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INCLUSIVE DISTRIBUTIONS FOR A $\textcircled{P}$ OMERON ABOVE ONE

M. Ciafaloni

Scuola Normale Superiore, Pisa

and

INFN, Sezione di Pisa

and

G. Marchesini ^{*)}

CERN - Geneva

ABSTRACT

We calculate the inclusive distributions for many-Pomeron exchange when the bare Pomeron intercept is higher than one. Rescattering is included via two-particle n Pomeron coupling constants of eikonal type, and Pomeron interactions are treated in the tree approximation. The inclusive spectra are related to the classical action of cut Reggeon field theory with mass insertions. In impact parameter space we obtain asymptotically constant density and correlation functions inside a disc whose radius increases logarithmically with energy. The resulting distributions automatically satisfy momentum sum rules and positivity constraints. The multiplicity grows asymptotically as $(\ln s)^3$ and the multiplicity moments satisfy formal KNO scaling.

^{*)} On leave of absence from the University of Parma, Italy.

1. INTRODUCTION

This paper deals with a non-perturbative calculation of inclusive distributions for the general Pomeron amplitude of Fig. 1 (generalized eikonal diagrams) and with the study of their consistency with s channel unitarity when the bare Pomeron intercept is above one. Therefore we consider the exchange of any number of Pomerons in the t channel, interacting via the Green's functions $G^{(n,m)}$ of Reggeon field theory (RFT) ¹⁾, with trilinear couplings. Following Amati, Caneschi and Jengo ²⁾, we are mainly concerned with the approximation in which Pomeron loops in $G^{(n,m)}$ are neglected (tree approximation), and will add only a few comments on quantum corrections.

The questions that arise are: why this particular model? and, why a non-perturbative calculation? The answer to the second question is that we are checking the model against s channel unitarity. It is in fact known ³⁾ that, due to the smallness of $\sigma_{el}/\sigma_{tot} \sim \sigma_{diff}/\sigma_{tot} \sim 1/5$, cut effects can be treated perturbatively up to ISR energies. However, the increase of σ_{tot} in this energy range can be better explained by assuming, as we do here, that the bare Pomeron intercept is $\alpha_P^0 = 1 + \mu$, with $\mu > 0$. In such a case each term of the perturbative series violates eventually the Froissart bound, and we need non-perturbative results in order to check the model for asymptotic consistency.

It is also known that loop effects -- that we eventually neglect here -- give rise to a negative intercept renormalization parametrized by $\mu_{crit}(\lambda)$ ^{4),5)} where λ is the triple-Pomeron coupling. If $\mu = \mu_{crit}(\lambda)$, the renormalized Pomeron intercept is $\alpha_P = 1$, while if $\mu < \mu_{crit}$ also $\alpha_P < 1$. In such cases the exchange of a renormalized Pomeron dominates the diagrams of Fig. 1, and perturbation theory is expected to be applicable for the renormalized skeleton expansion, if not for the unrenormalized one. However, a precise relation like $\mu = \mu_{crit}$ is not expected to hold in the topological expansion ⁶⁾, and moreover phenomenological estimates ⁷⁾ give $\mu \simeq 0.1$ and $\mu_{crit} \simeq 0.01$. Our non-perturbative results are therefore expected to be both applicable and interesting for $\mu \gg \mu_{crit}$. (The difference of our results with others recently obtained ^{8),9)} will be discussed at the end of the paper.)

There are several motivations for the choice of this particular model. First of all, Veneziano and the authors ⁶⁾ have shown that RFT is obtained from the asymptotic behaviour of the amplitude in the topological expansion of a unitarity theory. The question is then whether s channel unitarity is still contained in our model. The asymptotic form of unitarity for the elastic amplitude in the topological expansion is given in Fig. 2, where we have introduced the cut Pomeron P_c and the uncut ones $P_{\pm}$ above (below)

the cutting plane. More formally, we have

$$\sum_{n,m} \frac{g_n}{n!} G^{(n,m)} \frac{\tilde{g}_m}{m!} = \sum_{n_i, m_j} \frac{g\{n_i\}}{\pi_i n_i!} G^{(n_i, m_j)} \frac{\tilde{g}\{m_j\}}{\pi_j m_j!} \quad (1.1)$$

where $G^{(n_i, m_j)}$ are the Green's function of a cut Reggeon field theory (CRFT) introduced in Refs. 6) and 10), and $g\{n_i\}$ are computed according to the AGK rules ¹¹⁾ in terms of g_n .

Despite the fact that, in general, all n - m Pomeron couplings have to be included ^{*}) in $G^{(n,m)}$, we have shown in a previous paper ¹²⁾ that the discontinuity equation (1.1) is still satisfied if only the trilinear interaction is kept. Therefore our model keeps the unitarity constraints of the underlying topological expansion (in particular, elastic unitarity) in the form of Eq. (1.1). Unitarity for production amplitudes is not automatically incorporated in Eq. (1.1) because the latter is still satisfied by pure eikonal graphs ¹¹⁾ which correspond to $\lambda = 0$, in contrast with the discontinuity equation for the six-point function ¹³⁾. However, the hope is that once many-body unitarity is qualitatively included by having $\lambda \neq 0$ (besides $g_n \neq 0$), then the structure of Eq. (1.1) will make our model consistent with inelastic unitarity as well. Our results support this hope.

A phenomenological motivation for neglecting Pomeron loops in first approximation has been put forward by Schwimmer ¹⁴⁾, and by Amati, Caneschi and Jengo ²⁾. In fact, for baryon-baryon scattering, and especially for nuclei ¹⁴⁾, the two-particle-Pomeron coupling g is much larger than the triple-Pomeron coupling λ . This allows neglecting many-Pomeron couplings in $G^{(n,m)}$. Moreover, if we assume

$$g_n = g^n, \quad \tilde{g}_m = f^m \quad (1.2)$$

as in the pure eikonal model, we can approximate the completely connected contribution to $G^{(n,m)}$ by the terms $g^2(g\lambda)^n$. The sum of these terms corresponds to the tree approximation for $G^{(n,m)}$, while adding further powers of λ^2 leads to the loop expansion.

^{*}) Their order of magnitude, in topological expansion, is λ^{n+m-2} , if λ in the triple-Pomeron coupling.

The tree approximation itself is generated by replacing the classical equations of motion in the action of RFT, so that the asymptotic scattering matrix can be evaluated. The simplest result, obtained for zero transverse dimensions, is ²⁾ asymptotically

$$S(\infty) = \exp -g_{\min} \left(\frac{\mu}{\lambda} - \frac{1}{2} g_{\min} \right), \quad g_{\min} = \min(g, f) < \mu/\lambda \quad (1.3)$$

always for the case $\mu = \alpha_P^0 - 1 > 0$. Here we notice that the S matrix is correctly bounded, despite the "wrong" input. Quantum mechanical (loop) corrections do not drastically alter the picture ¹⁵⁾, and for small λ lead to a slowly vanishing asymptotic amplitude, $1 - S \simeq e^{-\Delta Y}$ with $\Delta \sim \mu \exp(-\mu^2/\lambda^2) \ll \mu$.

Having the result (1.3) for the amplitude, the interest in the calculation of inclusive distributions is strengthened, both phenomenologically and theoretically. On the one hand, a perturbative treatment of this model is likely to be in qualitative agreement with the rise of the plateau in the pionization region, by the famous AGK's cut cancellation in $d\sigma/dy$ (and not in σ_{tot}). On the other hand, we want to check inelastic unitarity, and this can be done by means of inclusive sum rules.

The particular case $\lambda = 0$ (pure eikonal model) is, strictly speaking ^{*)}, inconsistent with momentum conservation sum rules for $\mu > 0$. In fact, by the AGK cancellation for the single particle density (Fig. 3a), $d\sigma/dy$ behaves as s^μ , while σ_{tot} , by the eikonalization mechanism, behaves as $(\ln s)^2$ (black disc). It turns out that for $\lambda \neq 0$ no inconsistency arises at the level of single density, which for zero transverse dimension is given by ²⁾

$$\frac{d\sigma}{dy} \rightarrow \left(\frac{\mu}{\lambda} \right)^2 \quad (1.4)$$

However, an apparent contradiction arises at the level of double density because one of the contributions discussed by AGK (Fig. 3b) still behaves like s^μ , thus violating, by (1.4) the second momentum sum rule. Our calculation will eventually show that an additional class of diagrams contributing to the double density ¹²⁾ cancels the nasty s^μ behaviour and restores consistency with the sum rules.

*) The consistency can be restored by artificially cutting off $d\sigma/dy$ close to the fragmentation region, where the inelastic threshold constraint $\sqrt{s} > n_0 \sqrt{s_0}$ would no longer be satisfied (A. Kaidalov, private communication).

