or partially conserved currents for $r = 0$, or if Eq. (4.71) is applicable⁷⁾ with $r = 1$, the logarithmic power $\lambda(r,n)$ vanishes and the leading contribution to $\mathcal{C}_n^{(r)}$ (including the constant of proportionality) is given by the free-field result. [The last statement assumes that the complete operator O_n is generated in the expansion of ΠA_i in free-field theory. This need not be the

case for $O_n = \theta_{\mu\nu}$; e.g. if the operators A_i are chiral currents, only the fermionic part of $\theta_{\mu\nu}$ is generated. The result⁹³⁾ is that the constant of proportionality is computable and differs from the constant which normalizes the term $\theta_{\mu\nu}(\text{fermionic})$ in free-field theory.]

If we want to check these predictions for short-distance behaviour by measuring the amplitudes $\tilde{C}_n(q_i)$, it is essential that the momenta $\{q_i\}$ be non-exceptional. For example, consider the connected matrix element

$$\begin{aligned} W_{\mu\nu} &= (2\pi)^{-1} \int d^4x e^{iq \cdot x} \langle p | [J_\mu(x), J_\nu(0)] | p \rangle \\ &= -(g_{\mu\nu}/M) F_1(\xi, q^2) + (p_\mu p_\nu / p \cdot q) F_2(\xi, q^2) \\ &\quad + (i \varepsilon_{\mu\nu\lambda\eta} p^\lambda q^\eta / 2 p \cdot q) F_3(\xi, q^2) \\ &\quad + \text{other terms if } J_\mu \text{ not completely conserved} ; \end{aligned} \quad (4.75)$$

$$-1 \leq \xi = -q^2 / 2 p \cdot q = \omega^{-1} \leq 1 ; \quad F_i(-\xi, q^2) = \pm F_i(\xi, q^2) ;$$

$$|p\rangle = \text{nucleon, mass } M, \text{ momentum } p$$

For $0 \leq \xi \leq 1$, $W_{\mu\nu}$ is proportional to the total cross-section for deep inelastic lepton-nucleon scattering, where q is the momentum transferred by the current and F_i are the usual structure functions. As is well known^{94,95)}, the Bjorken limit $-q^2 \rightarrow \infty$ at fixed ξ corresponds to $x^2 \rightarrow 0$, not $x_\mu \rightarrow 0$. In the language of Eqs. (2.1) and (2.7), this is because the limit involves exceptional values of q :

$$q = \eta l + c ; \quad \eta \rightarrow \infty, \quad l^2 = 0, \quad p \cdot l \neq 0, \quad c \cdot l \neq 0. \quad (4.76)$$

Also well known⁹⁶⁾ is the fact that the leading Wilson coefficient function in the spin-J sector is asymptotically proportional to the moment

$$M_n^i(q^2) = \int_0^1 d\xi \xi^n F_i(\xi, q^2), \quad (n = J-1, J-2, J-1 \dots \text{ for } i = 1, 2, 3 \dots) \quad (4.77)$$

i.e. the $x_\mu \rightarrow 0$ limit corresponds to the limit $-q^2 \rightarrow \infty$ of M_n . It does not matter that the non-exceptional limit ($l^2 \neq 0$) is not kinematically consistent with the bound $|\xi| \leq 1$. This bound is relevant for the commutator amplitude (4.75), whereas the lack of exceptionality of the $\eta \rightarrow \infty$ limit should be checked for the time-ordered analogue of (4.75):

$$\begin{aligned} T &= T(\xi, q^2), \quad \xi = 2 p \cdot q / \sqrt{-q^2}, \\ -q^2 &\rightarrow \infty \quad \text{at fixed } \xi. \end{aligned} \quad (4.78)$$

(For simplicity, the indices μ, ν have been dropped.) Given a suitably normalized spin-J projection^{*}) of T ,

$$T_n(q^2) \propto (q^2)^{\text{power}} \int_{-1}^1 dv (1-v^2)^{1/2} C_n(v) T(-iv, q^2) \quad (4.79)$$

^{*}) Details of Eqs. (4.79) and (4.80), such as explicit formulas for μ_n , are given in Ref. 97. The functions $C_n(v)$ are Gegenbauer polynomials.

the $-q^2 \rightarrow \infty$ limit of T_n is non-exceptional and is determined by $\tilde{C}_n(q)$, and the commutator amplitude is recovered by taking the absorptive part:

$$\begin{aligned} \mu_n(q^2) &= \text{Abs } T_n(q^2) , \\ \mu_n/M_n &\longrightarrow 1 , \quad (-q^2 \rightarrow \infty) . \end{aligned} \quad (4.80)$$

The difference between μ_n and M_n is that only spin-J operators contribute to μ_n , whereas non-leading powers in the expansion of M_n involve a mixture of spins.

It would be desirable to directly measure the q -dependence of the moments in order to distinguish the alternatives

$$M_n(q^2) \sim \begin{cases} K_n & (\text{Bj. scaling}) \\ K'_n (\ln q^2)^{-\lambda(n)} & (\text{As. freedom, Eq. (4.73)}) \\ K''_n (-q^2)^{-a(n)/2} & (\text{Br. scale invariance, Eq. (4.70)}) \end{cases} \quad (4.81)$$

where K_n , K'_n , and K''_n are q -independent and the powers $\lambda(n)$, $a(n)$ are given by

$$\lambda(n) = c_{0n}/2b , \quad a(n) = \gamma_{0n}(q_\infty) \quad (4.82)$$

However, it has been observed⁹⁸⁾ that $M_n(q^2)$ and $\mu_n(q^2)$ (small n) differ significantly for $1 \leq -q^2 \leq 10 \text{ (GeV)}^2$, so there is an ambiguity caused by non-leading terms not being negligible. Thus it is not surprising that all three possibilities^{*}) (4.81) continue to be discussed in the literature. For example, Parisi¹⁰⁰⁾ and Nachtmann⁹⁸⁾ present the case for large anomalous dimensions $a(n)$ ($\approx 1/2$ or 1). Parisi's method [in which the moment equations (4.77) are inverted] has since been applied to asymptotically free theories^{101,102)}.

The most important qualitative feature of (4.81) is that, if Bjorken scaling is not valid, $F_i(\xi, q^2)$ should rise at small ξ and fall near $\xi = 1$ as $-q^2$ increases. This is because the smallest value of n ($n = 0$) corresponds to a conserved operator such as $\theta_{\mu\nu}$ ($\lambda(0) = 0 = a(0)$) and positivity requirements⁹⁷⁾ force $a(n)$ and $\lambda(n)$ to increase with n ($\lambda(n), a(n) > 0, n > 0$), so all moments decrease except for the asymptotically constant $n = 0$ moment. Eventually, F_i collapses to a δ -function:

$$F_i(\xi, q^2) \longrightarrow 2(K'_0 \text{ or } K''_0) \delta(\xi) \quad (4.83)$$

Recently, the FNAL muon-scattering group¹⁰³⁾ reported indications that the structure functions exhibit this behaviour.

^{*}) It has been argued^{38,99)} that Bjorken scaling is not consistent with the renormalization group. Note that the possibility $a(n) = 0$ (all n) for gauge theories with $g_\infty \neq 0$ has yet to be excluded.

4.3 Complications for Yang-Mills theories

In many cases, the operator O_m is not multiplicatively renormalized. Instead, there is a set of operators $\vec{O}_m$ which generate each other as renormalization counterterms

$$\vec{O}_m \longrightarrow \vec{O}_m \vec{Z}_m^{-1} \quad (4.84)$$

where the Z-factor and its γ -function

$$\vec{\gamma}_m = \lim_{n \rightarrow 4} (\mu \partial / \partial \mu \vec{Z}_m) \vec{Z}_m^{-1} \quad (4.85)$$

are matrices. [I am using dimensional regularization, as in (3.43), so O_m has been substituted for O_n in order to avoid confusion with the complex parameter n].

