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SOLITONS IN GAUGE THEORIES OF VECTOR AND SPINOR FIELDS

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A B S T R A C T

Non-linear c-number field equations are obtained from the operator field equations for gauge theories of vector and spinor fields by a simple variational ansatz. It is suggested that their soliton solutions characterize bound states in these theories. An example of such a static solution describing a bag-like quark-antiquark bound state in three dimensional space is exhibited.

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Three types of semi-realistic soliton-like images of hadrons have been proposed so far : a) field theoretical "bags" ¹⁾ and "bubbles" ²⁾ in models of interacting spinor and spinless fields; b) superconducting vortex lines in the abelian Higgs model ³⁾; c) extended magnetic monopoles in non-abelian Higgs models ⁴⁾. In all cases a fictitious "scalar glue" is present whose highly non-linear self-interaction is responsible for the spontaneous breakdown of symmetries and for binding forces holding the soliton together.

In this note we report the finding of soliton-like solutions in gauge theories of vector and spinor fields. In contrast to the previous examples, the binding in this case arises from the balance of what may be more easily regarded as fundamental physical forces. Our solutions may provide a natural basis for a new extended model of hadrons in canonical field theory.

The introductory discussion of this note will be confined to an abelian quark-gluon model. A more detailed discussion, including straightforward non-abelian extensions, will be presented elsewhere.

It is of course well known that the Bethe-Salpeter ladder approximation to electrodynamics exhibits the formation of bound states (H-atom, positronium,...). What we have to offer here is some developments of a different point of attack ^{*}) for the description of similar phenomena (the binding of "color" singlet states) which is characterized by : a) starting from non-linear c-number field equations, obtained from the fundamental operator field equations by a simple variational ansatz; b) describing bound states, in the lowest variational approximation, as soliton solutions to these equations. Thus our approach brings immediately out "extension" and other typical soliton properties which are reminiscent of hadrons. It should be stressed, however, that, although promising in our opinion, the relevance of such an approach to hadron physics remains to be established.

Let us consider the U(1) gauge theory of a massive quark field, $\psi(x)$, and of a massless gluon field, $B_\mu(x)$, with interaction defined by the canonical hamiltonian density (unrenormalized) ^{**)} :

$$\mathcal{H}(x) = \psi^\dagger \frac{\alpha}{i} \nabla \psi + m_0 \bar{\psi} \psi + \frac{1}{2} (G_{ij}^2 - G_{0i}^2) + g_0 \bar{\psi} \gamma^\mu \psi B_\mu, \quad (1)$$

^{*}) The two approaches begin from opposite ends. The conventional one starts from the classical theory, first-quantizes it to obtain a Schrodinger equation and then (in principle though not in practice) improves the approximation by summing classes of diagrams in the Bethe-Salpeter framework. We shall begin from the opposite end, the quantum field theory, and "de-quantize" it by chopping off from the Hilbert space all but a minuscule (but, as the case may be, the essential) portion (variational approximation).

^{**)} We use the conventions of Bjorken and Drell, Relativistic Quantum Fields (McGraw-Hill, New York, 1964)

where

$$G_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu . \quad (2)$$

The unrenormalized field equations are :

$$(i\not{D} - m_0) \psi(x) = g_0 \not{B} \psi(x) , \quad (3')$$

$$\partial_\nu G^{\mu\nu} = g_0 : \bar{\psi} \gamma^\mu \psi : , \quad (3'')$$

after normal-ordering the quark current to neutralize the vacuum state. As we shall not, for the moment, encounter other infinities, we may postpone the replacement of Eqs. (3) with renormalized, finite field equations ⁵⁾.

The charge operator is :

$$Q = \int d^3x (\bar{\psi} \gamma^0 \psi - \frac{1}{2} \{ \bar{\psi} \gamma^0, \psi \}) = \frac{1}{2} \int d^3x (\psi^\dagger \psi - \psi \psi^\dagger) . \quad (4)$$

We consider three states with charges +, -, and 0 :

$$Q |q\rangle = |q\rangle , \quad Q |\bar{q}\rangle = - |\bar{q}\rangle , \quad Q |q\bar{q}\rangle = 0 , \quad (5)$$

and the following transformation properties under charge conjugation ($\mathcal{C}$) :

$$\mathcal{C} |q\rangle = |\bar{q}\rangle , \quad \mathcal{C} |\bar{q}\rangle = |q\rangle , \quad \mathcal{C} |q\bar{q}\rangle = \pm |q\bar{q}\rangle . \quad (6)$$