Our method of calculation uses, in an essential way, the action of CRFT ^{6),10)} by means of which we generalize the AGK cutting rules to any Gribov diagram. According to Mueller's optical theorem, the observed particles in the intermediate states of the cut Pomeron P_c are singled out by adding a mass insertion vertex $\xi \psi_c^+ \psi_c$ to the CRFT Lagrangian. This allows us to compute directly the generating functional of inclusive distributions in impact parameter and rapidity $(\underline{b}, y)$ by replacing the P_c propagator D by the one in presence of the source $\xi(\underline{b}, y)$, i.e.,

$$D \rightarrow D_c(\xi) = D + D \xi D_c(\xi) , \quad (1.5)$$

$$D = e^{\mu y} e^{-b^2/4\alpha' y} / 4\alpha' y ,$$

where an operator product in $\underline{b}, y$ space is understood.

In order to gain some insight into this procedure, let us consider the simple eikonal case ($\lambda = 0$), for which the amplitudes in (1.1) become

$$1 - S = T = \sum_{n \geq 1} \frac{(gf)^n}{n!} (-D)^n ,$$

$$T_c(\xi) = \frac{1}{2} \sum_{n_i}' \frac{(gf)^{n_+ + n_- + n_c}}{n_+! n_-! n_c!} (-D)^{n_+ + n_-} (2D_c(\xi))^{n_c} , \quad (1.6)$$

where in the last sum the Σ' excludes the terms $n_c = n_- = 0$ ($n_c = n_+ = 0$) which are forbidden by the cutting rules, and the factor $\frac{1}{2}$ is needed to normalize T_c to the imaginary part. We have therefore

$$S(\underline{b}, y) = e^{-gf D(\underline{b}, y)} \rightarrow 0 \text{ for } b^2 < 4\alpha' \mu y^2 , \quad (1.7)$$

and

$$T_c(\xi, \underline{b}, y) = T(\underline{b}, y) + \frac{1}{2} \left(e^{-2gf[D - D_c(\xi)]} - 1 \right) . \quad (1.8)$$

It is clear that the relation $T_c(0) = T$, needed by the discontinuity Eq. (1.1), is satisfied. This is due to the fact that the free action of CRFT (which, in general, has different couplings $g_{\pm}, g_c$ to $P_{\pm}, P_c$), i.e.,

$$A_c(\xi) = g_+ f_+ D + g_- f_- D - 2g_c f_c D(\xi) , \quad (1.9)$$

vanishes for $\xi=0$ where all g_i 's and f_i 's are equal. Here we can see that the AGK cutting rules (i.e., equal g_i 's) are summarized in the property $A_c(0)=0$, which in turn ensures that Eq. (1.1) is satisfied. This condition plays an important role also in the interacting case for the choice of the solutions of the equations of motion.

The plan of the paper is as follows. In Section 2 we set the formalism, and generalize the relation (1.8) of the generating functional of inclusive distributions to the CRFT action. We show in Section 3 that the single particle distribution can be given in terms of the classical fields, and the double density in terms of their first functional derivatives. We then solve the equations of motion and give a graphical interpretation to the solutions. The physical inclusive distributions ($\alpha' \neq 0$) are given in Section 4, after a brief discussion of the simple case when there are no transverse dimensions ($d_t=0$). The main result is that the hadron looks like an expanding grey disc^{*)}, with asymptotically constant density and correlation functions inside the disc's radius. This proves the consistency with momentum sum rules. In the conclusive Section 5 we summarize and discuss our results. In particular, we compare them with other non-perturbative calculations of the multiplicity moments close to the critical point^{8),9)}. Our results compare better with the case $\mu \gg \mu_{\text{crit}}$ and give a clue to a good understanding of the shifted field method.

2. GENERAL FORMALISM

2.1 RFT and CRFT

The imaginary part of the amplitude in Fig. 1 in impact parameter representation is given by

$$S(\underline{B}, Y) - 1 = \sum_{n, m=1}^{\infty} \frac{1}{n! m!} \left(\prod_{i=1}^n db_i g(\underline{b}_i) \prod_{j=1}^m db'_j f(\underline{B} - \underline{b}'_j) \right) \times \\ \times \langle 0 | \psi(0, \underline{b}_1) \dots \psi(0, \underline{b}_n) \psi^\dagger(Y, \underline{b}'_1) \dots \psi^\dagger(Y, \underline{b}'_m) | 0 \rangle, \quad (2.1)$$

where $g(\underline{b})$, $f(\underline{b} - \underline{B})$ are real functions which parametrize the external particles form factors, so that an operator product in $\underline{b}$ space is usually understood in (2.1); if however $g(\underline{b}) = g\delta(\underline{b})$, $f(\underline{b} - \underline{B}) = f\delta(\underline{b} - \underline{B})$ (no form factors), we have simply

*) The calculation of the classical action and quantum corrections for $\alpha' \neq 0$ and $d_t=2$ will appear in a forthcoming paper of Amati, Le Bellac and the authors¹⁶⁾.

$$S(\underline{B}, Y) - 1 = \sum_{n,m=1}^{\infty} \frac{g^n}{n!} G^{(n,m)}(\underline{v}_0, V) \frac{f^m}{m!}, \quad (2.2)$$

where $\underline{v}_0 = (0, 0)$ and $V = (\underline{B}, Y)$ are impact parameter and rapidity of the external particles. Both in (2.1) and (2.2), we have absorbed the customary phase i^{n+m} into the fields' definition, so that following ACJ²⁾, the RFT Lagrangian is written as

$$\begin{aligned} \mathcal{L} &= \psi^+ K \psi + \lambda \psi^+ (\psi + \psi^+) \psi, \\ K &= -\frac{1}{2} \overset{\leftrightarrow}{\partial}_y^2 - \kappa' \nabla_\perp^2 - \mu, \end{aligned} \quad (2.3)$$

where, however, the field ψ and conjugate momentum ψ^+ satisfy anti-canonical commutation rules^{*)}

$$[\psi(y, \underline{b}_1), \psi^+(y, \underline{b}_2)] = -\delta(\underline{b}_1 - \underline{b}_2). \quad (2.4)$$

This condition embodies the absorptive character of the theory (a - sign for each loop)

It appears from (2.1) that the partial wave S matrix can be identified with the generating functional of Green's functions of RFT when the external currents $J(\tilde{J})$ coupled to $\psi(\psi^+)$ take the values

$$J(\underline{v}) = g(\underline{b}) \delta(y), \quad \tilde{J}(\underline{v}) = f(\underline{B} - \underline{b}) \delta(Y - y). \quad (2.5)$$

In the tree approximation we have therefore²⁾ that

$$S(\underline{B}, Y) = \exp - A(\underline{B}, Y; g, f), \quad (2.6)$$

where A is the classical action of the Lagrangian (2.3) coupled with the external sources (2.5).

*) Besides making the classical limit simpler, this convention allows the use of a Hermitean Hamiltonian in an indefinite metric space. For vanishing external current, we have therefore $\psi^+(y) = [\psi(-y)]$, thus making the $\psi = \psi^+$ symmetry more explicit. (The bar denotes Hermitean conjugation in this indefinite metric space.)

According to the unitarity equation (1.1) the imaginary part of the amplitude is also given in terms of the Green's functions $G^{(n_i, m_j)}(v_0, v)$ of CRFT, provided the two-particle n_1 Pomeron couplings $g(n_1)$ are given by the AGK relation

$$\begin{aligned} g(n_i) &= i(\sqrt{2})^{n_c-1} g_n = i(\sqrt{2})^{n_c-1} g^n, \quad \sum n_i = n, \\ f(m_j) &= i(\sqrt{2})^{m_c-1} f_m = i(\sqrt{2})^{m_c-1} f^m, \quad \sum m_j = m. \end{aligned} \quad (2.7)$$

The corresponding CRFT Lagrangian involves three fields ψ_c, ψ_+, ψ_- related to P_0, P_+, P_- , respectively, and can be written as

$$\begin{aligned} \mathcal{L}_c &= \sum_i \epsilon_i \psi_i^+ K \psi_i + \lambda \left[\psi_c^+ (\psi_+^+ + \psi_-^+) \psi_+ + (+ \leftrightarrow -) \right] + \\ &- 2\lambda \psi_c^+ \psi_c (\psi_+ + \psi_+^+ + \psi_- + \psi_-^+) + \sqrt{2}\lambda (\psi_c^+ \psi_+ \psi_- + \psi_+^+ \psi_-^+ \psi_c) + \sqrt{2}\lambda \psi_c^+ (\psi_c + \psi_c^+) \psi_c, \end{aligned} \quad (2.8)$$

where $\epsilon_+ = \epsilon_- = -\epsilon_0 = 1$, and the equal times commutation rules are

$$\left[\psi_c(v_1), \psi_c^+(v_2) \right]_{y_1=y_2} = \delta(\underline{b}_1 - \underline{b}_2), \quad \left[\psi_{\pm}(v_1), \psi_{\pm}^+(v_2) \right]_{y_1=y_2} = -\delta(\underline{b}_1 - \underline{b}_2). \quad (2.9)$$

Also in this case we have absorbed the phase factor $(i)^{n_+ + m_+ - n_- - m_-}$ of the external couplings into the definition of the fields in $G^{(n_i, m_j)}$, so that the Lagrangian of Refs. 6) and 10) is recovered by the replacement

$$\psi_c \rightarrow \psi_c, \quad \psi_+ \rightarrow i\psi_+, \quad \psi_- \rightarrow -i\psi_-, \quad (2.10)$$

and a similar one for the conjugate momenta.

