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CERENKOV AND TRANSITION RADIATION

FROM PARTICLES WITH SPIN

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INTRODUCTION

I am indebted to Professor F. Krienen for this problem : work out the basic formula for transition radiation in the case of particles with spin. In deriving this formula a corresponding result is obtained, in passing, for Cerenkov radiation. In that case comparison can be made with some results already in the literature ; there is sufficient agreement, except at one point, in connection with dispersion.

RESULTS

For unpolarized particles the results emerge in terms of two electromagnetic structure factors, $A(q^2)$ and $B(q^2)$, characterizing the particle involved. For a particle of mass M , charge Qe , spin j , and total magnetic moment $ge_j/(2M)$ (with $\hbar = c = 1$)

$$\left. \begin{aligned} A(0) &= Q^2 \\ B(0) &= g^2 j(j+1)/3 \end{aligned} \right\} \quad (1)$$

In the particular case of a spin one-half particle with Ernst-Sachs-Wali electric and magnetic form factors ^{*)} $G_E(q^2)$ and $G_M(q^2)$,

$$\left. \begin{aligned} A &= (G_E^2 + \frac{q^2}{4M^2} G_M^2) / (1 + \frac{q^2}{4M^2}) \\ B &= G_M^2 \end{aligned} \right\} \quad (2)$$

We do not explicitly consider polarized particles, except to note that an upper bound on the radiation rate for arbitrary polarization is given by $(2j+1)$ times the unpolarized rate.

^{*)} The notation is that of, for example, Källén ¹⁾.

For Cerenkov radiation the mean energy radiated per unit track length per unit photon energy is

$$\frac{d^2 \mathcal{E}}{d\ell d\omega} = \mu \alpha \omega \left(1 - \frac{\omega}{p_0}\right) \left\{ A(q^2) \sin^2 \theta + B(q^2) q^2 / (2\beta^2 p_0^2) \right\} \quad (3)$$

where

$$\begin{aligned} \hbar &= c = 1 & \alpha &\approx 1/137 \\ \ell &= \text{track length} \\ \omega &= \text{photon energy} \\ \mu &= \text{magnetic permeability at frequency } \omega \\ \beta &= \text{particle velocity} \\ p_0 &= \text{particle energy} \\ \cos \theta &= (n\beta)^{-1} \left(1 + \frac{1}{2}(n^2 - 1)\omega p_0\right) \\ n &= \text{refractive index at frequency } \omega \\ q^2 &= (n^2 - 1)\omega^2 \end{aligned}$$

Absorption has been ignored, i.e., μ and n are taken real. This result can be compared with various special cases collected by Zrelov²⁾. The agreement is in general adequate, taking into account approximations, $\omega/p_0 \ll 1$ and/or $m/p_0 \ll 1$, used by various authors, and the need sometimes to identify what they call magnetic moment with the "anomalous" part of the magnetic moment. We do not, however, obtain the factor (phase velocity/group velocity) included by Tidman³⁾. But this factor in Tidman's result should not be regarded as significantly different from unity, because his basic model is non-dispersive.

For transition radiation the mean energy radiated at a vacuum-dielectric interface, per unit photon energy, per unit solid angle Ω is

$$\frac{d^2 \mathcal{E}}{d\Omega d\omega} = \frac{\alpha}{\pi^2} \left(\frac{\omega_p}{\omega}\right)^4 \frac{\theta^2 A(q^2) + q^2 B(q^2) / (2 p_0^2)}{\left(\theta^2 + \frac{M^2}{p_0^2}\right)^2 \left(\theta^2 + \frac{M^2}{p_0^2} + \frac{\omega_p^2}{\omega^2}\right)^2} \quad (4)$$

where

$$\begin{aligned} \hbar &= c = 1 & \alpha &\approx 1/137 \\ \omega &= \text{photon energy} \\ \omega_p^2 &= (1-n^2)\omega^2 \\ n &= \text{refractive index at frequency } \omega \\ \theta &= \text{angle between photon and particle} \\ q^2 &= ((m^2/p_0^2) + \theta^2)\omega^2 \\ M &= \text{particle mass} \\ p_0 &= \text{particle energy} \\ \mu &= \text{magnetic permeability} = 1 \end{aligned}$$

and with the approximations, made in A and B contributions separately,

$$\begin{aligned} p_0 &\gg M \\ p_z &\gg k_z \gg \omega_p \\ p_z/p_0 &\gg \theta \gg \frac{p_0 \omega}{p_z k_z} \left(\frac{M^2}{2p_0^2} + \frac{\theta^2}{2} \right) \end{aligned}$$

where k_z and p_z are photon and particle momenta normal to the surface ^{*)}. It is only in these conditions, which exclude the case of either particle or photon moving almost parallel to the surface, that the direction of the surface appears. Very small angles θ are also excluded, but only over a region which contributes negligibly to the integrated effect if

$$1 \gg \frac{p_0 \omega}{p_z k_z} \frac{M}{2p_0}$$

^{*)} In this paper p_z and k_z are everywhere taken positive.