We shall solve the projection of the operator field equations (3) onto the Hilbert subspace spanned by any three such vectors. More specifically, we shall make the variational approximation of evaluating operator products by saturation with states within that subspace. It follows from the postulated charge and charge conjugation properties of these states, and of the quark field :

$$[Q, \psi(x)] = -\psi(x) , \quad \mathcal{C} \psi_\alpha(x) \mathcal{C}^{-1} = C_{\alpha\beta} \bar{\psi}_\beta(x) , \quad C = i\gamma^2 \gamma^0 , \quad (7)$$

that the only contributing quark field matrix elements are :

$$\langle \bar{q} | \psi(x) | q\bar{q} \rangle = u(x) = \langle q\bar{q} | \psi^\dagger(x) | \bar{q} \rangle^\dagger , \quad (8')$$

$$\langle q | \psi^\dagger(x) | q\bar{q} \rangle = v(x) = \langle q\bar{q} | \psi(x) | q \rangle^\dagger , \quad (8'')$$

and are related by :

$$u(x) = \pm i\gamma^2 v(x) . \quad (9)$$

Needless to say, the only non-vanishing matrix elements of the (abelian) gluon field within our subspace are diagonal ones.

Thus we arrive at the following system of c-number field equations :

$$(i \not{\partial} - m_0) u(x) = g_0 \langle \bar{q} | \not{A} | \bar{q} \rangle u(x) , \quad (10')$$

$$\partial_\nu \langle \bar{q} | G^{\mu\nu} | \bar{q} \rangle = -\frac{1}{2} g_0 \bar{u} \gamma^\mu u , \quad (10'')$$

$$\partial_\nu \langle q\bar{q} | G^{\mu\nu} | q\bar{q} \rangle = 0 , \quad (10''')$$

subject to the normalization condition :

$$\langle \bar{q} | \bar{q} | \bar{q} \rangle = -\frac{1}{2} \int d^3x u^\dagger u = -1 . \quad (11)$$

Soliton solutions to this system of equations will describe, in our variational approximation, quark-anti-quark bound states.

Before proceeding, it will perhaps be useful to attract the reader's attention to a number of points :

- a) The bound states that we are seeking all correspond to $|q\bar{q}\rangle$; which bound state in the complete Hilbert space of the theory is supposed to be approximated by $|q\bar{q}\rangle$ depends on which soliton solution of Eqs. (10)-(11) we are considering.
- b) $|q\rangle$ and $|\bar{q}\rangle$ are auxiliary states needed to construct the quark and anti-quark wave functions $u(x)$ and $v(x)$; physically one could think of them as virtual, constituent states, i.e., what remains of a quark-anti-quark bound state as one of the constituents is suddenly removed ; as such, they should not correspond, even approximately, to physical, on mass-shell particle states in the full blown theory.
- c) The c-number equations for the quark wave function, Eqs.(10')-(10''), exhibit an important change of sign with respect to the operator field equations (3) ; physically this mirrors the interaction of the quark charge distribution with the anti-quark charge distribution ; technically this change of sign is brought about by the normal-ordering of the quark current.
- d) The gluon amplitude in the bound state, $\langle q\bar{q} | G^{\mu\nu} | q\bar{q} \rangle$, is, according to Eq. (10'''), a free field ; this occurs because of the mutual cancellation between the constituent quark and anti-quark current densities in our (self-consistent) approximation ; we shall henceforth set :

$$\langle q\bar{q} | G^{\mu\nu} | q\bar{q} \rangle = 0 \quad (12)$$

We are mainly interested in static (up to a kinematical time dependence) soliton solutions. For simplicity, we shall furthermore focus our attention on systems with maximal, spherical symmetry. Thus, we make the customary ansatz :

$$u = \begin{pmatrix} u_1(r) \chi_k^m \\ i u_2(r) \chi_{-k}^m \end{pmatrix}, \quad (13)$$

where u_1 and u_2 are upper and lower component radial functions, χ_k^m are ordinary spin-angular functions, and $k = \pm (j + 1/2)$ identifies the parity of the two quark states of angular momentum j . Consistency with the requirement of spherical symmetry for the gluon field forces $j = 1/2$ for each of the constituents. The system of Eqs.(10)-(11) is then reduced to :

$$\frac{d}{dr} u_2(r) = \frac{k-1}{r} u_2(r) - (W - m_0 - g_0 V(r)) u_1(r), \quad (14')$$

$$\frac{d}{dr} u_1(r) = (W + m_0 - g_0 V(r)) u_2(r) - \frac{k+1}{r} u_1(r), \quad (14'')$$

$$\frac{1}{r^2} \frac{d^2}{dr^2} (r^2 V(r)) = \frac{1}{2} g_0 (u_1^2 + u_2^2), \quad (14''')$$

where $V(r)$ denotes the time component of the gluon field in the auxiliary anti-quark state

$$V(r) = \langle \bar{q} | B^0 | \bar{q} \rangle \quad (15)$$

and W is the spinor energy eigenvalue. The latter may also be regarded as a Lagrange multiplier to be varied to enforce the normalization condition (11). Boundary conditions of regularity at the origin and sufficiently rapid fall-off at infinity must also be imposed to ensure integrability of the soliton energy density. In particular, it is immediately established that asymptotically as $r \rightarrow \infty$ u_1, u_2 should fall off exponentially, while $V(r)$ should exhibit the Coulomb tail characteristic of the charge of the constituent $|\bar{q}\rangle$ state.