Let us recall that the unitarity equation (1.1) with the proviso (2.7) can be proved ¹²⁾ on the basis of an important linear relation among Green's functions of CRFT and RFT

$$\begin{aligned} \sum_{n_i}' (\sqrt{2})^{n_c-1} \frac{n!}{n_+! n_-! n_c!} G^{(n_i, m_j)}(v_0, v) &= (\sqrt{2})^{m_c-1} G^{(n, m)}(v_0, v), \\ \sum n_i &= n, \quad \sum m_j = m, \end{aligned} \quad (2.11)$$

where Σ' does not count the terms $n_+ = n$ and $n_- = n$ for any choice of the $\{m_j\}$, except when $m_+ = m$ and $m_- = m$. The proof of (2.11) has been given ⁶⁾ by a partial diagonalization of the Lagrangian (2.8) that will also be used in the following. We redefine the fields φ_α ($\alpha = +, -, 0$) and conjugate momenta π_α by a linear similarity transformation:

$$\begin{aligned} \psi_\pm &= \varphi_\pm, \quad \psi_c = -\frac{1}{\sqrt{2}} (\varphi_0 - \varphi_+ - \varphi_-), \\ \psi_\pm^+ &= (\pi_\pm + \pi_0), \quad \psi_c^+ = \sqrt{2} \pi_0, \end{aligned} \quad (2.12)$$

which transforms $\mathcal{L}_c$ into a new Lagrangian

$$\begin{aligned} \mathcal{L}_c' &= \sum_\alpha \pi_\alpha K \varphi_\alpha + \lambda \sum_\alpha \pi_\alpha (\pi_\alpha + \varphi_\alpha) \varphi_\alpha + \lambda \left[\pi_+ \pi_0 (\varphi_0 + \varphi_+ - \varphi_-) + \right. \\ &\quad \left. + \pi_- \pi_0 (\varphi_0 + \varphi_- - \varphi_+) + \pi_+ \pi_- (\varphi_+ + \varphi_- - \varphi_0) \right]. \end{aligned} \quad (2.13)$$

Since (2.13) allows off-diagonal transitions only in a definite time direction, the new Green's functions satisfy the identity (α not summed)

$$\langle 0 | \varphi_\alpha \dots \varphi_\alpha \pi_{\beta_1} \dots \pi_{\beta_m} | 0 \rangle = \delta_{\alpha\beta_1} \dots \delta_{\alpha\beta_m} G^{(n,m)}, \quad (2.14)$$

which turns out to be equivalent to (2.11).

If we take into account the identities

$$G^{(n_+, n_-, n_c; m_+, m_-, m_c)} = G^{(n, m)} \quad (2.15)$$

which are a particularly trivial case of (2.11), the unitarity equation (1.1) with the proviso (2.7) can be rewritten as

$$\sum_{n_i, m_j=0}^{\infty} \frac{g^{n_+ + n_-} (\sqrt{2}g)^{n_c}}{n_+! n_-! n_c!} G^{(n_i, m_j)}(v_0, V) \frac{f^{m_+ + m_-} (\sqrt{2}f)^{m_c}}{m_+! m_-! m_c!} = 1, \quad (2.16)$$

where we now count in the sum also the cases $n_- = n_c = 0$ ($m_- = m_c = 0$). The left-hand side of Eq. (2.16) has the simple interpretation of generating functional of Green's functions of CRFT, evaluated at the values

$$J_\pm(v) = \frac{1}{\sqrt{2}} J_c(v) = g(b) \delta(y), \quad \tilde{J}_\pm(v) = \frac{1}{\sqrt{2}} \tilde{J}_c(v) = f(b-b) \delta(y-y) \quad (2.17)$$

of the external sources coupled to the fields ψ_i and ψ_i^+ . Equation (2.16) means therefore that the action of CRFT, when the sources satisfy the AGK constraint (2.17), is identically zero.

2.2 Generating functional of inclusive distributions

Consider now the problem of calculating the discontinuity graphs appearing in Fig. 2, when all P_c propagators are replaced by the propagator $D_c(\xi)$ of Eq. (1.5), in the external source $\xi(b, y)$. This is the same as calculating the CRFT graphs of Fig. 2 with a Lagrangian modified by addition of a "mass" term, i.e.,

$$\mathcal{L}_c(\xi) = \mathcal{L}_c + \xi(\frac{1}{2}, y) \psi_c^+(\frac{1}{2}, y) \psi_c(\frac{1}{2}, y), \quad (2.18)$$

and, according to Mueller's optical theorem, gives directly the generating functional $T_c(\xi)$ of inclusive distributions.

We have therefore

$$T_c(\xi) = \frac{1}{2} \sum'_{n_i, m_i \geq 1} \frac{g^{n_+ + n_-} (\sqrt{2} g)^{n_c}}{n_+! n_-! n_c!} G_\xi^{(n_i, m_i)} \frac{f^{m_+ + m_-} (\sqrt{2} f)^{m_c}}{m_+! m_-! m_c!}, \quad (2.19)$$

where now $G_\xi^{(n_i, m_i)}$ is evaluated from the Lagrangian (2.18). By artificially adding and subtracting the terms $n_+ = n_c = 0$ and $m_- = m_c = 0$, and by using Eq. (2.15) (which is ξ independent) we get

$$T_c(\xi) = 1 - S + \frac{1}{2} \left(e^{-A_c(\xi, g, f, B, Y)} - 1 \right), \quad (2.20)$$

where S is given by (2.2) and $(-A_c)$ is the action of CRFT in interaction with the mass source (2.18) and the external currents (2.17). For $\lambda = 0$ this reduces to Eq. (1.8).

The unitarity equation (1.1) then implies $T_c(0) = 1 - S$, or as seen in Eq. (2.16)

$$A_c(\xi=0, g, f, B, Y) = 0. \quad (2.21)$$

This condition is important for the choice of the proper solutions of the classical equations of motion, and embodies the constraint of the AGK cutting rules.

A second general result, which -- together with (2.21) -- simplifies a lot the calculation of inclusive distributions, is the property

$$-\frac{\delta A_c}{\delta \xi(v)} = \psi_c^+(v, \xi) \psi_c(v, \xi), \quad (2.22)$$

where A_c is the classical action and ψ_c, ψ_c^+ are the classical fields in presence of the mass source ξ . In fact, by definition

$$-A_c = \int d\nu \mathcal{L}_c(\xi, \nu) = \int d\nu \mathcal{L}_c(\nu) + \int d\nu \xi(\nu) \psi_c^+(\nu) \psi_c(\nu), \quad (2.23)$$

so that the functional derivative is

$$\begin{aligned} -\frac{\delta A_c}{\delta \xi(v)} - \psi_c^+(v, \xi) \psi_c(v, \xi) &= \\ &= \int d\nu' \sum_i \left[\frac{\partial \mathcal{L}_c(\xi)}{\partial \psi_i} \frac{\delta \psi_i}{\delta \xi(v)} + \frac{\partial \mathcal{L}_c(\xi)}{\partial \dot{\psi}_i} \frac{d}{dy} \frac{\delta \psi_i}{\delta \xi(v)} + (\psi_i^+ \leftrightarrow \psi_i) \right] \end{aligned} \quad (2.24)$$

If we now use the equations of motion

$$\frac{d}{dy} \frac{\partial \mathcal{L}_c(\xi)}{\partial \dot{\psi}_i} = \frac{\partial \mathcal{L}_c(\xi)}{\partial \psi_i}, \quad \psi_i \leftrightarrow \psi_i^+, \quad (2.25)$$

the right-hand side of (2.24) can be reduced to the surface term

$$\sum_i \frac{1}{2} \int d\nu' \left[\psi_i^+ \frac{\delta \psi_i}{\delta \xi(v)} + \psi_i \frac{\delta \psi_i^+}{\delta \xi(v)} \right] \bigg|_{y'=-\infty}^{y'=+\infty} \quad (2.26)$$

which vanishes, because we solve the equations of motion under the boundary conditions

$$\psi_i(v, \xi) \bigg|_{y=+\infty} = \psi_i^+(v, \xi) \bigg|_{y=-\infty} = 0 \quad (2.27)$$

We shall see in the next Section that (2.22) makes the calculation of $d\sigma/d\nu$ a trivial one.