DERIVATION

Consider a slab of material with faces at $z=0$ and $z=L$. Let a particle incident with four-momentum p be scattered to four-momentum p' with emission of a photon of four-momentum k . To first order in the electromagnetic current j of the particle the quantum transition matrix element is

$$\mathcal{M} = \int d^4x A_\mu^*(x) (p's' | j_\mu(x) | ps) \quad (5)$$

where

s, s' label spin states

$$q = p - p'$$

$$A_\mu = a_\mu e^{ik \cdot x} + \text{scattered wave}$$

$$k \cdot x = \vec{k} \cdot \vec{x} - k_0 t, \quad k_0 = \sqrt{\vec{k}^2}$$

$$a_0 = 0, \quad |\vec{a}| = 1, \quad \vec{a} \cdot \vec{k} = 0$$

and the scattered wave, generated by the refracting power of the material, must satisfy incoming wave boundary conditions, because it represents here a final state.

Instead of the potential A_μ there is some advantage in using the gauge invariant electric field

$$\vec{E} = - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi$$

Then

$$-iq_0 \mathcal{M} = \int d^4x (p's' | \vec{E}^*(x) \cdot \vec{j}(x) | ps) \quad (6)$$

for this can be reduced to (5) by some partial integrations, invoking the continuity equation

$$0 = \partial_\mu j_\mu = \vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t}$$

and

$$(p's' | j_\mu(x) | ps) = e^{iq_\mu x_\mu} (p's' | j_\mu(0) | ps) \quad (7)$$

In calculating $\vec{E}$ we allow for a dielectric constant $\epsilon \neq 1$, and the associated refractive index $n = \sqrt{\epsilon}$, but ignore reflection. Then for k_z positive

$$\left. \begin{aligned} \vec{E} &= ik_0 \vec{a} e^{ik_\mu x_\mu}, & z > L \\ &= \sqrt{\frac{k_z}{k'_z}} ik_0 \vec{a}' e^{ik_\mu x_\mu} e^{i(k'_z - k_z)(z-L)}, & L > z > 0 \\ &= ik_0 \vec{a} e^{ik_\mu x_\mu} e^{i(k_z - k'_z)L}, & 0 > z \end{aligned} \right\} \quad (8)$$

where

$$\left. \begin{aligned} |\vec{a}'| &= 1 & \vec{a}' \cdot \vec{k}' &= 0 \\ \vec{k}' &= (k'_x, k'_y, k'_z) = (k_x, k_y, k'_z) \\ k'_z &= \sqrt{n^2 k_0^2 - k_x^2 - k_y^2} \end{aligned} \right\} \quad (9)$$

and $\vec{a}'$ is obtained from $\vec{a}$ by the same rotation about $\vec{k} \times \vec{k}'$ which takes $\vec{k}$ into $\vec{k}'$. The factor $\sqrt{k_z/k'_z}$ in the amplitude inside the slab is inferred from the balance of power, or by the appropriate form of Poynting's theorem

$$\text{Real } \vec{\nabla} \cdot (\vec{E}^* \times \vec{H}) = 0$$

In the case of a magnetic permeability $\mu \neq 1$, this gives an extra factor $\sqrt{\mu}$ in the amplitude of $\vec{E}$ inside the slab; this is where μ in (3) comes from.

Substituting (8) and (7) in (6) and integrating

$$M = (2\pi)^3 \delta(k_x - q_x) \delta(k_y - q_y) \delta(k_0 - q_0) \quad (10)$$

where

$$\eta = \left(e^{i(k'_z - k_z)L} - e^{i(q_z - k_z)L} \right) \times \left(\frac{(p's' | \vec{a} \cdot \vec{j}(0) | ps)}{i(q_z - k_z)} - \sqrt{\frac{k_z}{k'_z}} \frac{(p's' | \vec{a}' \cdot \vec{j}(0) | ps)}{i(q_z - k'_z)} \right) \quad (11)$$

The total transition probability is then

$$\int \frac{d^3 \vec{k}}{(2\pi)^3 2k_0} \frac{d^3 \vec{k}'}{(2\pi)^3 2k'_0} |\eta|^2$$