According to Eq. (1.8), the counterterms ΔQ of a vertex Q have canonical dimension $\dim(\Delta Q)$ less than or equal to $\dim(Q)$. Counterterms with $\dim(\Delta Q) < \dim(Q)$ are not generated unless masses or other dimensionful coupling constants are present; e.g. $\bar{\psi}\psi$ generates $m\bar{\psi}\psi$. So, if we restrict ourselves to the leading singularities G^{as} and $C_n(r=0)$, it is sufficient to compute all counterterms of the zero-mass theory:

$$\dim(\Delta Q) = \dim(Q) \quad (4.86)$$

In particular, the asymptotic behaviour of the moments $M_m(q^2)$ in (4.81) is controlled by a term $\vec{C}_m \cdot \vec{O}_m$ in the operator-product expansion of $J_\mu J_\nu$, where all operators $(O_m)_i$ in the set $\vec{O}_m$ have the same canonical dimension^{*)}

$$\dim O_m = J + 2, \quad (J = \text{spin of } O_m) \quad (4.87)$$

and where the coefficient functions $\vec{C}_m(r=0)$ satisfy the renormalization-group equation

$$[\eta \partial / \partial \eta - \beta(g) \partial / \partial g + \vec{\gamma}_m(g) - d(m)] \vec{\mathcal{C}}_m(\eta q; g, \mu) = 0 \quad (4.88)$$

$(d(m) = \text{mass dimensionality of } \mathcal{C}_m)$

The new feature of the solution of Eq. (4.88) [compared with the solution (2.34) of Eq. (2.33)] is that matrices occurring in the exponential must be ordered:

$$\begin{aligned} \vec{\mathcal{C}}_m(\eta q; g, \mu) &= \eta^{-d(m)} \left\{ \exp - \int_{\bar{g}}^{\bar{g}} dx \vec{\gamma}_m(x) / \beta(x) \right\}_{\text{ord.}} \vec{\mathcal{C}}_m(q; \bar{g}, \mu), \\ \left\{ \exp \dots \right\}_{\text{ord.}} &= \sum_{r=0}^{\infty} (-1)^r \int_{\bar{g}}^{\bar{g}} dx_1 \int_{x_1}^{\bar{g}} dx_2 \dots \int_{x_{r-1}}^{\bar{g}} dx_r \cdot \\ &\quad \cdot \vec{\gamma}_m(x_1) \vec{\gamma}_m(x_2) \dots \vec{\gamma}_m(x_r) / \prod_{i=1}^r \beta(x_i) \end{aligned} \quad (4.89)$$

*) In perturbation theory, the "twist" $\tau(Q)$ of an operator¹⁰⁴⁾ is defined as follows: $\tau(Q) = \dim(Q) - \text{spin}(Q)$. Thus Eq. (4.87) refers to twist-2 operators.

The limit $\eta \rightarrow \infty$ applied to (4.89) produces the following results:

- i) Broken scale invariance: anomalous dimensions are given by the eigenvalues of $\vec{\gamma}_m(g_\infty)$.
- ii) Asymptotic freedom⁹³⁾: the constant c_{0m} in Eq. (4.73) should be replaced by an eigenvalue of the matrix $\vec{c}_m$ given by

$$\vec{\gamma}_m(g) = g^2 \vec{c}_m + O(g^4) \quad (4.90)$$

In each case, the dominant contribution in Eq. (4.81) is obtained by substituting the smallest eigenvalue in Eq. (4.82).

The application of these rules to the Yang-Mills Lagrangian (2.65) is not entirely straightforward. The trouble is that, whatever the gauge, the mixing matrix $\vec{Z}$ for SU(3)-singlet twist-two operators is enormous. It would be an onerous task to compute all of the matrix elements of $\vec{c}_m$, extract the eigenvalues, and (by considering a sufficiently large class of gauges) determine which eigenvalues are physically relevant. Indeed, the original calculations⁹³⁾ are based on a simplified prescription which is assumed to be equivalent to the above rules. So the problem is to verify the correctness of the results for twist-two operators and more generally, to find labour-saving prescriptions for arbitrary-twist operators.

Let us begin with the tree approximation for the upper blob graphs $\mathcal{G}'$ in Figs. 2 and 14 (i.e. the zero-loop approximation for coefficient functions). Even for this case, the complete set of allowed operators is complicated. The intermediate sets $\{k_1, \dots, k_M\}$ may include Fadde'ev-Popov ghosts as well as fermions and gauge mesons, so some of the tree-approximation operators are ghost-dependent and hence not manifestly gauge-invariant. However, if we restrict ourselves to twist-two operators, it is easy to see that intermediate ghost lines cannot appear in the tree approximation for $\mathcal{G}'$. An analysis of Abelian gauge theories by Gross and Treiman¹⁰⁴⁾ can be readily generalized to show that the allowed operators are given by symmetrizing the gauge-invariant combinations

$$O^{\mu_1 \dots \mu_J} = \bar{\psi} \gamma^{\mu_1} D_f^{\mu_2} \dots D_f^{\mu_J} \{1 \text{ or } \lambda^i/2\} \psi \quad (4.91)$$

(λ^i = ordinary -SU(3) matrices)

in the Lorentz indices $\mu_1, \dots, \mu_J$ and removing traces. The symmetric-traceless part can be projected out by multiplying the operators in (4.91) by the source⁹³⁾

$$X_{\mu_1} X_{\mu_2} \dots X_{\mu_J}, \quad (X^2 = 0) \quad (4.92)$$

The result is a pair of spin-J operators, one an SU(3) singlet, the other an octet:

$$\begin{aligned} Q_1 &= \bar{\psi} X \cdot \gamma (iX \cdot D_f)^{J-1} \psi \\ Q(8) &= \bar{\psi} X \cdot \gamma (iX \cdot D_f)^{J-1} (\lambda^i/2) \psi \end{aligned} \quad (4.93)$$

The next step is to consider the renormalization of these operators so that the one-loop contributions $\vec{c}$ to the mixing matrix can be isolated. In this context, the one-loop approximation means that we are considering one-loop counterterms $\Delta_1 Q$ of Q , plus one-loop counterterms $\Delta_1 \Delta_1 Q$ of $\Delta_1 Q$, plus $\Delta_1 \Delta_1 \Delta_1 Q$, ..., and so on, until no new vertices are produced. All of these counterterms necessarily appear as operators in the Wilson expansion^{*)}; otherwise, the expansion would not transform consistently under renormalization-group transformations.

Renormalization produces complications because the Lagrangian (2.65) which generates the Feynman rules is not manifestly gauge-invariant. Instead, it is invariant under the following set of transformations, introduced by Becchi, Rouet and Stora^{43,50)}:

$$\begin{aligned} \delta A_\mu^a &= -D_\mu^{ab}(A) \phi^b \delta\lambda, & \delta\psi &= -ig \phi^a \delta\lambda \tau^a \psi \\ \delta \phi^{a*} &= C^a(A) \delta\lambda, & \delta \phi^a &= \frac{1}{2} c^{abc} \phi^b \phi^c \delta\lambda \end{aligned} \quad (4.94)$$

Observe that A_μ^a and ψ undergo a gauge transformation (2.68) with gauge function $\delta\omega^a$ given by $\phi^a \delta\lambda$. Since the ghost field obeys Fermi-Dirac statistics, the parameter $\delta\lambda$ should be treated as an anticommuting number¹⁰⁶⁾. The function $C^a(A)$ refers to the choice of gauge; according to the quantization rules^{45,49)}, the gauge-fixing term can be written in the form

$$\mathcal{L}_{g.f.} = -\frac{1}{2} \{C^a(A)\}^2 \quad (4.95)$$

with corresponding ghost Lagrangian given by

$$\mathcal{L}_{ghost} = \phi^{a*} \delta C^a(A) / \delta\lambda \quad (4.96)$$

As in Eqs. (2.65)-(2.72), Eq. (4.94) involves unrenormalized quantities.

The rules for counterterms ΔQ of a gauge-invariant operator Q are as follows:

- i) If a given order of perturbation theory produces no ghost-dependent terms in ΔQ , ΔQ is manifestly gauge-invariant. Roughly speaking, this is because $\phi \delta\lambda$ acts as a c-number for the set of Feynman diagrams being considered.
- ii) The appearance of ghost-dependent counterterms in ΔQ implies a lack of explicit gauge invariance for the accompanying meson-dependent counterterms. A non-trivial analysis¹⁰⁷⁻¹⁰⁹⁾ based on the Ward identities associated with (4.94) shows that ΔQ is symmetric under a source-dependent generalization of (4.94).

*) To simplify the discussion, I shall assume that operators not generated as counterterms of tree-approximation operators do not appear in the Wilson expansion. This problem is circumvented in Ref. 105.