3. SOLUTION OF EQUATIONS OF MOTION

The purpose of this Section is to calculate the classical action $A_c(\xi)$ of CRFT in the external sources (2.17) with the mass term (2.18). We are able to give an exact expression up to $(\delta^2 A_c / \delta \xi_1 \delta \xi_2) \big|_{\xi=0}$ and we shall discuss further terms only approximately in Section 4. By (2.20) the generating functional $T_c(\xi)$ is expressed in terms of $A_c(\xi)$ as

$$T_c(\xi) = 1 - S + \frac{1}{2} \left(e^{-A_c(\xi)} - 1 \right) \quad (3.1)$$

By using the important properties (2.21) and (2.22), i.e.,

$$A_c(\xi=0) = 0, \quad -\frac{\delta A_c(\xi)}{\delta \xi(v)} = \psi_c^+(v, \xi) \psi_c(v, \xi), \quad (3.2)$$

we find ^{*)}:

$$\begin{aligned} \frac{1}{\xi} \frac{d\sigma(v, v)}{dv} &= \left. \frac{\delta T_c(\xi)}{\delta \xi(v)} \right|_{\xi=0} = -\frac{1}{2} e^{-A_c(\xi)} \left. \frac{\delta A_c(\xi)}{\delta \xi(v)} \right|_{\xi=0} = \frac{1}{2} \psi_c(v, 0) \psi_c^+(v, 0), \\ \frac{1}{\xi^2} \frac{d\sigma}{dv_1 dv_2} &= \left. \frac{\delta^2 T_c(\xi)}{\delta \xi(v_1) \delta \xi(v_2)} \right|_{\xi=0} = \frac{e^{-A_c(\xi)}}{2} \left[\frac{\delta A_c}{\delta \xi(v_1)} \frac{\delta A_c}{\delta \xi(v_2)} - \frac{\delta^2 A_c}{\delta \xi(v_1) \delta \xi(v_2)} \right]_{\xi=0} \\ &= \frac{e^{-A_c}}{2} \left[\prod_{i=1,2} \psi_c^+(v_i, \xi) \psi_c(v_i, \xi) + \frac{\delta}{\delta \xi(v_1)} \psi_c^+(v_2, \xi) \psi_c(v_2, \xi) \right]_{\xi=0}, \end{aligned} \quad (3.3)$$

and so on. Here we see that in order to calculate the correlations, we need to know the classical fields and their first functional derivatives.

In order to exploit better the AGK relations (2.17), we shall write the equations of motion (2.25) of CRFT in terms of the quasi-diagonal fields φ_α and π_α introduced in Eq. (2.12). The corresponding currents $K_\alpha, \tilde{K}_\alpha$ are in fact simpler, and because of (2.17) given by

*) The normalized distributions in v space are found by dividing by $[1 - \exp(-A)] = (1 - S)$. ρ is the bare particle density.

$$k_0 = \frac{1}{\sqrt{2}} J_c = g(\frac{b}{2}) \delta(y) , \quad k_{\pm} = J_{\pm} - \frac{1}{\sqrt{2}} J_c = 0 ,$$

$$\tilde{K}_0 = -\sqrt{2} \tilde{J}_c + \tilde{J}_+ + \tilde{J}_- = 0 , \quad \tilde{K}_{\pm} = \tilde{J}_{\pm} = f(\frac{B-b}{2}) \delta(Y-y) .$$
(3.4)

Moreover, the equations are completely symmetrical with respect to $+$ and $-$ interchange. We can therefore write four (instead of six) coupled equations for the classical fields $\varphi_+ = \varphi_- = \varphi, \varphi_0$ and $\pi_+ = \pi_- = \pi, \pi_0$. After some algebra we find ^{*)}

$$\left(\frac{d}{dy} + \alpha' \nabla_b^2 - \mu \right) \pi = \bar{D}^{-1} \pi = -\lambda (2\pi\varphi + 2\pi^2) - \xi \pi_0 ,$$

$$\left(\frac{d}{dy} + \alpha' \nabla_b^2 - \mu \right) \pi_0 = \bar{D}^{-1} \pi_0 = -\lambda (2\pi_0\varphi_0 + \pi_0^2 + 2\pi_0\pi - \pi^2) + \xi \pi_0 + g \delta(y) ,$$

$$\left(-\frac{d}{dy} + \alpha' \nabla_b^2 - \mu \right) \varphi = \varphi \bar{D}^{-1} = -\lambda (4\pi\varphi + \varphi^2 + \varphi_0\pi_0 - \varphi_0\pi) + f \delta(Y-y) ,$$

$$\left(-\frac{d}{dy} + \alpha' \nabla_b^2 - \mu \right) \varphi_0 = \varphi_0 \bar{D}^{-1} = -\lambda (2\varphi_0\pi_0 + 2\varphi_0\pi + \varphi_0^2) + \xi (\varphi_0 - 2\varphi) ,$$
(3.5)

where we have introduced the notation

$$\bar{D}^{-1} = \frac{d}{dy} - \alpha' \nabla_b^2 - \mu$$
(3.6)

for the inverse Pomeron propagator (with the convention that $\vec{\partial}_y = -\overleftarrow{\partial}_y$). If we take

$$D(y, y') = e^{\mu(y-y')} \frac{e^{-\frac{(b-b')^2}{4\alpha'(y-y')}}}{4\alpha'(y-y')} \mathcal{I}(y-y') ,$$
(3.7)

for the operator inverse of (3.6), the equations of motion (3.5) are automatically solved with the boundary conditions

^{*)} Of course, the same result would be obtained by a proper reshuffling of (2.25).

$$\begin{aligned} \pi_0(v, \xi) = \pi(v, \xi) = 0 \quad \text{for } y < 0 ; \quad \Delta(\pi)_{y=0} = 0, \quad \Delta(\pi_0)_{y=0} = g, \\ \varphi_0(v, \xi) = \varphi(v, \xi) = 0 \quad \text{for } y > Y ; \quad \Delta(\varphi)_{y=Y} = -f, \quad \Delta(\varphi_0)_{y=Y} = 0. \end{aligned} \quad (3.8)$$

We now look for the solutions of (3.5) at $\xi = 0$. Since the 1st and 4th equations (3.5) do not involve the external currents, the only fields which can vanish identically for any g and f , in order to satisfy $A_0(0) = 0$, are π and φ_0 . The remaining equations thus decouple, giving

$$\pi(\xi=0, v) = \varphi_0(\xi=0, v) = 0 ; \quad (3.9)$$

$$\pi_0(\xi=0, v) = \chi(v, g) ; \quad \varphi(\xi=0, v) = \chi(v-v, f)$$

where χ is the solution of the differential equation

$$D^{-1} \chi = -\lambda \chi^2 + g \delta(y) , \quad (3.10)$$

which is the classical equation of motion of RFT when $f=0$. In other words, $\pi_0(v, \xi=0)$ is the sum of all free diagrams in Fig. 6 ("tadpole left"), while $\varphi(v, \xi=0)$ is given by the symmetric diagrams ("tadpole right").

Going back to the fields ψ_i and ψ_i^+ , we obtain by (2.12) and (3.3), for $\xi=0$, the solutions

$$\psi_c^+/\sqrt{2} = \psi_+^+ = \psi_-^+ = \chi(v, g) , \quad \psi_c/\sqrt{2} = \psi_+ = \psi_- = \chi(v-v, f) , \quad (3.11)$$

and

$$\frac{d\sigma}{dv} = \int \chi(v, g) \chi(v-v, f) . \quad (3.12)$$

The graphical interpretation of (3.12) is seen in Fig. 4 (AGK cancellation) where, however, the Green's functions $G^{(1,n)}$ are calculated according to Fig. 6 in the tree approximation. Referring to Section 4, the detailed study of the function χ , here we notice that χ has been exactly computed¹⁴⁾ for $d_t=0$, with the result

$$\chi^*(y, g) = \frac{g e^{\mu y}}{1 + g \frac{\lambda}{\mu} (e^{\mu y} - 1)} \rightarrow \frac{\mu}{\lambda} \quad \text{for } \mu y \gg 1. \quad (3.13)$$

Therefore, for $d_t = 0$ (3.12) reduces to the known ²⁾ result (1.4) which by (1.3) is in agreement with momentum sum rules ^{*}.

