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COLOUR SINGLET SOLITONS IN $SU(3)$ GAUGE THEORY
OF VECTOR AND SPINOR FIELDS

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A B S T R A C T

In this paper we set up a semi-classical framework for the description of hadrons in the context of a quark model based on a "colour" gauge theory of the strong interaction. The proposed description is in terms of wave functions of the hadron's constituents which are identified with solutions of non-linear soliton-type equations.

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INTRODUCTION

One of us ¹⁾ recently proposed a new description of hadrons as soliton solutions to gauge theories of vector (gluon) and spinor (quark) fields. The approach does away with the presence of fictitious self-interacting scalar fields or peculiar boundary conditions adopted in prototype "bag" models ²⁾, nominating a simpler and less arbitrary candidate for a fundamental description of hadrons.

In this paper we will further discuss this approach and extend it to a non-Abelian "colour" $SU(3)$ gauge theory. We investigate in particular the binding mechanism for colour singlet states and find solutions corresponding to mesons ($q\bar{q}$) and baryons (qqq). We also comment on the saturation of forces in colour singlet states and the absence of "exotics" such as ($q^2\bar{q}$).

In our picture of hadrons, the binding forces are simply the non-Abelian analogue of the electrostatic interaction, which leads to the formation of positronium in electrodynamics. The non-Abelian analogue of magnetic interactions remove spin degeneracies, splitting the π from the ρ , the nucleon from the Δ resonances, etc. We propose to implement these ideas, which are not new ³⁾, with the help of non-perturbative, semi-classical methods that we have developed, generalizing techniques used in popular soliton models. The end product will be a description of hadrons as extended systems in a canonical gauge theory.

Our approach to quark binding does not solve the problem of apparent quark and gluon "confinement". If this problem persists, we reserve the right of joining the party line which argues that the effective coupling constant in non-Abelian gauge theories grows at large distances ("infra-red slavery"). If such a phenomenon turns on at distances larger than 1 Fermi, it will not interfere with our binding mechanism in the interesting case of low-lying hadrons and may be ignored for our purposes.

The basis of our approach is the conventional quark model which depicts hadrons as composed of quarks carrying any of an ever increasing number of "flavours" and one of three "colours". Colour symmetry $[SU(3)]$ is exact and its gauging is the origin of the strong interactions. The global flavour symmetry is broken by differences in quark masses. These masses and the colour coupling constant will be the only free parameters of our model.

While we hope to produce in the near future a complete calculation of the hadronic spectrum in our model, the content of this paper is limited to a discussion of our methods with emphasis on the ideas, some technical points and the existence of the relevant soliton solutions. To this end, we find it convenient to begin the discussion with the weak coupling limit, as the ("colour") coupling constant $g \rightarrow 0$. In this limit the bound state dynamics is, in good approximation, non-relativistic and a quasi-static picture of the bound state emerges. We are then able to unambiguously derive soliton equations as the Euler equations of a quantum variational problem in Fock space. This derivation is reminiscent of the variational approach to field theoretical bag models ²⁾. There is, however, an important improvement in the structure of our trial states : instead of using a mean field approximation, we correlate the gluon field configuration and the position of the fermion. While such an improvement was not necessary to obtain the binding of quark constituents in a scalar bag ⁴⁾, it is in fact crucial in the present case because of the lack of a binding mean gluon field. The added sophistication also exposes the need for carrying out an infinite renormalization of the quark mass. Whereas previous treatments, in particular that of quarks bound by a scalar glue, ignored this need ⁴⁾, it is important for a correct description of the binding.

The soliton equations obtained by this procedure may also be derived by different methods. We consider in particular a scheme, that we refer to as "constituent method", where the operator field equations are saturated with a restricted set of states. In the non-relativistic limit this reproduces the same variational equations. More generally it enables us to treat the quarks relativistically and to take their spin interactions into account. Thus we arrive at different equations for bound states with different spins.

We then proceed to apply the constituent method to a derivation of soliton equations for $SU(3)$ ("colour") singlet bound states, in particular for spin 0 and 1 "mesons" and spin $\frac{3}{2}$ "baryons".

1. - MEAN FIELD APPROXIMATION

Let us consider the Abelian $U(1)$ gauge theory of a massive quark field, $\psi(x)$, and of a massless gluon field, $B_\mu(x)$, with Hamiltonian density

$$H(x) = \psi^\dagger \frac{\alpha \cdot \nabla}{i} \psi + m_0 \bar{\psi} \psi + \frac{1}{2} (G_{ij}^2 - G_{0i}^2) + g_0 \bar{\psi} \gamma^\mu \psi B_\mu \quad (1)$$

where

$$G_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu. \quad (2)$$

The unrenormalized field equations are :

$$(i \not{D} - m_0) \psi(x) = \chi(x) = g_0 \not{B} \psi(x), \quad (3)$$

$$\partial_\nu G^{\mu\nu} = j^\mu(x) = g_0 \bar{\psi} \gamma^\mu \psi(x). \quad (4)$$

In boson field theories soliton searches revolve around the field equations ⁵⁾. Unfortunately, because of the presence of a fermion field (whose classical counterpart is not a commuting c number), Eqs. (3)-(4) cannot be directly used to expose soliton structures. A way around the difficulty is provided by quantum variational methods.