Important special cases of rule (i) are:

- a) The operator $Q(8)$ is multiplicatively renormalized⁹³⁾ in Lorentz-invariant gauges, because octet twist-two operators cannot depend on ghosts and $Q(8)$ is the only vertex which is both Lorentz- and gauge-invariant.
- b) All counterterms ΔQ in the axial gauge¹¹⁰⁾

$$N^\mu A_\mu = 0, \quad (N^\mu = \text{fixed 4-vector}) \quad (4.97)$$

are manifestly gauge-invariant because the relevant gauge-fixing term

$$\mathcal{L}_{g.f.} = - \lim_{\alpha \rightarrow 0} \frac{1}{2\alpha} (N \cdot A)^2 \quad (4.98)$$

produces a vanishing interaction term $g\phi^*(N \cdot A \times \phi)$ in (4.96).

Since the renormalization of $Q(8)$ presents no difficulties, let us concentrate on the mixing matrix generated by the singlet operator Q_1 in Eq. (4.93). The one-loop counterterms $\Delta_1 Q_1$ include Q_1 itself plus a mesonic operator generated by the set of 1PI diagrams displayed in Fig. 17. Only fermion propagators are involved, so the new counterterm is proportional to the gauge-invariant combination

$$Q_2 = \chi^\mu F_{\mu\alpha} (i\chi \cdot D)^{J-2} F^{\alpha\nu} \chi_\nu \quad (4.99)$$

The next set of counterterms $\Delta_1 Q_2$ includes Q_1 and vertices given by the divergent parts of the diagrams in Fig. 18. As noted by Gross and Wilczek⁹³⁾, the ghost contributions from Fig. 18a do not vanish. This means that the accompanying mesonic terms (Fig. 18b) are necessarily not gauge-invariant¹¹¹⁾.

The prescription adopted in Ref. 93 is to construct a 2×2 submatrix $\overleftrightarrow{Z}(2)$ of $\overleftrightarrow{Z}$ corresponding to the gauge-invariant combinations Q_1, Q_2 . Three of the elements of $\overleftrightarrow{Z}(2)$ ($Q_1 \rightarrow Q_1$, $Q_1 \rightarrow Q_2$, $Q_2 \rightarrow Q_1$) are obtained from $\Delta_1 Q_1$ and $\Delta_1 Q_2$ without ambiguity. The diagonal element $Q_2 \rightarrow Q_2$ is defined to be the contribution from 1PI diagrams in which Q_2 is coupled to only *two* external lines, both of which are put on-shell. Thus the rows and columns of $\overleftrightarrow{Z}$ are chosen to refer to a basis vector with elements $\{Q_1, Q_2, O(A^2)$ operators which vanish on-shell, other operators, i.e. $O(A^3)$, $O(A^4)$, ..., $O(\phi^*\phi)$, $O(\phi^*A\phi)$, ...}, and $\overleftrightarrow{Z}(2)$ is taken to be the 2×2 submatrix in the upper left-hand corner of $\overleftrightarrow{Z}$. The answer is supposed to be given by the eigenvalues of $\overleftrightarrow{Z}(2)$. Gross and Wilczek⁹³⁾ checked their answer by repeating their on-shell prescription for the axial gauge (4.97) in the light-like case $N^2 = 0$.

This prescription is not obviously correct because in general, the eigenvalues of a submatrix do not coincide with any of the eigenvalues of the complete matrix. The fact that Gross and Wilczek obtain the same answer in two sets of gauges is not conclusive; it merely suggests that the prescription is gauge-invariant for twist-two operators. No matter how rigorous the proof of gauge invariance may be, the answer is not correct unless it is an *eigenvalue* of the complete mixing matrix $\overleftrightarrow{Z}$.

Indeed, Kluberg-Stern and Zuber¹¹²⁾ have observed that the prescription fails for the twist-4, spin-0 operator $(F_{\mu\nu})^2$ in the usual Fermi gauges (2.70). Subsequent analysis^{107,109)} has shown this to be the rule rather than the exception. However, the exceptional case turns out to be that of twist-two operators. For reasons which can be understood only by consulting the detailed analysis¹⁰⁷⁾, the eigenvalues of $\vec{Z}(2)$ are indeed eigenvalues of $\vec{Z}$. Instead of continuing in gauges with ghosts, I shall explain what happens for axial gauges¹⁰⁵⁾.

All Green's functions are homogeneous in the fixed vector N_μ , because the only source of N_μ -dependence is Eq. (4.98) which generates the free propagator for gauge mesons:

$$\Delta_{\mu\nu}(k, N) = -\frac{i}{k^2} \left\{ g_{\mu\nu} - \frac{k_\mu N_\nu + k_\nu N_\mu}{k \cdot N} + \frac{N^2 k_\mu k_\nu}{(k \cdot N)^2} \right\} \quad (4.100)$$

In order to carry out the renormalization program, it is essential that the light-like case $N^2 = 0$ chosen by Gross and Wilczek be avoided. The chief characteristic of integrals involving denominators $(k \cdot N)^{-\ell}$ for $N^2 = 0$ is that it is impossible to maintain power-counting¹¹³⁾; there are insufficient scalar products available. For example, a momentum-dependent scalar counterterm involving N_μ necessarily contains a factor

$$V(N, p, q) = \prod_{i=1}^r N \cdot p_i / N \cdot q_i, \quad (p, q = \text{momenta}, N^2 = 0) \quad (4.101)$$

because it must be homogeneous in N_μ . Since V is not a polynomial in p and q , power-counting does not work. In principle, renormalization is still permitted for the non-local vertex V , because the denominators $(N \cdot q_i)^{-1}$, $(N \cdot k)^{-\ell}$ are given principal-value singularities and hence have no absorptive part. However, explicit calculations¹⁰⁵⁾ of meson and fermion self-energies demonstrate the existence of divergent parts proportional to $\ln q^2$, where q is the momentum of the dressed propagator. The absorptive part does not vanish, so the $N^2 = 0$ gauge is not renormalizable. For $N^2 \neq 0$, these difficulties disappear^{105,114)}; the N_μ -dependence of counterterms always takes the form

$$\text{counterterm} = \left\{ N_{\mu_1} \dots N_{\mu_{2r}} / (N^2)^r \right\} \left\{ N_\mu\text{-independent} \right\}, \quad (4.102)$$

($r = \text{integer}$)

or some linear combination thereof, and all counterterms are polynomials in momentum space.

Consider the renormalization mixing matrix Z generated by the operator Q_1 in axial gauges ($N^2 \neq 0$). In view of the trouble caused by ghosts in Lorentz-invariant gauges, it is tempting to assume that the absence of ghosts in axial gauges means that no spurious mixing occurs. The simplest way to disprove this idea is to compute the divergent part of the diagram shown in Fig. 19 for the case spin $J = 2$ ¹⁰⁵⁾,

$$P.P. \Gamma_{ij}(J=2) = \left\{ g^2 \delta_{ij} C_2(R) / 4\pi^2(n-4) \right\} \left\{ -\frac{4}{3} \not{\chi} \chi.q + \not{N} \chi.N q \chi / N^2 - 2 \not{\chi} (N \chi)^2 / N^2 - \not{\chi} \chi.N q N / N^2 \right\} \quad (4.103)$$

where P.P. denotes the pole part of the 1PI amplitude Γ_{ij} and $C_2(R)$ is the value of the quadratic Casimir operator for the fermionic representation R of the gauge group:

$$(\tau^a \tau^a)_{ij} = \delta_{ij} C_2(R) \quad (4.104)$$

In addition to the matrix element $Q_2 \rightarrow Q_1$ given by the term proportional to $\not{\chi} \chi.q$, three N_μ -dependent vertices are generated. They appear in the following list of independent gauge-invariant operators P_r evaluated at zero momentum [i.e. $P_r \rightarrow$ zeroth-order 1PI amplitude $\langle \psi_j(-q) P_r \bar{\psi}_i(q) \rangle$]:

$$\begin{aligned} P_1 &= i \bar{\psi} \not{N} \chi.D_f \psi \chi.N / N^2 \longrightarrow \delta_{ij} \not{N} \chi.q \chi.N / N^2 \\ P_2 &= i \bar{\psi} \not{\chi} \psi (\chi.N)^2 / N^2 \longrightarrow \delta_{ij} \not{\chi} (\chi.N)^2 / N^2 \\ P_3 &= i \bar{\psi} \not{\chi} N.D_f \psi \chi.N / N^2 \longrightarrow \delta_{ij} \not{\chi} N.q \chi.N / N^2 \\ P_4 &= i \bar{\psi} \not{N} N.D_f \psi (\chi.N)^2 / (N^2)^2 \longrightarrow \delta_{ij} \not{N} N.q (\chi.N)^2 / (N^2)^2 \end{aligned} \quad (4.105)$$

(Here, and in the rules stated previously, the characterization "gauge-invariant" refers only to the operator-dependent parts of counterterms. The rules do not forbid dependence on gauge-dependent c-numbers such as N_μ .) Since we are considering the case $J = 2$, the list (4.105) is restricted to operators which are quadratic in χ . Equations (4.103) and (4.105) become very complicated for arbitrary spins J. Similar results can be obtained for the diagrams in Fig. 20; except for the case $J = 2$, P.P. $\Gamma(J)$ is N_μ -dependent.