Let us now consider the two-particle distribution. The first term in Eq. (3.3) is still given in terms of the "tadpole" functions

$$2 \int^2 \chi(v_1, g) \chi(v-v_1, f) \chi(v_2, g) \chi(v-v_2, f) = 2 \frac{d\sigma}{dv_1} \frac{d\sigma}{dv_2}. \quad (3.14)$$

The factor of 2 originates from the AGK couplings (2.7) and is also present in the pure eikonal case. The second term in (3.3) involves functional derivatives of the classical fields, such as $[\delta\psi_c(v_1)/\delta\xi(v_2)]|_{\xi=0}$, etc. The latter are obtained by functional derivation of the equations of motion (3.5) and satisfy the following set of equations:

$$\begin{aligned} (D^{-1} + 2\lambda \tilde{\chi}) \frac{\delta \pi(v')}{\delta \xi(v)} &= -\delta(v-v') \chi, \\ (D^{-1} + 2\lambda \chi) \frac{\delta \pi_0(v')}{\delta \xi(v)} &= \delta(v-v') \chi - 2\lambda \chi \left(\frac{\delta \varphi_0(v')}{\delta \xi(v)} + \frac{\delta \pi(v')}{\delta \xi(v)} \right), \\ \frac{\delta (2\varphi(v') - \varphi_0(v'))}{\delta \xi(v)} (D^{-1} + 2\lambda \tilde{\chi}) &= \delta(v'-v) 2\tilde{\chi} - 2\lambda \tilde{\chi} \left(4 \frac{\delta \pi(v')}{\delta \xi(v)} + \frac{\delta \varphi_0(v')}{\delta \xi(v)} \right), \\ \frac{\delta \varphi_0(v')}{\delta \xi(v)} (D^{-1} + 2\lambda \chi) &= -\delta(v-v') 2\tilde{\chi} \end{aligned} \quad (3.15)$$

where we have introduced the shorthand notation

^{*}) By this we mean that the s dependence of the left-hand side and right-hand side of the sum rule are consistent. The precise coefficients are, however, dependent on the shape of the fragmentation region, and cannot be reproduced correctly here.

$$\chi = \chi(v, g) \quad , \quad \tilde{\chi} = \chi(v-v, f) \quad , \quad (3.16)$$

and we have grouped together the terms $\delta(2\varphi - \varphi_0)/\delta\xi$ which directly occur in the expression

$$\begin{aligned} - \frac{\delta^2 A_c}{\delta \xi(v_1) \delta \xi(v_2)} \Big|_{\xi=0} &= \frac{\delta \psi_c^+(v_1)}{\delta \xi(v_1)} \psi_c(v_1) + \psi_c^+(v_1) \frac{\delta \psi_c(v_2)}{\delta \xi(v_2)} = \\ &= \frac{\delta \pi_0(v_1)}{\delta \xi(v_2)} 2\varphi(v_1) + \pi_0(v_1) \frac{\delta(2\varphi - \varphi_0)}{\delta \xi(v_2)} . \end{aligned} \quad (3.17)$$

Since $\delta\pi/\delta\xi$ and $\delta\varphi_0/\delta\xi$ are directly found from the 1st and 4th equation, all solutions of (3.15) can be formally given in terms of the resolvents

$$\begin{aligned} G &= (D^{-1} + 2\lambda \chi)^{-1} = D - 2\lambda D \chi G \quad , \\ \tilde{G} &= (D^{-1} + 2\lambda \tilde{\chi})^{-1} = D - 2\lambda \tilde{G} \tilde{\chi} D \quad , \end{aligned} \quad (3.18)$$

which are sort of Green's functions in the left (right) external fields $\chi(\tilde{\chi})$ (Fig. 6). They can be identified with the two-point functions of RFT (in the tree approximation) when only one of the two external sources is acting ("shifted" propagators). The simplest expression for (3.18) is still found for $d_t=0$ by solving the related differential equation with the expression (3.13) for χ . After some algebra we find

$$G^\circ(y, y', g) = e^{\mu(y-y')} \frac{[1 + \frac{\Delta}{\mu} g(e^{\mu y'} - 1)]^2}{[1 + \frac{\Delta}{\mu} g(e^{\mu y} - 1)]^2} g(y-y') \quad (3.19)$$

$$\tilde{G}^\circ(y, y', f) = G^\circ(y-y', y-y, f) .$$

Let us notice that for $\mu(y-y') \gg 1$ we have *)

$$G^\circ(y, y', g) \simeq e^{-\mu(y-y')} = \tilde{G}^\circ(y, y', f) \quad (3.20)$$

*) This property justifies somewhat the notation of "shifted" propagator (see also Section 5).

This is a simple consequence of $\chi^{(0)} \sim \mu/\lambda$ for large y , so that $[-\mu + 2\lambda \chi^{(0)}] \sim +\mu$.

By replacing (3.18) into (3.15) we get the solutions (G^T denotes the transposed matrix in v space)

$$\frac{\delta \varphi_0(v_1)}{\delta \xi(v_2)} = -2 \tilde{\chi}(v_2) G(v_2, v_1), \quad \frac{\delta \pi(v_1)}{\delta \xi(v_2)} = -\tilde{G}(v_1, v_2) \chi(v_2)$$

$$\frac{\delta \pi_0(v_1)}{\delta \xi(v_2)} = \frac{\delta \psi_c(v_1)}{\delta \xi(v_2)} = 4\lambda (G \chi G^T)_{v_1 v_2} \tilde{\chi}(v_1) + \left[G(v_1, v_2) + \right.$$

$$\left. + 2\lambda (G \chi \tilde{G})_{v_1 v_2} \right] \chi(v_2), \quad (3.21)$$

with a completely symmetrical expression for

$$\frac{\delta (2\varphi - \varphi_0)(v_1)}{\delta \xi(v_2)} = \frac{\delta \psi_c(v_1)}{\delta \xi(v_2)},$$

obtained by the replacements $v_1 \rightarrow V - v_1$, $g \rightarrow f$. Finally, by putting (3.21), (3.17) and (3.3) together, we obtain the result

$$\frac{d\sigma}{dv_1 dv_2} = 2 \frac{d\sigma}{dv_1} \frac{d\sigma}{dv_2} + \left[4\lambda \tilde{\chi}(v_2) \tilde{\chi}(v_1) (G \chi G^T)_{v_1 v_2} + (v_1 \leftrightarrow V - v_1; f \leftrightarrow g) \right]$$

$$+ \left[\chi(v_1) \tilde{G}(v_2, v_1) + 2\lambda (\tilde{G} \tilde{\chi} G)_{v_2, v_1} \tilde{\chi}(v_2) + (v_1 \leftrightarrow V - v_1; f \leftrightarrow g) \right]. \quad (3.22)$$

This expression looks quite complicated, but a simple graphical interpretation can be obtained by using Eq. (3.18) to recast the last term of (3.21) in a more symmetrical form. In fact we have

$$\tilde{G} + 2\lambda \tilde{G} \tilde{\chi} G = D + 2\lambda \tilde{G} \tilde{\chi} (G - D) =$$

$$= D - 4\lambda^2 \tilde{G} \tilde{\chi} D \chi G. \quad (3.23)$$

All terms of (3.22), with the replacement (3.23), are now shown in Fig. 7. Figures 7a and 7c give the usual eikonal-type terms, corresponding to Figs. 3b and 3c. Figure 7b represents a triple Pomeron graph and its absorptive corrections coming from rescattering with one of the colliding

particles. Finally, Fig. 7d gives the absorptive corrections to the contribution 7c which will be shown in the next Section to cancel the nasty s^μ behaviour.

4. INCLUSIVE DISTRIBUTIONS IN IMPACT PARAMETER SPACE

We now want to discuss the asymptotic form of the inclusive distributions (3.12) and (3.22) when all rapidities are large with respect to $1/\mu$. However, in two-transverse dimensions μY is not the only dimensionless parameter which can set the scale of asymptopia, and also $\alpha' Y/B^2$ and $\alpha' Y/\lambda^2$ are important. In previous treatments ^{2),14)} these extra parameters have been neglected, and the problem has been analyzed at $d_t = 0$. This is certainly correct for nuclei in the region $\alpha' Y \ll R^2$ (R is the nuclear radius), and is essentially correct for particles, provided we are in the asymptotic region. The boundaries in b space of the asymptotic region for the amplitude and the related role of α' are discussed elsewhere ¹⁶⁾. Here we first state the results for the inclusive distributions in the simple case $d_t = 0$, and then discuss the modifications which are needed in the real world. We shall show that the $d_t = 0$ results apply inside a hadronic radius $|B| < R(s)$ which is logarithmically increasing with energy.

4.1 No-transverse dimensions case

In this artificially simple case ($d_t = 0$, i.e., no impact parameter at all), we have the exact formulae (3.13) and (3.19) to be replaced in the inclusive distributions (3.12) and (3.22). When all rapidities are large, we can use the asymptotic expressions

$$\chi^\circ(y) \sim \frac{\mu}{\lambda}, \quad G^\circ(y-y') \sim e^{-\mu(y-y')} \sim \tilde{G}^\circ(y-y'). \quad (4.1)$$

Because of (4.1), Eq. (3.13) now shows how the absorptive corrections of Fig. 7d cancel the AGK contribution of Fig. 7c. We obtain, in fact, for the sum

$$G^\circ(y_1, y_2) + 2\lambda (G^\circ \chi^\circ \tilde{G}^\circ)_{y_1, y_2} \sim e^{-\mu(y_1-y_2)} [1 + 2\mu(y_1-y_2)] \quad (4.2)$$

when all rapidities are large with respect to $1/\mu$.

By inserting (4.1) into all the terms of (3.12) and (3.22), we finally get

$$\frac{1}{\sigma^{(0)}} \frac{d\sigma^{(0)}}{dy} \rightarrow \frac{g}{1-S(\infty)} \left(\frac{\mu}{\lambda}\right)^2 \quad (4.3)$$

and

$$\frac{1}{\sigma^{(0)}} \frac{d\sigma^{(0)}}{dy_1 dy_2} \rightarrow \frac{g^2}{1-S(\infty)} \left\{ 2 \left(\frac{\mu}{\lambda}\right)^4 + e^{-\mu|y_2-y_1|} \left[5 + 2\mu|y_2-y_1| \right] \right\} \quad (4.4)$$

where

$$S(\infty) = \exp - g_{\min} \left(\frac{\mu}{\lambda} - \frac{1}{2} g_{\min} \right), \quad g_{\min} < \frac{\mu}{\lambda}, \quad (4.5)$$

as anticipated in the Introduction.