When the $(2\pi)^3 \delta$ function factor in (10) is squared, one of the two factors can be interpreted as usual as the product of the infinite dimensions of the interaction region ; omitting it gives the transition probability per unit surface area of slab, per unit time. We use here the covariant normalization convention

$$(p's' | ps) = 2p_0 (2\pi)^3 \delta^3(\vec{p}' - \vec{p}) \delta_{s's}$$

This corresponds to $2p_z$ particles incident per unit area per unit time. Then the transition probability per particle is

$$\frac{1}{2p_z} \int \frac{d^3 \vec{k}}{(2\pi)^3 2k_0} \int \frac{d^3 \vec{k}'}{(2\pi)^3 2k'_0} |\eta|^2 (2\pi)^3 \delta(k_x - q_x) \delta(k_y - q_y) \delta(k_0 - q_0) \quad (12)$$

CERENKOV RADIATION

Consider first the case of Cerenkov radiation. This is characterized by the occurrence of the point $q_z = k'_z$ in the integration (12). Because for large L

$$\left| \frac{e^{i(k'_z - k_z)L} - e^{i(q_z - k_z)L}}{i(q_z - k'_z)} \right|^2 \rightarrow L 2\pi \delta(k'_z - q_z)$$

the transition probability then contains a term proportional to L . Retaining only this term, the transition probability per particle per unit track length $(= (|\vec{p}|/p_z)L)$ is

$$\begin{aligned} & \frac{1}{2|\vec{p}|} \int \frac{d^3 \vec{k}}{(2\pi)^3 2k_0} \int \frac{d^3 \vec{p}'}{(2\pi)^3 2p'_0} \frac{k_z}{k'_z} (2\pi)^4 \delta^4(k'_\mu - q_\mu) |(p's' | \vec{a}' \cdot \vec{j} | ps)|^2 \\ &= \int \frac{d^3 \vec{k}}{(2\pi)^3 2k_0} \frac{k_z}{k'_z} 2\pi \delta(k_0 - q_0) \frac{|(p's' | \vec{a}' \cdot \vec{j} | ps)|^2}{4|\vec{p}| p_0} \end{aligned} \quad (13)$$

with

$$\vec{p}' = \vec{p} + \vec{k}' \quad q_0 = p_0 - p'_0$$

Let us replace k_z as integration variable by k_0 :

$$\frac{dk_z}{2k_0} = \frac{dk_0}{2k'_z}$$

Then note that

$$\frac{dk_0}{2k'_z} = dk_0 dk'_z \delta(n^2 k_0'^2 - \vec{k}'^2)$$

from (5), with $k'_x = k_x$, $k'_y = k_y$, $k'_0 = k_0$. So (13) can be rewritten

$$\int \frac{d^4 k'}{(2\pi)^4} 2\pi \delta(n^2 k_0'^2 - \vec{k}'^2) 2\pi \delta(k_0 + p_0 - p'_0) \frac{|(p's' | \vec{a}' \cdot \vec{j} | ps)|^2}{4|\vec{p}| p_0} \quad (14)$$

Explicit reference to the faces of the material, making z a preferred direction, has finally disappeared here. We could have come more quickly to such an expression by considering from the start an unbounded medium.

Let (θ, ϕ) be polar co-ordinates giving the direction of $\vec{k}'$ with respect to $\vec{p}$, so that

$$d^3 \vec{k}' = d\phi d\cos\theta |\vec{k}'|^2 d|\vec{k}'|$$

Then integration over $|\vec{k}'|$ removes the first δ function in (14) and specifies elsewhere

$$|\vec{k}'| = nk'_0 \quad (15)$$

Integration over $\cos\theta$ then removes the second δ function, introducing a factor

$$\left| \frac{dp'_0}{d\cos\theta} \right|^{-1} = \frac{p'_0}{|\vec{p}| |\vec{k}'|} \quad (16)$$

from

$$p'_0 = \sqrt{M^2 + \vec{p}^2 + \vec{k}'^2 - 2\vec{p} \cdot \vec{k}'} \quad (17)$$

Then (14) becomes, for the transition probability per unit track length

$$\iint \frac{d\phi dk'_0}{8\pi^2} \left(\frac{p'_0}{p_0} \right) \left| (p's' | \frac{\vec{a}' \cdot \vec{j}}{2|\vec{p}|} | ps) \right|^2 \quad (18)$$

with

$$p'_0 = p_0 - k'_0 \quad \vec{p}' = \vec{p} - \vec{k}$$

and from (15), (17),

$$\cos\theta = \frac{1}{n\beta} \left(1 + \frac{n^2-1}{2} \frac{k'_0}{p_0} \right) \quad (19)$$