Evidently, the mixing matrix $\overleftrightarrow{Z}$ is very large. However, some simple features emerge if we define its rows and columns in terms of a well-chosen basis vector. Let us introduce operator classes C_p , where an operator Q belongs to C_p if it is of the form

$$Q = (\chi.N)^p \{ \chi.N - \text{independent terms} \}, \quad (p = 0, 1, 2, \dots, J) \quad (4.106)$$

The basis vector is partitioned in the following way: $\{(C_0 \text{ operators}), (C_1 \text{ operators}), (C_2 \text{ operators}), \dots, (C_J \text{ operators})\}$. The result is the partitioned matrix $\overleftrightarrow{Z}$ shown in Fig. 21. There is a series of submatrices $M_0, M_1, M_2, \dots, M_J$ which occupy the main diagonal, where M_p describes the mixing of operators within the class C_p .

All counterterms ΔQ of the operator Q in (4.106) contain the factor $(\chi.N)^p$, because $(\chi.N)^p$ commutes with loop integrals. Consequently, operators in the class C_p do not generate counterterms with smaller values of p:

$$C_p \longrightarrow C_{p'}, \quad (p' \geq p). \quad (4.107)$$

Equation (4.107) implies the identity

$$\det(\vec{Z} - \lambda \vec{I}) = \prod_{p=0}^J \det(M_p - \lambda I_p) \quad (4.108)$$

because all submatrices below the diagonal set $(M_0, M_1, M_2, \dots, M_J)$ vanish^{*}). Hence the eigenvalues of $\vec{Z}$ are given by the eigenvalues of $M_0, M_1, M_2, \dots, M_J$.

The only $\chi.N$ -independent gauge-invariant combinations are Q_1, Q_2 , and

$$Q_3 = (N^\alpha \chi^\beta F_{\alpha\beta}) (i\chi.D)^{J-2} (N^\gamma \chi^\delta F_{\gamma\delta}) / N^2 \quad (4.109)$$

so the matrix M_0 is 3×3 . The diagrams in Fig. 17 are N_μ -independent, so the matrix element $Q_1 \rightarrow Q_3$ vanishes. Also, an explicit calculation of the diagrams in Fig. 20 yields the result¹⁰⁵⁾

$$\begin{aligned} \text{P.P. } \Gamma^{ab}(J) &= \frac{g^2 C_2(G)}{2\pi^2(n-4)} \langle A_\alpha^a(q) Q_2 A_\beta^b(-q) \rangle_{\text{PI, bare}} \cdot \\ &\cdot \left\{ 1 - 1/J(J-1) - 1/(J+1)(J+2) + \sum_{j=2}^J j^{-1} \right\} + O(\chi.N) \end{aligned} \quad (4.110)$$

There are non-zero terms proportional to the bare vertices of Q_2 and (for $J \geq 4$) $\chi.N$ -dependent operators, but there is no term proportional to

$$\langle A_\alpha^a(q) Q_3 A_\beta^b(-q) \rangle_{\text{PI, bare}} = 2\delta^{ab} (N.q \chi_\alpha - \chi.q N_\alpha) (N.q \chi_\beta - \chi.q N_\beta) (\chi.q)^{J-2} / N^2, \quad (4.111)$$

$(J = 2, 4, 6, \dots)$

Hence the matrix element $Q_2 \rightarrow Q_3$ also vanishes. Furthermore, explicit calculations (with $N^2 \neq 0$) show that the 2×2 submatrix of M_0 generated by Q_1 and Q_2 is exactly the same as the submatrix $\vec{Z}(2)$ obtained in Ref. 93; for example, the $Q_2 \rightarrow Q_2$ matrix element can be checked by combining Eq. (4.110) with the one-loop result¹¹⁶⁾

$$Z_3(\text{axial gauge}) = 1 - (11C_2(G) - 4T(R)) g^2 / 24\pi^2(n-4) + O(g^4), \quad (N^2 \neq 0). \quad (4.112)$$

for the $\vec{Z}$ -factor of the gauge-meson field A_μ^a .

The results for M_0 are summarized in Fig. 22. Clearly, the eigenvalues of $\vec{Z}(2)$ are also eigenvalues of both M_0 and $\vec{Z}$. Therefore the results in Ref. 93 are correct, provided that the other eigenvalues of $\vec{Z}$ are unphysical. [A separate discussion is necessary in order to verify this last point¹⁰⁵⁾.]

The same procedure works for operators with arbitrary twist. Further simplification can be achieved by applying the following theorems¹⁰⁵⁾:

- i) The presence of a factor $D^\mu F_{\mu\nu}$, $\not{D}_f \psi$, or $\bar{\psi} \not{D}_f$ in the gauge-invariant combination Q means that ΔQ also contains one of these factors.

^{*}) That is, $\vec{Z}$ is block-triangular; see Fig. 21. The same property, involving just two operator classes, has been demonstrated for $\vec{Z}$ in Lorentz-covariant gauges^{107-109, 115)}.

ii) All N_μ -dependent counterterms contain either $D_{\mu\nu}^\mu$, $\bar{\psi}_f\psi$, or $\bar{\psi}\psi_f$ as a factor. Theorem (i) is the analogue of a theorem of Kluberg-Stern and Zuber¹⁰⁷⁾ for gauges with ghosts. Some consequences of theorem (ii) are evident in the twist-two case:

- a) The result $Q_2 \neq Q_3$, obtained in the one-loop approximation in Eq. (4.110), is true to all orders of perturbation theory.
- b) The operator P_4 , defined in Eq. (4.105), cannot contribute to Eq. (4.103), and the operators P_1 and P_3 necessarily appear in the combination $(P_1 - P_3)$.

4.4 $\eta \rightarrow 3\pi$ decay

Finally, let us consider a problem in current algebra¹¹⁷⁾, the soft-meson theorems for $\eta \rightarrow 3\pi$ decay. We shall see that it is essential that the analysis be carried out in terms of short-distance expansions. Only then does it become clear that there is an extra term¹¹⁸⁾ which must be added to the conventional predictions¹¹⁹⁻¹²¹⁾ for the decay amplitudes.

We begin by recalling the standard assumption¹²²⁾ for chiral symmetry-breaking terms in the energy density for strong interactions:

$$\theta_{00} = \bar{\theta}_{00} + u_0 + cu_8, \quad (c \simeq -1.25) \quad (4.113)$$

The operator $\bar{\theta}_{00}$ is $SU(3) \times SU(3)$ invariant. The scalar-density operators u_0, u_8 belong to the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation formed by the set $\{u_i, v_j$; $i, j = 0, 1, 2, \dots, 8$; u_i = scalar density, v_j = pseudoscalar density}. In gauge theories with explicit chiral symmetry breaking [$\mathcal{M} \neq 0$ in Eq. (2.65)], $(u_0 + cu_8)$ is given by the renormalized version of a mass term in $\bar{\psi}\mathcal{M}\psi$.