Several comments are in order. First of all, the energy dependence of (4.4) is strictly consistent with the momentum conservation sum rules, because it contains a constant term plus an integrable one (whose rapidity correlation length is, however, $\sim 1/\mu$, i.e., rather long range for plausible values of μ). Secondly, positivity of partial cross-sections requires that $(\langle n^2 \rangle - \langle n \rangle^2)$ be positive, and therefore that the constant term in the correlation function be positive also. By (4.4) and (4.3) we should therefore have

$$\frac{1}{1-S(\infty)} 2 \left(\frac{\mu}{\lambda}\right)^4 > \frac{1}{[1-S(\infty)]^2} \left(\frac{\mu}{\lambda}\right)^4, \quad (4.6)$$

or

$$S(\infty) < 1/2.$$

By using (4.5) we get the approximate condition

$$\lambda < \mu g_{\min} / \ln 2. \quad (4.7)$$

Strictly speaking, this means that positivity is not always satisfied in the tree approximation. However, if we look at the condition for the validity of such an approximation in the context of RFT, we find for $d_t = 0$

$$\frac{\lambda^2}{\mu} \ll \lambda g_{min} \quad \text{or} \quad \lambda \ll \mu g_{min} \quad (4.8)$$

for the loop contributions to be negligible. Therefore, due to (4.8), Eq. (4.6) is always satisfied, in agreement with the general positivity requirements for the topological expansion of unitarity ⁶⁾.

4.2 Physical single-particle distribution

According to Eq. (3.11) we have

$$\frac{d\sigma}{dv}(v, v) = \chi(v, g) \chi(v-v, f) \quad , \quad (4.9)$$

where χ is the solution of the differential equation

$$\frac{d}{dy} \chi = \alpha' \nabla_b^2 \chi + \mu \chi - \lambda \chi^2 + g F(\underline{b}) \delta(y) \quad , \quad (4.10)$$

in which we have introduced a normalized form factor, e.g.,

$$F(\underline{b}) = \frac{1}{A^2} e^{-b^2/A^2} \quad , \quad \int \frac{d^2b}{\pi} F(b) = 1 \quad . \quad (4.11)$$

In the region of v space where $\chi \ll \mu/\lambda$, the quadratic term of (4.10) can be dropped to yield the perturbative solution

$$\chi \rightarrow \chi^{(0)} = Dg = g e^{\mu y} \frac{e^{-b^2/A^2 + \alpha' y}}{A^2 + 4\alpha' y} \quad . \quad (4.12)$$

This region is characterized by the condition $\chi^{(0)} \ll \mu/\lambda$, i.e.,

$$b^2 \gg R^2(y) = \mu y (A^2 + 4\alpha' y) \quad (4.13)$$

If, however, $b^2 \lesssim R^2(y)$, $\chi^{(0)}$ is no longer small (it increases as $e^{\mu y}$ at any fixed $\underline{b}$) and the quadratic term in (4.10) is essential. The problem is: starting from a small $\chi^{(0)}$ at fixed $\underline{b}$, what happens to the solution χ of the homogeneous equation

$$\frac{d}{dy} \chi = \alpha' \nabla_b^2 \chi + \mu \chi - \lambda \chi^2 \equiv 0 \chi, \quad (4.14)$$

when $\mu y \rightarrow \infty$? This problem is analyzed by looking at the fixed points of (4.14), i.e., at the solutions of $0\chi = 0$. Now, there are two trivial stationary solutions, i.e., $\chi = 0$ and $\chi = \mu/\lambda$. We know already that $\chi = 0$ is an unstable fixed point (χ increases away from it). Moreover, it can be shown (see Appendix A) that there are no other stationary solutions $\chi = \bar{\chi}(b)$ with $0 < \bar{\chi} < \mu/\lambda$ and $\bar{\chi}(b) \rightarrow 0$ as $|b| \rightarrow \infty$. In absence of this extra fixed point $\bar{\chi}(b)$, the solution $\chi(b, y)$ can only increase, at any fixed b , to the asymptotic value $\chi_{as} = \mu/\lambda$. This value will be reached for $y \gg y_{as}(|b|)$ such that $R^2(y_{as}(|b|)) = b^2$.

The resulting picture for $\chi(b, y)$ is therefore that of an expanding plateau, of height $\chi_{as} = \mu/\lambda$, inside a disc of radius $R(y)$. Outside the disc, the Regge tail $\chi^{(0)}(b, y)$ of Eq. (4.12) comes out. As a consequence the schematic picture of the particle density is

$$\frac{d\sigma}{d\nu}(Y, \nu) \rightarrow \rho \left(\frac{\mu}{\lambda} \right)^2 \vartheta(R^2(Y) - b^2) \vartheta(R^2(Y - \nu) - (B - b)^2). \quad (4.15)$$

By integrating over b and B we finally get

$$\frac{d\sigma}{dY} \rightarrow \rho \left(\frac{\mu}{\lambda} \right)^2 2\pi R^2(Y) R^2(Y - \nu). \quad (4.16)$$

In order to see the behaviour of the normalized density (and check the inclusive sum rules) it remains to be seen what the total cross-section does. It can be shown ¹⁶⁾ that, for large enough $\alpha' Y$, a similar picture arises in the sense that the $d_t = 0$ result (4.5) is the asymptotic value ³⁾ of the partial wave inside a disc of radius $R(Y)$. We have therefore

$$\begin{aligned} T(B, Y) &\rightarrow [1 - S(\infty)] \vartheta(R^2(Y) - B^2), \\ \sigma_T &\rightarrow [1 - S(\infty)] 2\pi R^2(Y), \\ \frac{1}{\sigma_T} \frac{d\sigma}{dY} &\rightarrow \rho \frac{(\mu/\lambda)^2}{1 - S(\infty)} \frac{R^2(Y) R^2(Y - \nu)}{R^2(Y)}. \end{aligned} \quad (4.17)$$

*) In Eq. (4.5) g and g^2 mean in this case the integrated values of $g(b)$ and $g^2(b)$, as defined in (4.10).

It appears from Eq. (4.17) that momentum sum rules are satisfied ($d\sigma/dy$ and σ_{tot} increase with $\ln s$ in the same way for finite $Y-y$). Moreover, the average multiplicity is

$$\langle n \rangle \sim \int \frac{(\mu/\lambda)^2}{1-s(\infty)} Y R^2(Y) ,$$

and therefore increases as $(\ln s)^3$ for $\alpha'Y \gg A^2$. This behaviour is in sharp contrast with models of purely absorptive type which also have $\mu > 0$ and $\sigma_{\text{tot}} \sim R^2(Y)$, but have $\langle n \rangle \sim \ln s$ ¹⁷⁾. The difference is due to events of multiple multiplicity which in the present model occur when n_c Pomerons are cut according to AGK.

In fact the quantity

$$\frac{1}{s} \frac{1}{\sigma_T} \int \frac{d\sigma}{d^2v} (v, v) d^2B$$

is just the average number $\langle n_c(v) \rangle$ of P_c 's at a point v for given Y . Then (4.15) means that such an average is constant within the discs' radius. The result $\langle n \rangle \sim (\ln s)^3$ is also in qualitative agreement with the rise of the (normalized) plateau ¹⁸⁾ predicted at perturbative level because of the AGK cut cancellation in $d\sigma/dy$ and not in σ_{tot} .

4.3 Higher order distributions

In order to evaluate asymptotically the two-particle density, we need to know the analogue of (4.1) for the resolvents $G(v_1, v_2, g)$ and $\tilde{G}(v_1, v_2, f)$, which are also V and v_0 dependent through χ and $\tilde{\chi}$. We have, for instance,

$$\left[\alpha_1 - \alpha' \nabla_b^2 - \mu + 2\lambda \chi(V-v, f) \right] \tilde{G}(v, v') = \delta(y-y') \delta^2(b-b'). \quad (4.18)$$

We are unable to solve this equation in general, but we know that for $\mu(Y-y) \gg 1$ and $(\underline{B}-b)^2 < R^2(Y-y)$, $\chi \rightarrow \mu/\lambda$ reversing the signs of μ in the square brackets. We then have

$$\tilde{G}(v, v') \xrightarrow{\mu(y_1) \gg 1} \mathcal{D}(v-v') \mathcal{D}(R^2(y_1) - (\underline{B}-\underline{b}_1)^2) ,$$

$$G(v, v') \xrightarrow{\mu(y_1) \gg 1} \mathcal{D}(v-v') \mathcal{D}(R^2(y_1) - b_1^2) ,$$

$$\mathcal{D}(v) \equiv e^{-\mu y} \frac{e^{-b^2/4\alpha' y}}{4\alpha' y} \mathcal{D}(y) , \quad (4.19)$$

where (b_1, y_1) is any of the two co-ordinates v and v' . [Note that distinguishing between v and v' is immaterial, due to the factor $\mathcal{D}(v-v')$.] In other words $G[\tilde{G}]$ reduces inside discs centred at $v_0 = (0,0)$ [$V = (\underline{B}, Y)$] to Pomeron bare propagators with μ shifted to $(-\mu)$.