Averaging over initial spins s and summing over final spins s' we need in (18) the space part of the tensor

$$T_{\mu\nu} = \sum_s \sum_{s'} (p^s | j_\mu | p'^{s'}) (p'^{s'} | j_\nu | p^s) \quad (20)$$

for which Lorentz invariance and current continuity dictate the form

$$T_{\mu\nu} = e^2 (A(q^2) (p+p')_\mu (p+p')_\nu + B(q^2) (q^2 \delta_{\mu\nu} - q_\mu q_\nu)) \quad (21)$$

(plus an antisymmetric term, if parity is violated, that we do not need if we do not consider circular photon polarization) with

$$q = p - p'$$

In (18), $q_\mu = k'_\mu$, and $q_\mu a'_\mu = 0$. So we have only to calculate

$$e^2 (4 A(q^2) \vec{a} \cdot \vec{p} \vec{p} \cdot \vec{a}' + q^2 B(q^2) \vec{a}' \cdot \vec{a})$$

On summing over the two photon polarization states, both perpendicular to $\vec{k}'$, respectively in and out of the $(\vec{k}', \vec{p})$ plane, this becomes

$$e^2 (4 |\vec{p}|^2 A(q^2) \sin^2 \theta + 2 q^2 B(q^2))$$

whence (3), with $(e^2/4\pi) = \alpha$, putting $k'_0 = \omega$, and including a factor ω to convert transition probability to mean energy.

TRANSITION RADIATION

When the point $q_z = k'_z$ does not occur in the integration, the second factor of (11) has no longer a singularity giving rise to an L proportional contribution. The radiation is then a surface effect. If L is sufficiently large that

$$(q_z - k'_z) L \gg 1$$

we can ignore a rapidly oscillating interference term between the two exponentials in (11), taking

$$|N|^2 = 2 \left| \frac{(p's' | \vec{a} \cdot \vec{j} | ps)}{q_z - k_z} - \sqrt{\frac{k_z}{k'_z}} \frac{(p's' | \vec{a}' \cdot \vec{j} | ps)}{q_z - k'_z} \right|^2$$

The transition probability per face is then

$$\int \frac{d^3 \vec{k}}{(2\pi)^3 2k_0} \left(\frac{k_0}{q_z - k_z} \right)^2 \left(\frac{k_0}{q_z - k'_z} \right)^2 \left(\frac{k_z - k'_z}{k_0} \right)^2 |F|^2 \quad (22)$$

where

$$2\sqrt{p'_z p_z} F = (p's' | \vec{a} \cdot \vec{j} | ps) + \frac{q_z - k_z}{k_z - k'_z} (p's' | \vec{a} \cdot \vec{j} - \vec{a}' \cdot \vec{j} \sqrt{k_z/k'_z} | ps) \quad (23)$$

In performing the $\vec{p}'$ integration in (12) we have used

$$dp'_z / p'_0 = dp'_0 / p'_z$$

In (23), because of the δ functions in (12),

$$p' = p + q$$

with

$$q_0 = k_0 = k'_0 \quad q_x = k_x = k'_x \quad q_y = k_y = k'_y$$

and

$$\begin{aligned} q_z &= p_z - p'_z \\ &= p_z - \sqrt{((p_0 - k_0)^2 - (p_x - k_x)^2 - (p_y - k_y)^2 - M^2)} \\ &= p_z - \sqrt{(p_z - k_z)^2 + 2\vec{p} \cdot \vec{k} - 2p_0 k_0} \end{aligned} \quad (24)$$

Following Durand ^{5),*)} we now simplify the working by going at once to the case of high energy. From (9), with

$$n^2 = 1 - \omega_p^2 / k_0^2$$

we have

$$k_z' = \sqrt{k_z^2 - \omega_p^2}$$

or

$$\frac{k_z - k_z'}{k_0} \approx \frac{\omega_p^2}{2k_z k_0} \quad (25)$$

if

$$k_z \gg \omega_p \quad (26)$$

From (24)

$$\frac{q_z - k_z}{R_0} \approx \frac{p_0}{p_z} \left(\frac{M^2}{2p_0^2} + \frac{\theta^2}{2} \right) \quad (27)$$

if

$$(p_0/M) \gg 1 \quad (28)$$

$$\theta \ll 1 \quad (29)$$

$$k_z \ll p_z \quad (30)$$

and

$$\frac{k_0 p_0}{p_z^2} \left(\frac{M^2}{2p_0^2} + \frac{\theta^2}{2} \right) \ll 1 \quad (31)$$

*) This paper and that of Artru et al. ⁶⁾ list the standard references on transition radiation.