It is natural to assume that the strong-interaction energy density induced by second-order electromagnetic effects is^{*})

$$\mathcal{H}_{em}(\text{pure}) = -e^2 \int d^4x D^{\mu\nu}(x) T\{J_\mu(x) J_\nu(0)\} \quad (4.114)$$

where $D^{\mu\nu}(x)$ is the free photon propagator in coordinate space and J_μ is the electromagnetic current for hadrons; (see Fig. 23). However, the consequences of (4.114) disagree with experiment:

- a) Sutherland¹²³⁾ showed that the decay amplitudes $A(\eta \rightarrow \pi^+\pi^-\pi^0)$ and $A(\eta \rightarrow 3\pi^0)$ vanish in the $SU(2) \times SU(2)$ approximation in which one of the pions (charged or uncharged) is soft.
- b) Dashen¹²⁴⁾ obtained the rule

$$\begin{aligned} \langle K^+ | \mathcal{H}_{em}(\text{pure}) | K^+ \rangle &= \langle K^0 | \mathcal{H}_{em}(\text{pure}) | K^0 \rangle \\ &\simeq \langle \pi^+ | \mathcal{H}_{em}(\text{pure}) | \pi^+ \rangle = \langle \pi^0 | \mathcal{H}_{em}(\text{pure}) | \pi^0 \rangle \end{aligned} \quad (4.115)$$

^{*}) Added note: To normalize $\mathcal{H}_{em}$ correctly, set $D_{\mu\nu}(x)$ equal to $(ig_{\mu\nu}/8\pi^2)(x^2 - i\epsilon)^{-1}$.

which is not consistent with the observed $K^+ - K^0$ and $\pi^+ - \pi^0$ mass differences.

The standard remedy^{119,121)} is to include a u_3 tadpole¹²⁵⁾ in the expression for the energy density:

$$\text{total energy density} = \theta_{00} + \mathcal{H}_{em}(\text{tadpole}) \quad (4.116)$$

$$\mathcal{H}_{em}(\text{tadpole}) = \mathcal{H}_{em}(\text{pure}) + c_3 u_3, \quad (c_3 = O(e^2)). \quad (4.117)$$

In this scheme, Eq. (4.115) is retained and the tadpole is supposed to be responsible for the deviation

$$\delta m^2 = m^2(K^+) - m^2(K^0) - m^2(\pi^+) + m^2(\pi^0) \quad (4.118)$$

from Dashen's rule; e.g. the SU(3) approximation implies

$$\begin{aligned} \langle \pi | u_3 | \pi \rangle &\simeq 0, \\ \langle K^+ | u_3 | K^+ \rangle &\simeq - \langle K^0 | u_3 | K^0 \rangle; \end{aligned} \quad (4.119)$$

$$\delta m^2 \simeq 2c_3 \langle K^+ | u_3 | K^+ \rangle \simeq c_3 \sqrt{3} \langle \pi^0 | u_3 | \eta \rangle. \quad (4.120)$$

Similarly, Sutherland's null result for $\mathcal{H}_{em}(\text{pure})$ is retained and the decay amplitudes

$$T = - \langle 3\pi | \mathcal{H}_{em}(\text{tadpole}) | \eta \rangle \quad (4.121)$$

are supposed to be given by

$$T \simeq - c_3 \langle 3\pi | u_3 | \eta \rangle \quad (4.122)$$

in the SU(2) $\times$ SU(2) approximation. The amplitude $T(\eta \rightarrow \pi^+ \pi^- \pi^0)$ still vanishes when a charged pion becomes soft, but the soft- π^0 limit produces a new equal-time commutator

$$i[F_5^3, u_3] = (\sqrt{2} v_0 + v_8)/\sqrt{3} = v \quad (4.123)$$

$$T(\eta \rightarrow \pi^0(\text{soft}) 2\pi) = (c_3/F_\pi) \langle 2\pi | v | \eta \rangle \quad (4.124)$$

where F^i, F_5^j ($i, j = 1, \dots, 8$) are the charges which generate chiral SU(3) $\times$ SU(3)¹²²⁾,

$$F^i = \int d^3x \mathcal{F}_0^i(x), \quad F_5^i = \int d^3x \mathcal{F}_{50}^i(x) \quad (4.125)$$

the pion decay constant^{*)} $F_\pi \simeq 94$ MeV is given by

$$\langle \pi^0(q) | \mathcal{F}_{5\mu}^3(0) | 0 \rangle = - i q_\mu F_\pi \quad (4.126)$$

and the matrix element $\langle 2\pi | v | \eta \rangle$ ($= \langle 2\pi^0 | v | \eta \rangle = \langle \pi^+ \pi^- | v | \eta \rangle$) is evaluated at zero momentum transfer. If $\langle 2\pi | v | \eta \rangle$ is not negligibly small, the energy dependence of the decay spectra is given by the formulas^{120,126)}

*) Note the extra factor 2 in the definitions of chiral currents and F_π in Ref. 119: e.g. $F_\pi(\text{Ref. 119}) = 2F_\pi(4.126)$.

$$\begin{aligned} T(\eta \rightarrow 3\pi^0) &\simeq \text{constant} = (c_3/F_\pi) \langle 2\pi | v | \eta \rangle, \\ T(\eta \rightarrow \pi^+\pi^-\pi^0) &\simeq (1 - 2E_0/m_\eta) (c_3/F_\pi) \langle 2\pi | v | \eta \rangle, \end{aligned} \quad (4.127)$$

($E_0 = \pi^0$ energy with η at rest)

in agreement with experiment¹²⁷). Contracting another pion yields the equal-time commutator

$$i[F_5^3, v] = -u_3 \quad (4.128)$$

$$\langle 2\pi | v | \eta \rangle \simeq F_\pi^{-1} \langle \pi^0 | u_3 | \eta \rangle \quad (4.129)$$

Equations (4.120), (4.127), and (4.129) yield the result^{119,120})

$$\begin{aligned} T(\eta \rightarrow 3\pi^0) &\simeq \delta m^2/\sqrt{3} F_\pi^2 \\ T(\eta \rightarrow \pi^+\pi^-\pi^0) &\simeq (1 - 2E_0/m_\eta) \delta m^2/\sqrt{3} F_\pi^2 \end{aligned} \quad (4.130)$$

This corresponds to a partial decay width¹¹⁹) $\simeq 65$ eV for $\eta \rightarrow \pi^+\pi^-\pi^0$ which is considerably smaller than the experimental value¹²⁸), 204 ± 29 eV. It can be argued that the prediction (4.130) is not expected to be as accurate as (4.127) because it involves approximate $SU(3) \times SU(3)$, not $SU(2) \times SU(2)$. Even so, the agreement with experiment is not impressive.

Now we consider the short-distance approach to $\eta \rightarrow 3\pi$ decay. Wilson⁴) observed that the Sutherland analysis and its generalization to include tadpoles assume that the expression (4.114) for $\mathcal{J}\mathcal{C}_{em}$ (pure) converges. For the theory of broken scale invariance or asymptotically free theories, this is not true^{4,7}). When the operator-product expansion

$$T\{J_\mu(x) J_\nu(0)\} \sim \sum_n \mathcal{C}_{n\mu\nu}(x) O_n(0), \quad (x_\alpha \rightarrow 0) \quad (4.131)$$

is substituted in Eq. (4.114) to test the ultraviolet convergence of the integral $\int d^4x$, Lorentz invariance implies

$$\int d^4x \mathcal{C}_{n\mu\nu}(x) D^{\mu\nu}(x) = 0 \quad (4.132)$$

for all coefficient functions except those which multiply scalar operators. In the isospin-1 sector^{*)}, we have

$$T\{J_\mu(x) J_\nu(0)\} \sim \mathcal{C}_{\mu\nu}(x) u_3(0), \quad (x_\alpha \rightarrow 0, (J, I) = (0^+, 1) \text{ sector}) \quad (4.133)$$

This is an example of the special case (4.71), so the formula

$$\mathcal{C}_{\mu\nu}(x) \sim (\text{constant}) (g_{\mu\nu} x^2 + 2x_\mu x_\nu) / (x^2)^2, \quad (x_\alpha \rightarrow 0) \quad (4.134)$$

is valid for both broken scale invariance⁴) (irrespective of the anomalous dimension d of u_3 , $1 \leq d < 4$) and asymptotic freedom⁷). Since $D^{\mu\nu}(x)$ goes as x^{-2} , we see that the integral (4.114) diverges logarithmically.

*) Singularities in the $SU(3)$ -singlet sector also have to be removed, but this is irrelevant for $\eta \rightarrow 3\pi$ decay and electromagnetic mass differences.

The integral must be properly subtracted so that the result is a convergent expression to which current-algebraic techniques can be applied. The divergence is logarithmic, so a single subtraction suffices⁴⁾:

$$\mathcal{H}_{em} = -e^2 \int d^4x D^{\mu\nu}(x) \left[T\{J_\mu(x) J_\nu(0)\} - \mathcal{C}_{\mu\nu}(x) u_3(0) \right] + f u_3(0) \quad (4.135)$$

The finite constant f is introduced to adjust the finite part of the u_3 contribution to fit electromagnetic mass differences.