It is well known that boson field equations coincide with the Euler equations of the quantum variational problem defined by the minimization of the expectation value of the (normal ordered) Hamiltonian within coherent states of the boson fields ⁶⁾. Semi-classical equations in theories with fermion fields may be obtained along similar lines, by a suitable definition of a quantum variational problem. Let us recall the simple procedure used for bag models ²⁾.

The field operator $\psi(x)$ is expanded in terms of creation and annihilation operators for quarks $(b^\dagger, b)$ and anti-quarks $(d^\dagger, d)$

$$\psi(x) = \sum_n (b_n u_n(x) + d_n^\dagger v_n(x)) \quad (5)$$

in a general orthonormal basis of spinor functions

$$(u_i, u_j) = (v_i, v_j) = \delta_{ij}, \quad (u_i, v_j) = 0, \quad (6)$$

$$(u_i, u_j) = \int d^3x u_i^\dagger(x) u_j(x), \quad \dots \quad (7)$$

The equal time canonical anticommutation relations of the field imply

$$\{b_m, b_m^\dagger\} = \{d_m, d_m^\dagger\} = \delta_{mm}, \quad 0 = \{b_m, d_m\} = \dots \quad (8)$$

and a Fock space is constructed in the usual way. It should be noted that, given the arbitrariness of the spinor basis, the no-particle state $|0\rangle$, defined by

$$b_m |0\rangle = d_m |0\rangle = 0 \quad (9)$$

and the ensuing Fock space construction will in general not be Poincaré invariant. A Fock space for the boson field $\phi(x)$ (a spinless field in the case of bag models) is also constructed in terms of an ordinary plane wave basis. The state $|0\rangle$ will henceforth denote the no-particle state of the product of the fermion and boson Fock spaces.

Trial states are then constructed by applying to $|0\rangle$ the product of a boson coherent operator with creation operators for quarks and antiquarks :

$$|s\rangle = e^{-i\pi(f)} b_i^\dagger \dots d_j^\dagger |0\rangle, \quad (10)$$

where

$$\pi(f) = \int d^3x f(x) \pi(x), \quad \pi(x) = \dot{\phi}(x). \quad (11)$$

The expectation value of the Hamiltonian (normal ordered with respect to $|0\rangle$) within the trial state $|s\rangle$ is easily evaluated. Its minimization with respect to the trial functions $f(x)$, $u_i(x)$, $v_j(x)$ [subject to the

normalization constraints (6) leads to static semi-classical field equations. Soliton solutions to these equations are taken to approximately describe bag states ⁴⁾.

It will be important to clarify the nature of this approximation. As implied by the factorized character of the trial state (10), it is a self-consistent mean field approximation. The constituent's (quark and antiquark) wave functions are made to interact with an average gluon field built from all of the constituents. Physical correlations between the quantum fluctuations of the constituents and the quantum fluctuations of the glue are not taken into account. Mass renormalizations mirroring the partially unphysical character of the self-interaction of constituents with their own gluon field are also not taken into account.

2. - CORRELATIONS

These simplifications do not prevent the manifestation of bound states in bag models. In fact the mean scalar gluon field of a bag is always non-vanishing, because the contributions of quarks and antiquarks as sources for this field are equal in sign. However, this is obviously not the case for gauge theories. The correlations between the fluctuations of the constituents and of the gluon fields then become essential to the binding of charge (colour) singlet states. In a quantum variational approach they should be taken into account by an ansatz for the trial states which is more sophisticated than Eq. (10). In the weak coupling limit, what is needed is a trial state describing a quantum mechanical superposition of states of point-like constituents with their associated Coulomb fields.

For the purpose of handling some technical points, let us first consider such a trial state for a one-quark system. We shall work in a translational invariant Fock space associated with an ordinary plane wave expansion of the quark field, Eq. (5). We shall find it convenient to introduce operators ^{*)}

$$b^+(\underline{x}) = \sum_n b_n^+ u_n^+(\underline{x}), \quad d^+(\underline{x}) = \sum_n d_n^+ v_n(\underline{x}), \quad (12)$$

*) The time label will be dropped, as we work at a fixed time.

for creating a quark or an anti-quark at the point x . Let us then consider the following trial state

$$|q\rangle = \int d\underline{x} \ u(\underline{x}) \ b^\dagger(\underline{x}) \ C(\underline{x}) |0\rangle, \quad (13)$$

where $u(x)$ is a positive energy spinor function and $C(x)$ is a coherent (unitary) operator creating the Coulomb field generated by a point-like quark located at x :

$$\langle 0 | C^\dagger(\underline{y}) \ B_\mu(\underline{x}) \ C(\underline{y}) | 0 \rangle = \delta_{\mu 0} \frac{g_0}{4\pi} \frac{1}{|\underline{x} - \underline{y}|} \quad (14)$$