A simple variational estimate indicates that stable solutions to the system of Eqs. (14) indeed exist and predicts some of their properties. Sandwiching the canonocal hamiltonian operator between the $|q\bar{q}\rangle$ state and inserting the constituent states as intermediate states, we obtain an expression for the energy, $\mathcal{E}$, of $|q\bar{q}\rangle$ in terms of the quark wave functions and the gluon amplitudes in the constituent states :

$$\begin{aligned} \mathcal{E} = \frac{1}{2} \int d^3x & \left[u^\dagger \left(\frac{\alpha \cdot \nabla}{i} + m_0 \gamma^0 \right) u - v^\dagger \left(\frac{\alpha \cdot \nabla}{i} + m_0 \gamma^0 \right) v \right. \\ & \left. + g_0 u^\dagger u \langle \bar{q} | B^0 | \bar{q} \rangle - g_0 v^\dagger v \langle q | B^0 | q \rangle \right]. \end{aligned} \quad (16)$$

For systems of characteristic size R , $\mathcal{E}$ is estimated to be

$$\mathcal{E} \simeq 2 \sqrt{m_0^2 + p^2} - 2 \frac{g_0^2}{4\pi} \frac{1}{R}, \quad p \simeq \frac{1}{R}, \quad (17)$$

where the first and second terms approximate respectively the rest mass and kinetic energy of the constituents and the interaction energy between constituents.

$\mathcal{E}$, as given by Eq.(17), is minimized by (Fig. 1) :

$$R \approx \frac{\sqrt{1-\beta^2}}{\beta} m_0^{-1}, \quad \beta = \frac{g_0^2}{4\pi} \quad (18)$$

and its minimum is (Fig. 1) :

$$\mathcal{E} \approx \sqrt{1-\beta^2} 2m_0. \quad (19)$$

Note that Eqs.(18)-(19) cease to be meaningful for strong coupling, i.e., as $\beta \rightarrow 1$. This may be interpreted to suggest a phase transition of the vacuum state to a condensate of pairs.

A numerical analysis of the system of Eqs.(14) supports the estimates presented above, after yielding the precise form of the constituent's wave function and of the gluon amplitudes in the auxiliary constituent states. An example of such a solution for weak coupling is reported in Fig.2 . We call the reader's attention to the smoothness of the gluon amplitude, which plays the role of an effective binding potential, near the origin. The picture of bound states that emerges is not dissimilar, especially after introducing a gluon mass term (see below), to the bag picture of Ref. 1 . On the other hand, the absence of couplings to fictitious self-interacting scalar fields makes the present picture a much less arbitrary candidate for a fundamental description of hadrons.

Let us conclude with a few observations :

- a) In addition to static soliton solutions, the system of Eqs.(10)-(11) admits a rich spectrum of solitons with dynamical time dependence (as opposed to trivial kinematical time dependences associated with translational modes in space-time and implied by the relativistic invariance of the system of Eqs. (10)-(11)), which we shall not discuss here.
- b) In the abelian U(1) model adopted in this note, we were only able to construct bound states of the $q\bar{q}$ variety. This is simply because $(q\bar{q})^n$ is the only quark combination leading to U(1) singlets. The extension of our approach to non-abelian gauge models will also allow for bound states of the qqq variety.
- c) A preliminary investigation of the effect of adding a gluon mass term, $\frac{1}{2}M_0^2 B_\mu B^\mu$, to the U(1) gauge Lagrangian indicates that soliton solutions persist provided the ratio of gluon to quark mass does not exceed a critical value, function of the coupling constant,

$$\frac{M_0}{m_0} < f(\beta) \quad (20)$$

The function $f(\beta)$ depends upon the soliton solution under consideration.

- d) The bound states we have constructed in the $U(1)$ gauge model (without a gluon mass term) are obviously unstable against radiative decays. Technically, this instability would emerge if we enlarged our family of trial states to allow for radiation. Stability may be simply regained in the $U(1)$ model by giving the gluon a mass in excess of an appropriate fraction of the bound state mass. Alternatively, in the context of non-abelian gauge models, one may want to keep the gluons massless and appeal to gluon confinement by infrared "slavery" (assuming such a thing really exists...) to provide the necessary stability.
- e) An obvious question, for which we have no immediate answer, is the quality of our variational approximation. In its crudest form adopted in this note (with only three states), we have taken the overly optimistic attitude that what is crucial to the description of our bound states is the bound state itself and two virtual constituent states.

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FIGURE CAPTIONS

Fig. 1 Estimate of the soliton's characteristic radius and energy.

Fig. 2 Example of soliton solution for weak coupling.

FIG.1

FIG. 2