Previously, we saw that the tadpole term was introduced as an extra parameter in response to the difficulties associated with Eq. (4.114). Here, the u_3 -dependence of the electromagnetic interaction arises naturally. Indeed, the distinction between an electromagnetic term $\mathcal{H}_{em}$ (pure) and a strong isospin-breaking term $c_3 u_3$, which can be made for Eqs. (4.116) and (4.117), no longer exists.

Wilson^{4,129)} concludes that the interaction (4.135) yields the same results for $\eta \rightarrow 3\pi$ decay as the tadpole interaction (4.117). He identifies the $f u_3$ term as the Coleman-Glashow tadpole¹²⁵⁾ $c_3 u_3$, and argues that the finiteness of the other term means that Sutherland's argument applies to it, in the same way that the Sutherland argument applies to $\mathcal{H}_{em}$ (pure) if $\int d^4x$ in Eq. (4.114) happens to converge. However, for reasons which will now be explained, I believe that this conclusion has to be modified.

First, we note that the quantities $\mathcal{C}_{\mu\nu}(x)$, f in Eq. (4.135) are not absolute; they can be specified only with respect to a particular *subtraction condition*. The choice of a subtraction condition from the infinite set of possibilities is purely a matter of convenience. A label i will be added to a symbol to indicate its dependence on a given subtraction condition i :

$$f, \mathcal{C}_{\mu\nu}(x) \rightarrow f_i, \mathcal{C}_{\mu\nu}^i(x) \quad (4.136)$$

$$I_i = \int d^4x T\{J_\mu(x) J_\nu(0) - \mathcal{C}_{\mu\nu}^i(x) u_3(0)\} D^{\mu\nu}(x)$$

Of course, the total electromagnetic energy density $\mathcal{H}_{em}$ is observable, so it does not depend on i :

$$\mathcal{H}_{em} = -e^2 I_i + f_i u_3(0) \quad (4.137)$$

For example, we may decide to define the subtraction condition $i = 7$ by imposing the constraint

$$\langle p | I_7 | p \rangle = \langle n | I_7 | n \rangle \quad (4.138)$$

in which case f_7 is proportional to the proton-neutron mass difference:

$$f_7 \{ \langle p | u_3 | p \rangle - \langle n | u_3 | n \rangle \} = m_p - m_n \quad (4.139)$$

Note

- i) As this example shows, the principles of renormalization are not confined to perturbation theory.
- ii) It is *not* possible to substitute the leading x^{-2} singularity in Eq. (4.134) for $C_{\mu\nu}(x)$, because that would introduce an infrared singularity in $\int d^4x$ at $x = \infty$.
- iii) As a rule (e.g. in perturbation theory), $C_{\mu\nu}^i(x)$ should include the complete leading power of $T\{J_\mu(x)J_\nu(0)\}$ as $x \rightarrow 0$. In particular, if the theory is asymptotically free, terms in $T\{J_\mu(x)J_\nu(0)\}$ which are only logarithmically less singular than x^{-2} can also produce logarithmic infinities in $\int d^4x$, so they must be included in the subtraction.
- iv) For a given prescription i , $C_{\mu\nu}^i(x)$ remains unspecified up to functions $\Delta_{\mu\nu}$ which satisfy

$$\int d^4x D^{\mu\nu}(x) \Delta_{\mu\nu}(x) = 0 ; \quad \mathcal{C}_{\mu\nu}^i \longrightarrow \mathcal{C}_{\mu\nu}^i + \Delta_{\mu\nu} \quad (4.140)$$

Now consider the amplitudes for $\eta \rightarrow 3\pi$ decay:

$$A = - \langle 3\pi | \mathcal{H}_{em} | \eta \rangle \quad (4.141)$$

{The symbol A is used instead of T [Eq. (4.121)] to avoid confusing the results implied by the interactions (4.117) and (4.135).} The u_3 subtraction does not affect the soft- $\pi^\pm$ limit because it commutes with the corresponding axial charges. Therefore we recover the desirable result [SU(2) $\times$ SU(2) approximation]

$$\begin{aligned} A(\eta \rightarrow 3\pi^0) &\simeq A_0 \\ A(\eta \rightarrow \pi^+\pi^-\pi^0) &\simeq (1 - 2E_0/m_\eta) A_0 \end{aligned} \quad (4.142)$$

where the constant A_0 is given by

$$\begin{aligned} A_0 &= A(\eta \rightarrow \pi^0(\text{soft}) 2\pi) \\ &= e^2 \langle 2\pi, \pi^0(\text{soft}) | I_i | \eta \rangle + (f_i/F_\pi) \langle 2\pi | v | \eta \rangle \end{aligned} \quad (4.143)$$

and is assumed not to be small.

It is evident that the question of whether Sutherland's argument is applicable or not has nothing to do with the convergence of the integral in (4.135). All of the integrals I_i are finite. If we consider a pair I_1, I_2 corresponding to inequivalent subtraction conditions ($f_1 \neq f_2$), Eq. (4.137) implies

$$e^2(I_1 - I_2) = (f_1 - f_2)u_3 \quad (4.144)$$

$$e^2 \langle 2\pi, \pi^0(\text{soft}) | (I_1 - I_2) | \eta \rangle = -(f_1 - f_2) \langle 2\pi | v | \eta \rangle / F_\pi \quad (4.145)$$

so Sutherland's argument cannot be valid for both of the convergent integrals I_1 and I_2 . In fact, we can define a subtraction prescription $i = S$ by requiring the validity of the "Sutherland condition"

$$\langle 2\pi, \pi^0(\text{soft}) | I_S | \eta \rangle = 0 \quad (4.146)$$

or equivalently,

$$A_0 = (f_S/F_\pi) \langle 2\pi | v | \eta \rangle \quad (4.147)$$

The same reasoning can be applied to Dashen's rule, Eq. (4.115). Obviously, its derivation assumes the convergence of $\int d^4x$ in Eq. (4.114). For the energy density (4.135), the problem is that the u_3 term in Eq. (4.144) does not satisfy the rule; according to Eq. (4.119), it contributes to the total mass difference δm^2 defined in Eq. (4.118). So there is no analogue of Dashen's rule for the subtracted integrals I_1 except for a special subtraction prescription $i = D$ in which the "Dashen condition"

$$\begin{aligned} \langle K^+ | I_D | K^+ \rangle - \langle K^0 | I_D | K^0 \rangle \\ = \langle \pi^+ | I_D | \pi^+ \rangle - \langle \pi^0 | I_D | \pi^0 \rangle \end{aligned} \quad (4.148)$$

is true *by definition*. The formula for the corresponding constant f_D is

$$\begin{aligned} \delta m^2 \\ = f_D [\langle K^+ | u_3 | K^+ \rangle - \langle K^0 | u_3 | K^0 \rangle - \langle \pi^+ | u_3 | \pi^+ \rangle + \langle \pi^0 | u_3 | \pi^0 \rangle] \end{aligned} \quad (4.149)$$

The conclusion is that the interaction (4.135) produces the same results for $\eta \rightarrow 3\pi$ decay as the tadpole Hamiltonian (4.117) only if the subtraction prescriptions S and D happen to coincide [either exactly, or within the $SU(3) \times SU(3)$ approximation]. There is no reason to suppose that this is the case: in general, there will be a *renormalization mismatch*:

$$f_S \neq f_D \quad (4.150)$$

The analogue of the previous $SU(3) \times SU(3)$ analysis can be carried out by observing that the $SU(3)$ approximation to Eq. (4.149) looks like Eq. (4.120):

$$\delta m^2 \simeq 2 f_D \langle K^+ | u_3 | K^+ \rangle \simeq f_D \sqrt{3} \langle \pi^0 | u_3 | \eta \rangle \quad (4.151)$$