The canonical equal time (anti-) commutation relations of the fields imply :

$$\langle q | q \rangle = \int d\underline{x} \ u^\dagger u(\underline{x}) + O(p/m), \quad (15)$$

$$\langle q | : \bar{\psi} \Gamma \psi : | q \rangle = \bar{u} \Gamma u(\underline{x}) + O(p/m), \quad (16)$$

$$\langle q | B_\mu(\underline{x}) | q \rangle = \delta_{\mu 0} \frac{g_0}{4\pi} \int d\underline{y} \ \frac{u^\dagger u(\underline{y})}{|\underline{x} - \underline{y}|} + O(p/m), \quad (17)$$

where p is a typical momentum in the Fourier analysis of $u(x)$. In the weak coupling (non-relativistic) limit, in which we shall be working, terms of $O(p/m)$ will be neglected. As evidenced by Eqs. (16)-(17), the state $|q\rangle$ then describes a bare, point-like quark with wave function $u(x)$ and its correlated Coulomb-like gluon field.

3. - QUARK MASS RENORMALIZATION

Since the state (13), in contrast to uncorrelated trial states such as (10), accounts for the local self-interaction of such a system, it is obvious that the divergence structure of the field theory will now partially come into play. The evaluation of the operator product $: \bar{\psi} \gamma_\mu \psi(x) : B^\mu(x+\xi)$ within $|q\rangle$ leads to :

$$\langle q | : \bar{\psi} \gamma_\mu \psi(\underline{x}) : B^\mu(\underline{x} + \underline{\xi}) | q \rangle = u^\dagger u(\underline{x}) \frac{g_0}{4\pi} \frac{1}{|\underline{\xi}|}, \quad (18)$$

which is singular in the limit $\underline{\xi} \rightarrow 0$. Thus a renormalization of the interaction term $:\bar{\psi}\gamma_{\mu}\psi(x):B^{\mu}(x)$ in the Hamiltonian density, Eq. (1), must be carried out to remove that singular part of the operator product which has been exposed by our improved trial state.

It is apparent that only a quark mass renormalization is required to subtract off the infinite ("electrostatic") self-mass of the quark ^{*)}. Technically, we regulate the interaction term with the help of a cut-off function $c(\underline{\xi})$,

$$g_0 \bar{\psi}\gamma_{\mu}\psi(x)B^{\mu}(x) \longrightarrow g_0 \bar{\psi}\gamma_{\mu}\psi(x) \int d\underline{\xi} B^{\mu}(x+\underline{\xi})c(\underline{\xi}), \quad (19)$$

while we define a physical quark mass, m , in terms of the bare mass, m_0 , and the cut-off function, according to :

$$m = m_0 + \frac{1}{2} \frac{g_0^2}{4\pi} \int d\underline{\xi} c(\underline{\xi}) \frac{1}{|\underline{\xi}|}. \quad (20)$$

As we remove the cut-off [i.e., $c(\underline{\xi})$ approaches a three-dimensional δ function], we let the bare mass m_0 vary to keep the physical mass, m , fixed and finite. In the limit, the expectation value of the renormalized Hamiltonian, H_R , within $|q\rangle$, reduces to :

$$\langle q | H_R | q \rangle \sim \int d\underline{x} [u^{\dagger} \underline{\alpha} \cdot \underline{\nabla} u(\underline{x}) + m \bar{u} u(\underline{x})], \quad (21)$$

i.e., the one-particle Hamiltonian of a free quark of mass m . This conclusion is intuitively satisfactory since $|q\rangle$ was constructed to represent such a state of one physical ("dressed") quark.

Needless to say, the construction analogous to Eq. (13) for the state of an anti-quark ($\bar{q}$) is :

$$|\bar{q}\rangle = \int d\underline{x} v(\underline{x}) d^{\dagger}(\underline{x}) C^{\dagger}(\underline{x}) |0\rangle, \quad (22)$$

^{*)} Our trial states for the weak coupling (non-relativistic) limit do not expose infinities requiring wave function or coupling constant renormalization.

where $v(x)$ is a negative energy spinor function. The equations corresponding to Eqs. (14)-(17) are :

$$\langle 0 | C(\underline{y}) B_\mu(\underline{x}) C^\dagger(\underline{y}) | 0 \rangle = -\delta_{\mu 0} \frac{g_0}{4\pi} \frac{1}{|\underline{x}-\underline{y}|}, \quad (23)$$

$$\langle \bar{q} | \bar{q} \rangle = \int d\underline{x} v^\dagger v(\underline{x}) + O(p/m), \quad (24)$$

$$\langle \bar{q} | : \bar{\psi} \Gamma \psi(\underline{x}) : | \bar{q} \rangle = -\bar{v} \Gamma v(\underline{x}) + O(p/m), \quad (25)$$

$$\langle \bar{q} | B_\mu(\underline{x}) | \bar{q} \rangle = -\delta_{\mu 0} \frac{g_0}{4\pi} \int d\underline{y} \frac{v^\dagger v(\underline{y})}{|\underline{x}-\underline{y}|} + O(p/m). \quad (26)$$

Notice some crucial changes of sign which are automatically brought about by the (normal) ordering of the operators. After renormalization, the expectation value of the Hamiltonian within $|\bar{q}\rangle$ reduces to :

$$\langle \bar{q} | H | \bar{q} \rangle \sim -\int d\underline{x} \left[v^\dagger \frac{\alpha \cdot \nabla}{i} v(\underline{x}) + m \bar{v} v(\underline{x}) \right], \quad (27)$$

i.e., a (positive-definite) one-particle Hamiltonian for a free antiquark of mass m .