We can now replace (4.19) into (3.22) and do the integrations by noting that the width in impact parameter of $\mathcal{D}$ is much smaller than the disc's radius where all μy 's are large. Since the $\mathcal{D}$ functions have the simple property that

$$\int \frac{d^2 b}{\pi} \mathcal{D}(v_1 - v) \mathcal{D}(v - v_2) = \mathcal{D}(v_1 - v_2) ,$$

we finally get the simple results

$$\begin{aligned} \frac{d\sigma}{d\sigma_1 d\sigma_2} &\rightarrow 2\pi \sum_{i=1,2} \chi(v_i) \chi(V-v_i) + \\ &+ \mathcal{D}(R^2(y_1) - b_1^2) \mathcal{D}(R^2(y_2) - (\underline{B}-\underline{b}_2)^2) \mathcal{D}(v_1 - v_2) (5 + 2\mu(y_2 - y_1)) , \\ \chi(v) &\rightarrow \frac{\mu}{\lambda} \mathcal{D}(R^2(y) - b^2) . \end{aligned} \quad (4.20)$$

The same argument can now go through to higher order distributions, as far as the leading behaviour is concerned. The crucial observation is that all functional derivatives of the classical fields, with respect to ξ , are given by (3.5) in terms of the "shifted" propagators G and $\tilde{G}$ such as in Eq. (3.15). Since the latter have the asymptotic behaviour (4.19), if we neglect terms behaving as $s^{-\mu}$, we can dispose of all functional derivatives of $A_c(\xi)$ beyond the first. In this rough approximation we get by (3.3)

$$\frac{d\sigma}{dv_1 \dots dv_n} \rightarrow 2^{n-1} \prod_{i=1}^n \chi(v_i) \chi(V-v_i) \quad (4.21)$$

and therefore

$$\begin{aligned} -A_c(\xi) &\rightarrow 2 \int dv \chi(v) \chi(V-v) \xi(v) \\ &\rightarrow 2 \left(\frac{\mu}{\lambda}\right)^2 \int dv \Theta(R^2(y) - b^2) \Theta(R^2(y) - (B-b)^2) \xi(v). \end{aligned} \quad (4.22)$$

This shows that all momentum sum rules are satisfied to leading order.

From (4.21) the asymptotic multiplicity moments can be calculated. In the regime when $R(Y) \sim R_0 Y$, $R_0 = 2\sqrt{\alpha' \mu}$, we get the particularly simple results (Appendix B)

$$\langle n_k(Y, B) \rangle \sim [2(1-S(\infty))]^{k-1} \langle n(Y, B) \rangle^k, \quad (4.23)$$

$$\langle n(Y, B) \rangle \sim \left(\frac{\mu}{\lambda}\right)^2 \frac{R_0^2 Y^3}{12(1-S(\infty))} \left[1 - \frac{B^2}{R_0^2 Y}\right]^{3/2} \Theta(R_0^2 Y - B^2),$$

thus confirming the condition $S(\infty) < \frac{1}{2}$ for the positivity constraint of $\langle n^2(B, Y) \rangle - \langle n(B, Y) \rangle^2$ inside the disc. For the integrated multiplicity moments we get instead

$$\begin{aligned} \langle n(Y) \rangle &\sim \frac{2}{5} \left(\frac{\mu}{\lambda}\right)^2 \frac{R_0^2 Y^3}{12(1-S(\infty))}, \\ \langle n_k(Y) \rangle &\sim \langle n(Y) \rangle^k \frac{(5/2)^k}{1 + \frac{3}{2}k} [2(1-S(\infty))]^{k-1}, \end{aligned} \quad (4.24)$$

which gives a formal KNO scaling at asymptotic energies.

5. DISCUSSION

In the preceding Section we have calculated the asymptotic particle density in impact parameter and the corresponding correlation functions for the generalized eikonal diagrams in the tree approximation.

The picture that emerges is consistent with the one of Ref. 16), according to which the hadron for $\mu = \alpha_P^0 - 1 > 0$ (or better $\mu > \mu_{\text{crit}}$) is a grey disc of opacity $1 - S(\infty)$ [Eq. (4.5)] and expanding radius, which in

the classical case is $R^2(s) \approx 4\alpha' \mu (\ln s)^2$. We find in fact:

- 1) The particle density and correlation functions are asymptotically constant inside the expanding disc. The rapidity correlation length of the next-to-leading terms is $1/\mu$. The total multiplicity increases as $(\ln s)^3$ at asymptotic energies, and the multiplicity moments are $\langle n_k \rangle \sim c_k \langle n \rangle^k$.
- 2) Within the tree approximation we have asymptotic consistency with both positivity requirements and momentum conservation sum rules. In contrast to this, for pure eikonal diagrams (i.e., $\lambda \rightarrow 0$), we have a black disc, but the sum rules are no longer automatically fulfilled.

The question now arises of whether our tree approximation results survive loop corrections and how they compare with other non-perturbative calculations. The addition of loops (actually an infinite number of them) leads to the critical intercept renormalization (4.5) $\mu_{\text{crit}}(\lambda)$, such that if $\mu \leq \mu_{\text{crit}}$, $\alpha_P(\text{renormalized}) \leq 1$.

Our calculation is then meaningful only for $\mu \gg \mu_{\text{crit}}$ as mentioned in the Introduction. However, contrasting estimates leading to vanishing cross-sections and purely logarithmic multiplicity have been given in the literature ^{8),9)} for the case $\mu \gtrsim \mu_{\text{crit}}$. We believe, however, that if the theory is defined for $\mu > \mu_{\text{crit}}$ instead according to the treatment of Refs. 2), 15) and 16), our classical asymptotic results will stay essentially correct also in the full quantum theory. The approach of Refs. 2), 15) and 16) differs from that of Refs. 8) and 9) in some important interpretative points that we now briefly discuss, to the extent that they pertain to the calculation of inclusive distributions.

5.1 Tadpole contributions

The asymptotic constants contributing inside the disc to the amplitude ¹⁶⁾ and to the particle density are essentially "tadpoles", i.e., ground state expectation values of the fields ψ, ψ^+ (Fig. 8). The simplest example is given by the classical field discussed in subsection 4.1 which, for $d_t = 0$, has the closed form ¹⁴⁾

$$\chi(y) = \frac{g e^{\mu y}}{1 + g \frac{\lambda}{\mu} (e^{\mu y} - 1)} \quad (5.1)$$

This expression gives the formal sum of the power series

$$g e^{\mu y} \sum_n \left[-\frac{\lambda}{\mu} g (e^{\mu y} - 1) \right]^n$$

but becomes g independent ($\chi \rightarrow \mu/\lambda$) when $\mu y \gg 1$.

From the point of view of the shifted field method, the contribution $f(\mu/\lambda)$ to the amplitude comes from coupling the external current $\tilde{J}$ with the vacuum expectation value of the field ψ , and it appears therefore as a disconnected contribution to the amplitude and neglected as such. However, the previous remark shows that this contribution comes instead from the sum of connected diagrams to the formal perturbative sum. Since only formal perturbation theory can be derived from a basic underlying theory (e.g., in the topological expansion), we conclude that tadpoles are to be kept, at least at the semiclassical level ^{*)} (e.g., $\mu \gg \mu_{\text{crit}}$).

Similarly, the constant contribution $(\mu/\lambda)^2$ to the particle density comes from the vacuum expectation values of ψ_c and ψ_c^+ (Fig. 8). Note also that in this case we have assumed that the mass insertion $\xi(b,y)$ couples to $\psi_c^+ \psi_c$, and not to the shifted fields in contrast to Ref. 9). We presume in fact that only formal perturbation theory admits particle interpretation, i.e., is the shadow of physical production amplitudes.

4.2 AGK cutting rules and CRFT tadpoles

It is shown in Refs. 15) and 16) that for $\mu > \mu_{\text{crit}}$, RFT admits two degenerate vacua which have to be treated symmetrically so that one cannot really define the theory by "shifting" the fields (ψ, ψ^+) by one of the two tadpoles $(0, \mu/\lambda)$ or $(\mu/\lambda, 0)$. Although no similar treatment is yet available for CRFT, our classical results confirm the necessity of the $\psi_i = \psi_i^+$ symmetry for satisfying the s channel discontinuity equation with AGK cutting rules. In fact, the procedure that we have followed in Sections 2 and 3 determines the tadpoles which contribute to the particle distributions in a way consistent with the symmetry. This can be seen as follows.