Combining Eqs. (4.129), (4.147), and (4.151), we obtain the result

$$A_0 \simeq (f_S/f_D) \delta m^2 / \sqrt{3} F_\pi^2 \quad (4.152)$$

Apart from the extra factor f_S/f_D , the answer is the same as Eq. (4.130). The obvious conclusion is that this factor is responsible for the discrepancy between the conventional answer (4.130) and experiment, with

$$|f_S/f_D|_{\text{expt.}} \simeq 1.8 \quad (4.153)$$

At present, an independent theoretical estimate of f_S/f_D is not available, but it is easy to see that a connection with operator-product expansions exists. A combination of Eqs. (4.143) and (4.147) yields the result

$$\begin{aligned} F_\pi \langle 2\pi, \pi^0(\text{soft}) | I_D | \eta \rangle &= e^{-2}(f_S - f_D) \langle 2\pi | v | \eta \rangle \\ &= \lim_{q \rightarrow 0} i \int d^4x e^{iq \cdot y} T \langle 2\pi | \partial^\alpha \mathcal{F}_{5\alpha}^3(y) I_D | \eta \rangle \end{aligned} \quad (4.154)$$

or equivalently,

$$\begin{aligned} \partial_y^\alpha T \{ \mathcal{F}_{5\alpha}^3(y) I_D \} &= T \{ \partial^\alpha \mathcal{F}_{5\alpha}(y) I_D \} + ie^{-2}(f_S - f_D) \delta^4(y) v(0) \\ &\quad + Q_\alpha \partial^\alpha \delta^4(y) + Q_{\alpha\beta} \partial^\alpha \partial^\beta \delta^4(y) + \dots \end{aligned} \quad (4.155)$$

where the series of derivative terms $Q_\alpha \partial^\alpha \delta^4(y)$, $Q_{\alpha\beta} \partial^\alpha \partial^\beta \delta^4(y)$, ..., is finite, and does not contribute to the $q \rightarrow 0$ limit. The term proportional to $v(0) \delta^4(y)$ is *not* an equal-time commutator because the operator I_D [defined by Eq. (4.136)] is *bilocal*, not local. Indeed, Sutherland's method, which works for the bilocal $\mathcal{J}\mathcal{C}_{\text{em}}$ (pure) in (4.114) when $\int d^4x$ converges, cannot be applied to I_D . If we suppose that the derivative ∂_y^α can be commuted with $\int d^4x$ in (4.136), the left-hand side of (4.155) becomes

$$\begin{aligned} \int d^4x D^{\mu\nu}(x) (\partial/\partial y)^\alpha T \{ \mathcal{F}_{5\alpha}^3(y) \{ J_\mu(x) J_\nu(0) - \mathcal{C}_{\mu\nu}^D(x) u_3(0) \} \} \\ = T \{ \partial^\alpha \mathcal{F}_{5\alpha}^3(y) I_D \} + i \delta^4(y) v(0) \int d^4x D^{\mu\nu}(x) \mathcal{C}_{\mu\nu}^D(x) \end{aligned} \quad (4.156)$$

because the equal-time commutators with J_μ, J_ν vanish, and the commutator with u_3 is given by Eq. (4.123). The factor $\int d^4x D^{\mu\nu} \mathcal{C}_{\mu\nu}^D$ is infinite, so the method fails.

The simplest way to understand why the $\delta^4(y) v(0)$ term appears is to note that it must correspond to a singularity $O(y^{-3})$ in the trilocal $T\{\mathcal{F}_{5\alpha}^3 I_D\}$. Equation (4.155) and the identity

$$\partial^\alpha [y_\alpha / (y^2 - i\varepsilon)^2] = -2\pi^2 i \delta^4(y) \quad (4.157)$$

imply

$$\begin{aligned} \int d^4x D^{\mu\nu}(x) T \{ \mathcal{F}_{5\alpha}^3(y) \{ J_\mu(x) J_\nu(0) - \mathcal{C}_{\mu\nu}^D(x) u_3(0) \} \} \\ \sim - (2\pi^2 e^2)^{-1} (f_S - f_D) \{ y_\alpha / (y^2 - i\varepsilon)^2 \} v(0), \quad (y_\alpha \rightarrow 0) \end{aligned} \quad (4.158)$$

for the pseudoscalar isoscalar sector of the expansion of the trilocal. The left-hand side of (4.158) can be analysed in the same manner as the example discussed in Eqs. (4.17)-(4.27) and Fig. 12. The following regions must be considered:

- i) The limit $y \rightarrow 0$ for the product of two operators associated with the subtraction term:

$$T \{ \mathcal{F}_{5\alpha}^3(y) u_3(0) \} \sim \{ y_\alpha / 2\pi^2 (y^2 - i\varepsilon)^2 \} v(0) \quad (4.159)$$

The normalization is fixed by Eqs. (4.123) and (4.157).

ii) The limit $x, y \rightarrow 0$ for the product of three operators

$$T\{\mathcal{F}_{5\alpha}^3(y) J_\mu(x) J_\nu(0)\} \sim \mathcal{C}_{\alpha\mu\nu}(x, y) v(0) \quad (4.160)$$

Equation (4.160) is yet another example of the special case (4.71), so the equation

$$\mathcal{C}_{\alpha\mu\nu}(\rho x, \rho y) = O(\rho^{-5}) \quad , \quad (\rho \rightarrow 0) \quad (4.161)$$

is valid for either broken scale invariance or asymptotic freedom. In general, equal-time commutators from the regions $x - y \ll x, y$ and $y \ll x$ also have to be considered, but for the present case, there is no contribution because F_5^3 commutes with J_μ and J_ν at equal times.

Substituting Eqs. (4.159) and (4.160) into Eq. (4.158), we arrive at the desired result:

$$\begin{aligned} \int d^4x D^{\mu\nu}(x) [\mathcal{C}_{\alpha\mu\nu}(x, y) - \mathcal{C}_{\mu\nu}^D(x) y_\alpha / 2\pi^2(y^2 - i\varepsilon)^2] \\ \sim - (2\pi^2 e^2)^{-1} (f_s - f_v) y_\alpha / (y^2 - i\varepsilon)^2 \quad , \quad (y \rightarrow 0) \end{aligned} \quad (4.162)$$

In contrast with the situation in Eq. (4.156), the integral $\int d^4x$ converges at $x = 0$, because the x^{-2} singularity of $\mathcal{C}_{\alpha\mu\nu}(x, y)$ in the limit $x \ll y$ is precisely cancelled off by the subtraction term. Observe that if we count powers in the integral (-5 for $\mathcal{C}_{\alpha\mu\nu}$, -2 for $D^{\mu\nu}$, and +4 for $\int d^4x$), the result $-5 - 2 + 4 = -3$ agrees with the power $O(y^{-3})$ which appears on the right-hand side of Eq. (4.162).

It must be emphasized that the only similarity between this analysis and the short-distance analysis^{4, 89)} of $\pi^0 \rightarrow 2\gamma$ decay is that both involve coefficient functions for products of three operators¹³⁰⁾. The anomalous constant¹³¹⁾ S for $\pi^0 \rightarrow 2\gamma$ decay can be produced by the function $\mathcal{C}_{\alpha\beta\gamma}(x, y)$ in

$$T\{J_\alpha(x) J_\beta(0) \mathcal{F}_{5\gamma}^3(y)\} \sim \mathcal{C}_{\alpha\beta\gamma}(x, y) I \quad , \quad (x, y \rightarrow 0) \quad (4.163)$$

if it is sufficiently singular⁴⁾:

$$\lim_{\rho \rightarrow 0} \rho^q \mathcal{C}_{\alpha\beta\gamma}(\rho x, \rho y) \neq 0 \quad , \quad (x, y \neq 0) \quad (4.164)$$

Thus, when the operator $(\partial/\partial y)_\gamma$ is applied to the left-hand side of (4.163), the leading power $\mathcal{C}_{\alpha\beta\gamma}$ can generate a contact term which scales as ρ^{-10} *):

$$\partial_y^\gamma \mathcal{C}_{\alpha\beta\gamma}(x, y) = - \frac{S}{2\pi^2} \varepsilon_{\alpha\beta\mu\nu} \partial_x^\mu \partial_y^\nu \delta^4(x) \delta^4(y) \quad (4.165)$$

[compare with Eq. (4.157)]. On the other hand, the analysis of $\eta \rightarrow 3\pi$ decay does not involve $\delta^4(x)\delta^4(y)$ or its derivatives explicitly, and should not be understood

*) Conserved δ -function ambiguities in $\mathcal{C}_{\alpha\beta\gamma}$ [e.g. $\varepsilon_{\alpha\beta\mu\nu}(\partial_x)_\gamma \partial_x^\mu \partial_y^\nu \delta^4(x) \delta^4(y)$] are irrelevant: they cannot contribute to the $\partial_x^\mu \partial_y^\nu \delta^4(x) \delta^4(y)$ term. In the literature, beware of incorrect statements that the anomaly is controlled by short-distance singularities of $T\{J_\alpha J_\beta \partial^\gamma \mathcal{F}_{5\gamma}^3\}$.

purely in terms of the failure of a naive manipulation $[\partial_y, \int d^4x D^{\mu\nu}(x)]$. The important point for $\eta \rightarrow 3\pi$ decay (irrelevant for $\pi^0 \rightarrow 2\gamma$ decay) is that one must keep track of the initial subtraction prescription.