4. - SOLITON EQUATIONS

Having learned how to handle technical points (such as the renormalization of the Hamiltonian in our variational approximation) in the one-fermion sectors, we are now ready to attack the quark-antiquark bound state problem. We construct trial states for a $q\bar{q}$ system by taking direct products of a q state (13) with a $\bar{q}$ state (22) :

$$|q\bar{q}\rangle = \int d\underline{x} u(\underline{x}) b^\dagger(\underline{x}) C(\underline{x}) \int d\underline{y} v(\underline{y}) d^\dagger(\underline{y}) C^\dagger(\underline{y}) | 0 \rangle. \quad (28)$$

The expectation value of the renormalized Hamiltonian, ~~H~~ , within $|q\bar{q}\rangle$ is :

$$\langle \bar{q} | q | \bar{q} \rangle \sim \int d\underline{x} \left[u^\dagger \frac{\underline{\alpha} \cdot \underline{\nabla}}{i} u + m \bar{u} u - v^\dagger \frac{\underline{\alpha} \cdot \underline{\nabla}}{i} v - m \bar{v} v \right] \\ - \frac{g_0^2}{4\pi} \int d\underline{x} \int d\underline{y} \frac{u^\dagger u(\underline{x}) v^\dagger v(\underline{y})}{|\underline{x} - \underline{y}|} . \quad (29)$$

As in the one-fermion sectors, the static self-interaction of each constituent has been entirely reabsorbed into a mass renormalization. The cross term is finite (unaffected by the renormalization procedure) and describes the static interaction of the quark with the antiquark. The Euler equations of the variational problem are :

$$(i \underline{\gamma} \cdot \underline{\partial} - m - \gamma_0 E_u) u(\underline{x}) = - \frac{g_0^2}{4\pi} \int d\underline{y} \frac{v^\dagger v(\underline{y})}{|\underline{x} - \underline{y}|} \gamma_0 u(\underline{x}) , \quad (30)$$

$$(i \underline{\gamma} \cdot \underline{\partial} + m - \gamma_0 E_v) v(\underline{x}) = \frac{g_0^2}{4\pi} \int d\underline{y} \frac{u^\dagger u(\underline{y})}{|\underline{x} - \underline{y}|} \gamma_0 v(\underline{x}) , \quad (31)$$

where the spinor energy eigenvalues E_u, E_v have been introduced as Lagrange multipliers to enforce the normalization constraints :

$$\langle q | q \rangle = \int d\underline{x} u^\dagger u = \langle \bar{q} | \bar{q} \rangle = \int d\underline{x} v^\dagger v = 1 . \quad (32)$$

The requirement that $|q\bar{q}\rangle$ be an eigenstate of charge conjugation, $\mathcal{C}$,

$$\mathcal{C} |q\bar{q}\rangle = \pm |q\bar{q}\rangle , \quad \mathcal{C} \psi_\alpha(x) \mathcal{C}^{-1} = C_{\alpha\beta} \bar{\psi}_\beta(x) , \quad C = i\gamma^2 \gamma^0 , \quad (33)$$

implies

$$u(\underline{x}) = \pm i\gamma^2 v(\underline{x}) , \quad -E_v = +E_u = E \quad (34)$$

and reduces the system of Eqs. (30)-(31) to :

$$(i \underline{\gamma} \cdot \underline{\partial} - m - \gamma_0 E) u(\underline{x}) = - \frac{g_0^2}{4\pi} \int d\underline{y} \frac{u^\dagger u(\underline{y})}{|\underline{x} - \underline{y}|} \gamma_0 u(\underline{x}) . \quad (35)$$

Introducing the c number potential

$$B^0(\underline{x}) = - \frac{g_0}{4\pi} \int d\underline{y} \frac{u^+ u(\underline{y})}{|\underline{x} - \underline{y}|}, \quad (36)$$

Equation (35) may be rewritten as a differential system

$$(i \underline{\gamma} \cdot \underline{\partial} - m - \gamma_0 E) u(\underline{x}) = g_0 \gamma_0 B^0(\underline{x}) u(\underline{x}), \quad (37)$$

$$\partial^2 B^0(\underline{x}) = -g_0 u^+ u(\underline{x}). \quad (38)$$

Soliton solutions to this system of equations have been exhibited in Ref. 1). They are solutions with maximal, spherical symmetry obtained from the ansatz

$$u(\underline{x}) = \begin{pmatrix} u_1(r) \chi_{\kappa}^{\mu} \\ u_2(r) \chi_{-\kappa}^{\mu} \end{pmatrix}, \quad (39)$$