- 1) We have shown in Ref. 12) that, if the theory is properly defined ($\mu < \mu_{\text{crit}}$), the two-body unitarity equation holds in the form of Eq. (1.1).
- 2) In subsection 2.2 we have proved that the validity of Eq. (1.1) implies that

$$A_c \left(J_+ = J_- = \frac{1}{\sqrt{2}} J_c = g ; \tilde{J}_+ = \tilde{J}_- = \frac{1}{\sqrt{2}} \tilde{J}_c = f \right) = 0, \quad (5.2)$$

i.e., that the action of CRFT vanishes when evaluated at external currents induced by the scattering particles, according to the AGK rules.

^{*)} At the quantum level our classical explanation of why tadpoles have to be kept is better understood as due to the symmetric treatment of two degenerate ground states [cf. Refs. 15) and 16) for a discussion of this point].

- 3) In Section 3 we have also shown that the vanishing of A_c for any g and f determines which fields assume non-vanishing expectation values, and precisely we have, by (3.11), in the limit $y, Y-y \gg 1/\mu$,

$$\langle \Psi_{\pm} \rangle = \frac{1}{\sqrt{2}} \langle \Psi_c \rangle = \langle \Psi_{\pm}^+ \rangle = \frac{1}{\sqrt{2}} \langle \Psi_c^+ \rangle = \frac{\mu}{\lambda} \quad (5.3)$$

Therefore, if we want to keep s channel discontinuity equation for $\mu > \mu_{\text{crit}}$, we cannot use asymmetrically shifted fields ^{*)}, and we have to couple $\xi(b, y)$ to $\psi_c^+ \psi_c$.

To sum up, we have presented in this paper a semiclassical calculation valid for $\mu \gg \mu_{\text{crit}}$ (tree approximation) and we have suggested that our results can survive at the quantum level if RFT is defined for $\mu \gtrsim \mu_{\text{crit}}$ in a way consistent with s channel unitarity. This completes, from our point of view, the arguments of Ref. 16). The results presented above are phenomenologically satisfactory, but have the usual troubles with the J plane structure (hard cuts) typical of an expanding grey disc. It is not known at present to what extent loop corrections can cure this disease by modifying the t dependence of the J plane singularities.

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^{*)} The violation of the AGK rule in Ref. 9) is due to a shift of this kind.

POSSIBLE FIXED-POINT SOLUTIONS OF THE FIELD

In Section 4 we have reduced the study of the asymptotic behaviour of $\chi(\underline{b}, y)$ to that of the fixed-point solutions of Eq. (4.14), i.e., functions $\chi(\underline{b})$ satisfying

$$0\chi = \alpha' \nabla_{\underline{b}}^2 \chi + \lambda \chi \left(\frac{\mu}{\lambda} - \chi \right) = 0. \quad (\text{A.1})$$

Trivial solutions of (A.1) are $\chi = 0$ and $\chi = \mu/\lambda$. We now want to show that no other solution $\chi(|\underline{b}|)$ exists such that $0 < \chi < \mu/\lambda$ and $\chi(|\underline{b}|) \rightarrow 0$ (or μ/λ) as $|\underline{b}| \rightarrow \infty$. In fact by integrating (A.1) inside a disc of radius R , we get

$$\alpha' R \frac{\partial}{\partial R} \chi(R) = -\lambda \int_0^R b \, db \, \chi \left(\frac{\mu}{\lambda} - \chi \right). \quad (\text{A.2})$$

It is now easily seen that (a) for $R \rightarrow \infty$ only $\chi \rightarrow 0$ (or $\chi \rightarrow \mu/\lambda$) are possible. A different finite limit would imply $\chi' \rightarrow -\infty$ as $R \rightarrow \infty$, and therefore $\chi < 0$ from somewhere on. (b) Even if χ or $\mu/\lambda - \chi$ vanish as $R \rightarrow \infty$, no asymptotic behaviour reproduces correctly on the right-hand side if somewhere χ or μ/λ are non-vanishing. If, in fact χ (or $\mu/\lambda - \chi$) $\sim R^{-\alpha-1}$, the right-hand side of (A.2) behaves as $R^{1-\alpha}$ if $\alpha > 1$ and as a constant if $\alpha < 1$. The left-hand side instead behaves as $R^{-\alpha-1}$ in both cases. This proves the absence of an "intermediate" fixed point $\chi(\underline{b})$.

Let us finally remark that $0\chi > 0$ for any constant function $0 < \chi < \mu/\lambda$ and $0\chi < 0$ for $\chi > \mu/\lambda$. Moreover, 0χ is a fortiori positive at any point for which χ is convex ($0 < \chi < \mu/\lambda$), while it can be negative if χ is sufficiently concave, depending on the values of $\chi(\mu/\lambda - \chi)$. This shows that, starting from

$$\chi^0 = e^{\mu y} \frac{e^{-b^2/4\alpha' y}}{4\alpha' y},$$

the field $\chi(\underline{b}, y)$ increases rapidly towards $\chi_{as} = \mu/\lambda$ at the centre of the disc $|\underline{b}|^2 < R^2(y) = 4\alpha' \mu y^2$, while it can stay close to $\chi^{(0)}$ for $|\underline{b}| \gg R(y)$.

APPENDIX B

CALCULATION OF MULTIPLICITY MOMENTS

In order to estimate the asymptotic multiplicity moments $\langle n(n-1) \dots (n-k+1) \rangle = \langle n_k \rangle$ we have to integrate the asymptotic inclusive distributions (4.21) over phase space. This can be done in the regime $R \sim R_0 Y$, $R_0 = 2\sqrt{\alpha' \mu}$, by noting that ¹⁹⁾

$$\chi(\underline{b}, y) \rightarrow \left(\frac{\mu}{\lambda}\right) \vartheta(R_0^2 y^2 - b^2) = \frac{\mu}{\lambda} \vartheta_+(x^2) \quad (\text{B.1})$$

where we have defined a Lorentz 3-vector $x = (R_0 y, \underline{b})$.

By substituting (B.1) into (4.21) we then have

$$\sigma_T \langle n_k(Y) \rangle \simeq 2^{k+1} \left(\frac{\mu}{\lambda}\right)^{2k} \int d\underline{B} \left[\int \prod_{i=1}^k \frac{d^3 x_i}{\pi R_0} \vartheta_+(x_i^2) \vartheta_+((X-x_i)^2) \right], \quad (\text{B.2})$$

where $X = (R_0 Y, \underline{B})$. Since the integral in square brackets is invariant under x rotations, we can evaluate it in the system where $X = (R_0 Y, \underline{B}, 0)$ and we get

$$\begin{aligned} \langle n(\underline{B}, Y) \rangle &= \left(\frac{\mu}{\lambda}\right)^2 \int \frac{d^3 x}{\pi R_0} \vartheta_+(x^2) \vartheta_+((X-x)^2) = \left(\frac{\mu}{\lambda} R_0\right)^2 \frac{(Y^2 - \underline{B}^2/R_0^2)^{3/2}}{12(1-S(\infty))} \vartheta_+(X^2), \\ \langle n_k(\underline{B}, Y) \rangle &= \langle n(\underline{B}, Y) \rangle^k [2(1-S(\infty))]^{k-1}. \end{aligned} \quad (\text{B.3})$$

By finally doing the $\underline{B}$ integrations in (B.2) we obtain

$$\begin{aligned} \langle n(Y) \rangle &= \frac{2}{5} \left(\frac{\mu}{\lambda}\right)^2 \frac{R_0^2 Y^3}{12(1-S(\infty))} \\ 2\pi R_0^2 Y^2 \langle n_k(Y) \rangle &= [2(1-S(\infty))]^{k-1} \left(\frac{5}{2} \langle n \rangle\right)^k 2\pi \int d\underline{B}^2 \left(1 - \frac{\underline{B}^2}{R_0^2 Y^2}\right)^{\frac{3}{2}k}, \end{aligned}$$

from which Eq. (4.14) follows immediately.

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FIGURE CAPTIONS

- Fig. 1 : Generalized eikonal diagram with Pomeron interactions.
 G represents the full Pomeron Green's function of RFT.
- Fig. 2 : Representation of the discontinuity equation [see Eq. (1.1)].
 Solid lines represent the cut Pomeron propagators.
- Fig. 3 : The Mueller diagrams for the single (3a) and double (3b, c) particle distribution for $\lambda = 0$ (pure eikonal model).
- Fig. 4 : Representation of the function X which satisfies Eq. (3.10).
 $v_0 \equiv (0,0)$ and $v \equiv (b,y)$ denote impact parameter and rapidity of the external particle and of X , respectively.
- Fig. 5 : Single-particle inclusive distribution of Eq. (3.12).
- Fig. 6 : Representation of the resolvent operator $G(X)$ in Eq. (3.18).
 Open circles represent the sum of all the graphs in Fig. 4.
- Fig. 7 : The classes of graphs contributing to the two-particle distribution.
- Fig. 8 : The origin of the asymptotic constants contributing
 (a) to the particle density, and
 (b) to the amplitude inside the impact parameter disc.
 The sum of graphs on the left become asymptotically independent of the external particles' couplings. The black dot denotes the constant μ/λ .

FIG.1

FIG.2

FIG.3

FIG.4

FIG.5

$$G_{vv'}(\chi) = \Sigma$$

FIG.6

FIG.7