Unfortunately, there is no analogue of the fact that the answer for $\pi^0 \rightarrow 2\gamma$ decay can be computed directly from the leading-power coefficient function $C_{\alpha\beta\gamma}(x,y)$. Whatever function is chosen for the leading power*) $C_{\alpha\mu\nu}(x,y)$ in the integrand of (4.162), there is no way of sidestepping the difficulty that we do not know the correct expression for $C_{\mu\nu}^D$. Since this function refers explicitly to the subtraction prescription D, there is no hope of calculating it in a reasonably model-independent fashion: it depends on complicated non-asymptotic details of the strong interactions. Instead, one should look for measurable amplitudes involving integrals which can be related to the integral in Eq. (4.162).

So far, the U(1) problem¹³²⁾ of quark models has been ignored. For a long time, it has been known that free-quark models (quark-parton model and its abstractions) are not consistent with approximate chiral symmetry. Such models contain an observable axial baryonic current

$$\mathcal{F}_{5\alpha}^0 = \bar{\psi} i \gamma_\alpha \gamma_5 \psi \quad (4.166)$$

[e.g. generated by $U(6) \times U(6)$ algebra¹³³⁾] which is partially conserved. The consequences of this are disastrous*):

i) Glashow¹³⁴⁾ obtained the result

$$m_\eta^2 \simeq m_\pi^2 \quad (4.167)$$

which corresponds to the fact that a linear combination $K_{5\alpha}$ of $\mathcal{F}_{5\alpha}^0$ and $\mathcal{F}_{5\alpha}^8$ becomes conserved in the $SU(2) \times SU(2)$ limit ($c \rightarrow -\sqrt{2}$):

$$\partial^\alpha K_{5\alpha} = -(\sqrt{2} + c) v / \sqrt{3} \quad (4.168)$$

More generally, there must be an observable pseudoscalar isoscalar particle L with mass m_L given by^{119, 132)}

$$m_L^2 \lesssim 3 m_\pi^2 \quad (4.169)$$

ii) Brandt and Preparata¹³⁵⁾ pointed out that Eq. (4.168) implies

$$\langle 2\pi | v | \eta \rangle = 0 \quad (4.170)$$

*) For asymptotically free theories, the leading singularity of $C_{\alpha\mu\nu}$ (but not the entire leading power) coincides with the result for free-field theory.

**) This is the motivation for not assuming $U(6) \times U(6)$ algebra in Ref. 89.

because the matrix element is evaluated at zero-momentum transfer. Their solution is to abandon approximate chiral symmetry, but I think that too many good chiral-symmetric predictions [e.g. for $K_{\ell 3}$ decay¹³⁶⁾] have to be dismissed as accidents for this approach to be believable.

For gauge theories, the divergence of $\mathcal{F}_{5\alpha}^0$ is anomalous, but there is a gauge non-invariant symmetry current

$$\mathcal{F}_{5\alpha}^S = \mathcal{F}_{5\alpha}^0 - (g^2 S / 4\pi^2) \varepsilon_{\alpha\beta\mu\nu} A_a^\beta (\partial^\mu A_a^\nu + \frac{1}{3} g c^{abc} A_b^\mu A_c^\nu) \quad (4.171)$$

which is partially conserved, so v is still equal to a total divergence:

$$v \propto \partial^\alpha K_{5\alpha}^S \quad (4.172)$$

In order to obtain a non-vanishing result for $\langle 2\pi | v | \eta \rangle$, Kogut and Susskind¹²⁹⁾ proposed the formula

$$\langle 2\pi | \mathcal{F}_{5\alpha}^S(0) | \eta \rangle \sim (\text{constant}) q_\alpha / q^2, \quad (q \rightarrow 0) \quad (4.173)$$

where q is the momentum carried by $\mathcal{F}_{5\alpha}^S$. To ensure that the zero-mass pole is unobservable experimentally, they suppose that it is a linear combination of positive and negative metric propagators, i/q^2 and $-i/q^2$, which cancel for observable operators but not for unobservable operators such as $\mathcal{F}_{5\alpha}^S$. Weinberg¹¹⁹⁾ has shown that the Kogut-Susskind mechanism also invalidates the bad result (4.169).

Remarks at the end of Chapters 2 and 3 about the infrared behaviour of current amplitudes can be extended to amplitudes involving $\mathcal{F}_{5\mu}^S$. Since $\mathcal{F}_{5\mu}^S$ is partially conserved, it is not multiplicatively renormalized, so the form factor $\pi_1(q^2)$ for the amplitude

$$i \int d^4x e^{iq \cdot x} T \langle 0 | \mathcal{F}_{5\alpha}^S(x) \mathcal{F}_{5\beta}^S(0) | 0 \rangle = (q_\alpha q_\beta - g_{\alpha\beta} q^2) \pi_1(q^2) + g_{\alpha\beta} \pi_2(q^2) \quad (4.174)$$

satisfies the same renormalization-group equations as $\pi(q^2)$ in Eq. (2.46) [but with a different subtraction function $K(x) \rightarrow K_1(x)$]. To obtain a pole in $\pi_1(q^2)$ at $q^2 = 0$,

$$\lim_{\eta \rightarrow 0} \eta^2 \pi_1(q^2; \bar{g}, \bar{\mathcal{M}}, \mu) = \text{finite} \neq 0 \quad (4.175)$$

it is necessary to assume $\bar{g} \rightarrow \infty$, or $\bar{\mathcal{M}} \rightarrow \infty$, or both; otherwise the basic assumptions of the renormalization-group method are violated. If we assume the infrared-singularity theorems^{63, 65)} mentioned at the end of Chapter 2 to be applicable here, the effective coupling constant $\bar{g}'$ obtained by omitting massive fields should also diverge as η tends to zero.

*) The gauge-invariant charge $\int d^3x \mathcal{F}_{50}^S(x)$ may be observable.

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Figure captions

- Fig. 1 : Divergent part of an internally subtracted LPI graph, pp. 2-3.
- Fig. 2 : Subgraph $\mathcal{G}'$ of a graph $\mathcal{G}$ for the amplitudes $\mathcal{A}$, $\mathcal{A}'$ in Eqs. (2.1), (4.33), and (4.34). The large momenta q flow through $\mathcal{G}'$.
- Fig. 3 : Graphs for Eq. (2.3).
- Fig. 4 : Subgraph illustrating the existence of exceptional momenta, p. 15.
- Fig. 5 : Graph for Eqs. (2.5)-(2.6').
- Fig. 6 : Dominant contributions for graphs with a single mass insertion Δ , Eqs. (2.13)-(2.15).
- Fig. 7 : Dominant graphs with two mass insertions, Eq. (2.20).
- Fig. 8 : Graphs generating the $(\ln \eta)^{n-1}$ terms ($n \geq 2$) in Eq. (2.28).
- Fig. 9 : Conditions for a fixed point to be ultraviolet or infrared stable, p. 23.
- Fig. 10 : Artist's conception of a Callan-Symanzik β -function for which Eq. (2.38) is relevant.
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- Fig. 16 : Overlapping subgraphs, Eq. (4.51).
- Fig. 17 : One-loop graphs with divergent part proportional to Q_2 , Eq. (4.99).
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- Fig. 20 : Simplest examples of graphs in Fig. 18b. See p. 64 and Eq. (4.110).
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Fig. 1

Fig. 2

Fig. 3

Fig. 4

$K \left(\frac{1}{2}\phi^2\right)$ INSERTIONS

Fig. 5

(a)

} g' {

(b)

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

(a)

(b)

(c)

Fig. 11

Fig. 12

Fig. 13

Fig. 14

Fig. 15

Fig. 16

Fig. 17

(a)

(b)

Fig. 18

Fig. 19

Fig. 20

Fig. 21

Fig. 22

Fig. 23