where u_1 and u_2 are upper and lower component radial functions, χ_{κ}^{μ} are ordinary spin-angular functions, and $\kappa = \pm(j + \frac{1}{2})$ identifies the parity of the two-quark states of angular momentum j . Consistency with the requirement of spherical symmetry forces $j = \frac{1}{2}$. The system of Eqs. (37)-(38) is reduced to the system of radial equations

$$\frac{d}{dr} u_2(r) = \frac{\kappa-1}{r} u_2(r) - (E - m - g_0 B^0(r)) u_1(r), \quad (40)$$

$$\frac{d}{dr} u_1(r) = (E + m - g_0 B^0(r)) u_2(r) - \frac{\kappa+1}{r} u_1(r), \quad (41)$$

$$\frac{1}{r^2} \frac{d^2}{dr^2} (r^2 B^0(r)) = g_0 (u_1^2 + u_2^2). \quad (42)$$

These were solved numerically with boundary conditions of regularity at the origin ($r=0$) and sufficiently rapid fall-off at infinity ($r \rightarrow \infty$) to ensure integrability of the soliton energy density. In particular, asymptotically as $r \rightarrow \infty$ u_1, u_2 fall off exponentially, while $B^0(r)$ exhibits the Coulomb tail characteristic of the charge of the constituent $|\bar{q} >$ state. Since the effective potential $B^0(r)$ acting on the wave function $u(x)$ corresponds to a negative charge distribution, Eq. (38), only positive energy ($E > 0$) soliton solutions exist, consistently with the hypothesis which led to Eqs. (37)-(38). An example of a solution is reported in Fig. 1. Only the upper component $u_1(r)$ is shown because the lower component $[u_2(r)]$ of the spinor wave function is negligibly small, consistently with our weak coupling approximation.

5. - STABILITY

These solutions correspond to minima of the energy and are therefore stable. In fact, introducing a parameter R to characterize the "radius" of the solution, its energy $\mathcal{E}$ as a function of R is, on the basis of Eq. (29), easily estimated to be

$$\mathcal{E} \sim 2(m^2 + p^2)^{1/2} - \frac{g_0^2}{4\pi} \frac{3}{5} \frac{1}{R}, \quad p \sim \frac{\pi}{R}, \quad (43)$$

where the first and second terms approximate respectively the rest mass and kinetic energy of the constituents and the interaction energy between constituents. A balance between these contrasting effects obtains for

$$mR \sim \frac{\pi(1-x^2)^{1/2}}{x}, \quad x = \frac{3}{10\pi} \frac{g_0^2}{4\pi}, \quad (44)$$

which determines the minimum energy to be

$$\mathcal{E} \sim 2m(1-x^2)^{1/2} \quad (45)$$

Note that Eqs. (44)-(45), derived in our weak coupling approximation, cease to be meaningful for strong coupling, i.e., as $x \rightarrow 1$. At this point, our effective Hamiltonian, Eq. (29), becomes unbounded below. This may be interpreted to suggest a phase transition of the vacuum state to a condensate of quark-antiquark pairs corresponding to a critical (strong) coupling.

Stability in the soliton sense should, of course, not be confused with stability in the particle sense. Our variational approximation is a zero-width approximation since we have not included in our set of trial states "radiation states" with the same quantum numbers of $|q\bar{q}\rangle$ and in which $|q\bar{q}\rangle$ could decay. This quantum mechanical instability may be cured by the introduction of a gluon mass term $(\frac{1}{2}M^2 B_\mu B^\mu)$ in our Hamiltonian, Eq. (1). In fact this addition will not destroy soliton solutions provided the ratio of gluon to quark mass does not exceed a critical value, function of the coupling constant

$$M/m < f(x). \quad (46)$$

On the other hand, particle stability is simply regained if the gluon mass exceeds an appropriate fraction of the bound state mass. These conditions are compatible for sufficiently large values of the coupling constant. In any case later, when we come to the relevant non-Abelian theory, we may appeal to gluon confinement by infra-red slavery to provide the necessary particle stability against radiation of gluons.

6. - ABSENCE OF EXOTICS

The binding mechanism responsible for the existence of our soliton solutions in the present Abelian $U(1)$ gauge model is ordinary exchange of soft gluons between oppositely charged constituents :

In our weak coupling approximation, the corresponding binding forces saturate in charge singlet states formed from minimal combinations of constituents, i.e. (in the present Abelian case), from a $q\bar{q}$ pair. In other words, "exotic" combinations such as $q^2\bar{q}^2$ are not bound :