From (27) and (25)

$$\frac{q_z - k_z}{k_0} \approx \frac{p_0}{p_z} \left(\frac{M^2}{2p_0^2} + \frac{\theta^2}{2} + \frac{\omega_p^2}{2k_0^2} \frac{k_0 p_z}{k_z p_0} \right) \quad (32)$$

The factors multiplying $|F|^2$ in (22) peak strongly for small θ ; see (27) and (32). Over this small region the second term in (23) is negligible compared with the first. The leading contribution to $\vec{j}$ is

$$(p's' | \vec{j} | ps) \approx \delta_{s's} 2\vec{p}$$

whence the first contribution to F in (23) is (for the appropriate photon polarization)

$$\sim \theta$$

The second contribution contains instead of θ a factor

$$\sim \frac{q_z - k_z}{k_z} \approx \frac{p_0 k_0}{p_z k_z} \left(\frac{M^2}{2p_0^2} + \frac{\theta^2}{2} \right) \quad (33)$$

from (27). This can exceed the first term if p_z , or k_z , or θ , are sufficiently small. But excluding the cases in which either particle or photon go almost parallel to the surface, and for typical θ ($\gtrsim M/p_0$) we have

$$\frac{p_0 k_0}{p_z k_z} \left(\frac{M^2}{2p_0^2} + \frac{\theta^2}{2} \right) \ll \theta \quad (34)$$

so that the second term in (23) is negligible. Dropping this term, using (27) and (32) in (22), and multiplying by k_0 , the mean energy radiated per face, per unit solid angle, per unit photon energy, is

$$\frac{d^2 \mathcal{E}}{d\Omega dk_0} = \frac{1}{4\pi^3} \left(\frac{\omega_p}{k_0} \right)^4 \frac{|(p's' | \vec{a} \cdot \vec{j} / (2p_0) | ps)|^2}{(\theta^2 + M^2/p_0^2)(\theta^2 + M^2/p_0^2 + \omega_p^2/k_0^2)^2} \quad (35)$$

using at some places the small θ approximation

$$\theta \ll \frac{p_z}{p_0}, \quad \frac{k_z}{k_0} \approx \frac{p_z}{p_0} \quad (36)$$

With unpolarized particles we are again concerned with the tensor (20), (21). The A contribution involves the squared modulus of

$$\begin{aligned} & \vec{a} \cdot (\vec{p} + \vec{p}') / (2p_0) \\ &= \vec{a} \cdot (\vec{p}/p_0 + \vec{q}/(2p_0)) \\ &= \vec{a} \cdot \left(\frac{\vec{p}}{p_0} \right) + \left(\frac{M^2}{2p_0^2} + \frac{\theta^2}{2} \right) \frac{k_0}{2p_z} a_z \end{aligned} \quad (37)$$

where we have used $\vec{a} \cdot \vec{k} = 0$, and that $\vec{q}$ differs from $\vec{k}$ only in the z component, as given by (27). The second term is quite negligible in view of (34). We are left with, summing over the two-photon polarizations,

$$\sum |\vec{a} \cdot \vec{p}/p_0|^2 = |\vec{p}/p_0|^2 \sin^2 \theta \approx \theta^2 \quad (38)$$

The B contribution involves

$$\begin{aligned} & q^2 |\vec{a}|^2 - |\vec{q} \cdot \vec{a}|^2 \\ &= q^2 |\vec{a}|^2 - (q_z - k_z)^2 a_z^2 \\ &= q^2 (|\vec{a}|^2 - \frac{q_z - k_z}{q_z + k_z} a_z^2) \end{aligned} \quad (39)$$

using

$$\begin{aligned} q^2 &= q_z^2 + q_x^2 + q_y^2 - q_0^2 \\ &= q_z^2 - k_z^2 \end{aligned}$$

Now in view of (33) and (34)

$$q_z - k_z \ll k_z$$

so the second term in (39) is negligible, and

$$\begin{aligned} q^2 &\approx 2 k_z^2 (q_z - k_z) / k_z \\ &\approx \frac{p_0}{p_z} k_z k_0 \left(\frac{M^2}{p_0^2} + \theta^2 \right) \\ &\approx k_0^2 \left(\frac{M^2}{p_0^2} + \theta^2 \right) \end{aligned} \quad (40)$$

Finally

$$q^2 \sum |\vec{a}|^2 / (4 p_0^2) = q^2 / (2 p_0^2) \quad (41)$$

Substituting (41) and (38) in (35), and with $e^2/4\pi = \alpha$, we have the result (4).

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• 1. The first part of the text
 