To exhibit this phenomenon we construct trial exotic states

$$|q^2 \bar{q}^2\rangle = \prod_{j=1}^2 \int d\underline{x}_j u_j(\underline{x}_j) b^\dagger(\underline{x}_j) C(\underline{x}_j) \int d\underline{y}_j v(\underline{y}_j) d^\dagger(\underline{y}_j) C^\dagger(\underline{y}_j) |0\rangle, \quad (47)$$

where u_i, v_i are charge conjugate wave functions, and carry out variational procedure as before. The effective (renormalized) Hamiltonian is :

$$\begin{aligned} \langle q^2 \bar{q}^2 | H | q^2 \bar{q}^2 \rangle \sim & \sum_{j=1}^2 \int d\underline{x}_j \left[u_j^\dagger \underline{\alpha}_j \cdot \nabla u_j + m \bar{u}_j u_j - v_j^\dagger \underline{\alpha}_j \cdot \nabla v_j - m \bar{v}_j v_j \right] \\ & - \frac{g_0^2}{4\pi} \sum_{j=1}^2 \int d\underline{x}_j \int d\underline{y}_j \frac{u_j^\dagger u_j(\underline{x}_j) v_j^\dagger v_j(\underline{y}_j)}{|\underline{x}_j - \underline{y}_j|} \end{aligned} \quad (48)$$

and leads to independent Euler equations for u_1 and u_2 . This factorization property of the model is, of course, intimately connected to the vector nature of its interaction. In fact bag models based on the exchange of scalar gluons ^{*)} like to bind any combination of quarks and antiquarks. Consequently their spectrum of states has a tendency to be very rich, in conflict with the actual hadronic spectrum.

7. - THE CONSTITUENT METHOD

Our derivation of c number soliton equations in the weak coupling (non-relativistic) limit, Eqs. (37)-(38), was based on a Hamiltonian quantum variational procedure in Fock space with explicit trial states as defined by Eq. (28). For later convenience, we will now derive a relativistic generalization of those equations by saturating the operator field equations, Eqs. (3)-(4), with a limited family of states in an abstract Hilbert space. We shall refer to this approach as the "constituent method" ^{**) .}

^{*)} Even if not explicitly, as in the case of MIT bag model ⁷⁾.

^{**) To simplify the presentation, in this section we will ignore the spin labels of the states. They will be introduced and properly taken into account in the following section, where soliton equations for spin 1 and spin 0 $q\bar{q}$ bound states will be derived.}

Let us consider three states with charges g_0 , $-g_0$ and 0 :

$$Q|q\rangle = g_0|q\rangle, \quad Q|\bar{q}\rangle = -g_0|\bar{q}\rangle, \quad Q|q\bar{q}\rangle = 0, \quad (49)$$

$$Q = \int dx j_0(x). \quad (50)$$

Here $|q\bar{q}\rangle$ represents the bound state that we are seeking while $|q\rangle$ and $|\bar{q}\rangle$ are auxiliary states : they will be needed to construct quark and antiquark wave functions $[u(x)$ and $v(x)]$ and can be thought of as virtual, "constituent states", i.e., what remains of the bound state as one of the constituents is suddenly removed. The only non-vanishing quark field matrix elements are

$$\langle \bar{q} | \psi(x) | q\bar{q} \rangle = u(x) = \langle q\bar{q} | \psi^\dagger(x) | \bar{q} \rangle^\dagger, \quad (51)$$

$$\langle q | \psi^\dagger(x) | q\bar{q} \rangle = v^\dagger(x) = \langle q\bar{q} | \psi(x) | q \rangle^\dagger. \quad (52)$$

We approximate expectation values of the currents $\chi(x)$, Eq. (3), and $j_\mu(x)$, Eq. (4), as follows :

$$\begin{aligned} & \langle q\bar{q} | j^\mu(x) | q\bar{q} \rangle = \\ & \simeq \sum_{ij} \left[\langle q\bar{q} | \psi_i^\dagger(x) | \bar{q} \rangle \alpha_{ij}^\mu \langle \bar{q} | \psi_j(x) | q\bar{q} \rangle - \langle q\bar{q} | \psi_j(x) | q \rangle \alpha_{ij}^\mu \langle q | \psi_i^\dagger(x) | q\bar{q} \rangle \right] \\ & \simeq \langle q | j^\mu(x) | q \rangle + \langle \bar{q} | j^\mu(x) | \bar{q} \rangle, \end{aligned} \quad (53)$$

$$\langle \bar{q} | \chi(x) | q\bar{q} \rangle \simeq g_0 \langle \bar{q} | \beta(x) | \bar{q} \rangle \langle \bar{q} | \psi(x) | q\bar{q} \rangle, \quad (54)$$

$$\langle q | \chi^\dagger(x) | q\bar{q} \rangle \simeq g_0 \langle q | \beta(x) | q \rangle \langle q | \psi^\dagger(x) | q\bar{q} \rangle, \quad (55)$$

where

$$\alpha_{ij}^\mu = g_0 (\gamma^0 \gamma^\mu)_{ij}. \quad (56)$$

This approximation leads to the following system of c number field equations :

$$(i\not{D} - m) u(x) = g_0 \langle \bar{q} | \not{B}(x) | \bar{q} \rangle u(x), \quad (57)$$

$$\partial_\nu \langle \bar{q} | G^{\mu\nu}(x) | \bar{q} \rangle = -g_0 \bar{v} \gamma^\mu v(x), \quad (58)$$

$$(i\not{D} - m) v(x) = g_0 \langle q | \not{B}(x) | q \rangle v(x), \quad (59)$$

$$\partial_\nu \langle q | G^{\mu\nu}(x) | q \rangle = g_0 \bar{u} \gamma^\mu u(x). \quad (60)$$

The requirement that $|q\bar{q}\rangle$ be an eigenstate of charge conjugation, as in Eq. (33), together with the assumption that

$$C|q\rangle = |\bar{q}\rangle, \quad C|\bar{q}\rangle = |q\rangle, \quad (61)$$

relates the quark and antiquark wave functions, $u(x)$ and $v(x)$, and reduces the system of equations to

$$(i\not{D} - m) u(x) = g_0 \langle \bar{q} | \not{B}(x) | \bar{q} \rangle u(x), \quad (62)$$

$$\partial_\nu \langle \bar{q} | G^{\mu\nu}(x) | \bar{q} \rangle = -g_0 \bar{u} \gamma^\mu u(x). \quad (63)$$

The time-independent restriction of Eqs. (62)-(63) in the non-relativistic limit (neglecting magnetic interactions) reproduces Eqs. (36)-(38). The saturation of the Hamiltonian expectation value within $|q\bar{q}\rangle$ yields :

$$\begin{aligned} \langle q\bar{q} | H | q\bar{q} \rangle \sim \int dx & \left[u^\dagger \frac{\alpha_i}{i} \nabla_i u + m \bar{u} u - v^\dagger \frac{\alpha_i}{i} \nabla_i v - m \bar{v} v + \right. \\ & \left. + \frac{1}{2} (\langle q\bar{q} | G_{ij} | q\bar{q} \rangle^2 - \langle q\bar{q} | G_{0i} | q\bar{q} \rangle^2 + g \bar{u} \gamma_\mu u B_\nu^\mu - g \bar{v} \gamma_\mu v B_\mu^\nu) \right] \end{aligned} \quad (64)$$

where

$$B_u^\mu(\underline{x}) = \langle q | B^\mu(\underline{x}) | q \rangle, \quad B_v^\mu(\underline{x}) = \langle \bar{q} | B^\mu(\underline{x}) | \bar{q} \rangle, \quad (65)$$

or, in the case of time-independent solutions,

$$B_u^\mu(\underline{x}) = \frac{g_0}{4\pi} \int d\underline{y} \frac{\bar{u} \gamma^\mu u(\underline{y})}{|\underline{x} - \underline{y}|}, \quad B_v^\mu(\underline{x}) = - \frac{g_0}{4\pi} \int d\underline{y} \frac{\bar{v} \gamma^\mu v(\underline{y})}{|\underline{x} - \underline{y}|}. \quad (66)$$

Again, the time-independent restriction of Eq. (64) in the non-relativistic limit (neglecting magnetic interactions) reproduces Eq. (29).

The system of equations (57)-(60) has an intuitive interpretation : the quark wave function, $u(\underline{x})$, in Eq. (57), "feels" the gluon field generated by the antiquark, i.e., the gluon field in the antiquark constituent state, $\langle \bar{q} | \not{B}(\underline{x}) | \bar{q} \rangle$; and vice versa. These interactions are attractive and lead to soliton solutions with the large components of the spinor functions $u(\underline{x})$ and $v(\underline{x})$ being the upper and the lower components, respectively. We have already presented an example of such a solution for weak coupling in connection with Eqs. (37)-(38).

Increasing the value of the coupling constant g_0 , the magnetic interactions of the quark-antiquark spins become non-negligible. In analogy with atomic physics, we expect two (previously degenerate) $|q\bar{q}\rangle$ states to emerge, depending on the relative orientation of the quark and antiquark spins. To account for these phenomena, we must first introduce spin labels for the states.

8. - SOLITON EQUATIONS FOR SPIN 0 AND SPIN 1 $q\bar{q}$ BOUND STATES

We shall consider for simplicity only spin 0 and spin 1 $q\bar{q}$ bound states and spin $\frac{1}{2}$ constituent, q or $\bar{q}$, states and adopt the following notation :

$$J^2 |q\bar{q}, j, m\rangle = j(j+1) |q\bar{q}, j, m\rangle, \quad (67)$$

$$J_z |q\bar{q}, j, m\rangle = m |q\bar{q}, j, m\rangle, \quad -j \leq m \leq j, \quad (68)$$

$$J^2 |q, \frac{1}{2}, m\rangle = \frac{3}{4} |q, \frac{1}{2}, m\rangle, \quad (69)$$

$$J_z |q, \frac{1}{2}, m\rangle = m |q, \frac{1}{2}, m\rangle, \quad m = \pm 1/2. \quad (70)$$

By the Wigner-Eckart theorem, matrix elements of the gluon field in constituent states are expressed in terms of reduced matrix elements

$$\langle q, \frac{1}{2}, m | B_0(x) | q, \frac{1}{2}, m' \rangle = \delta_{mm'} \langle q | B_0(x) | q \rangle, \quad (71)$$

$$\langle q, \frac{1}{2}, m | B_i(x) | q, \frac{1}{2}, m' \rangle = \delta_{m+i, m'} C(\frac{1}{2} 1 \frac{1}{2}; m i) \langle q | B(x) | q \rangle, \quad (72)$$

where $C(\frac{1}{2} 1 \frac{1}{2}; m i)$ are $SU(2)$ Clebsch-Gordan coefficients in the notation of Ref. 8). Similarly, quark field matrix elements between bound and constituent states are reduced as follows :

$$\langle \bar{q}, \frac{1}{2}, m' | \psi_{1,2}(x) | q\bar{q}, j, m \rangle = \delta_{m \pm \frac{1}{2}, m'} C(\frac{1}{2} \frac{1}{2} j; m \pm \frac{1}{2}) \langle \bar{q} | \psi_u(x) | q\bar{q} \rangle, \quad (73)$$

$$\langle \bar{q}, \frac{1}{2}, m' | \psi_{3,4}(x) | q\bar{q}, j, m \rangle = \delta_{m \pm \frac{1}{2}, m'} C(\frac{1}{2} \frac{1}{2} j; m \pm \frac{1}{2}) \langle \bar{q} | \psi_d(x) | q\bar{q} \rangle, \quad (74)$$

where the labels u and d refer to upper and lower component.

With the help of these and analogous reductions, we then retrace our steps of the previous section. In evaluating operator products by saturation, we must now sum over intermediate states with different spin orientations. After some algebra, soliton equations for spin 0 and spin 1 $q\bar{q}$ bound states may be cast in the form :

$$\text{spin } 0 \left\{ \begin{aligned} (i \not{\nabla} - m) u_{\uparrow}(x) &= g_0 (B_0 \gamma^0 - 3 B_3 \gamma^3) u_{\uparrow}(x), & (75) \\ \square B_{\mu}(x) &= -g_0 \bar{u}_{\uparrow} \gamma_{\mu} u_{\uparrow}(x), & (76) \\ u_{\uparrow}(x) &= \sqrt{2} \langle \bar{q}, \frac{1}{2}, -\frac{1}{2} | \psi(x) | q \bar{q}, 0, 0 \rangle, & (77) \end{aligned} \right.$$

and

$$\text{spin } 1 \left\{ \begin{aligned} (i \not{\nabla} - m) u_{\uparrow}(x) &= g_0 (B_0 \gamma^0 + B_3 \gamma^3) u_{\uparrow}(x), & (78) \\ \square B_{\mu}(x) &= -g_0 \bar{u}_{\uparrow} \gamma_{\mu} u_{\uparrow}(x), & (79) \\ u_{\uparrow}(x) &= \langle \bar{q}, \frac{1}{2}, \frac{1}{2} | \psi(x) | q \bar{q}, 1, 1 \rangle. & (80) \end{aligned} \right.$$

These equations must be solved with the normalization condition for the quark wave function $u_{\uparrow}(x)$:

$$\int d\mathbf{x} \ u_{\uparrow}^{\dagger} u_{\uparrow}(x) = 1. \quad (81)$$

as implied by Eq. (49). Corresponding to stationary solutions, their energy is given by :

$$\mathcal{E} = 2 \mathcal{E}_{km} + \mathcal{E}_{el} + \mathcal{E}_{mg} \begin{cases} 3 \text{ spin } 0 \\ (-1) \text{ spin } 1 \end{cases} \quad (82)$$

$$2 \mathcal{E}_{km} = \int d\mathbf{x} \left[u_{\uparrow}^{\dagger} \frac{\boldsymbol{\alpha} \cdot \nabla}{i} u_{\uparrow} + m \bar{u}_{\uparrow} u_{\uparrow} - v_{\uparrow}^{\dagger} \frac{\boldsymbol{\alpha} \cdot \nabla}{i} v_{\uparrow} - m \bar{v}_{\uparrow} v_{\uparrow} \right], \quad (83)$$

$$\mathcal{E}_{el} = - \frac{g_0^2}{4\pi} \int d\underline{x} \int d\underline{y} \frac{\bar{u}_\uparrow \gamma^0 u_\uparrow(\underline{x}) \bar{v}_\uparrow \gamma^0 v_\uparrow(\underline{y})}{|\underline{x} - \underline{y}|}, \quad (84)$$

$$\mathcal{E}_{mg} = \frac{g_0^2}{4\pi} \int d\underline{x} \int d\underline{y} \frac{\bar{u}_\uparrow \gamma^3 u_\uparrow(\underline{x}) \bar{v}_\downarrow \gamma^3 v_\downarrow(\underline{y})}{|\underline{x} - \underline{y}|}. \quad (85)$$

9. - COLOUR SU(3) MODEL : EQUATIONS FOR "MESONS" AND "BARYONS"

We now proceed to extend the previous considerations to non-Abelian gauge theories, taking SU(3) as an example. The Hamiltonian density is

$$\mathcal{H}(x) = \psi_j^\dagger \frac{\alpha \cdot \nabla}{i} \psi_j + m_0 \bar{\psi}_i \psi_i + \frac{1}{2} \sum_a (G_{aij}^2 - G_{a0j}^2) + g_0 \bar{\psi} \frac{\lambda^a}{2} \gamma^\mu \psi B_\mu^a, \quad (86)$$

where

$$G_{\mu\nu}^a = \partial_\nu B_\mu^a - \partial_\mu B_\nu^a + g_0 f^{abc} B_\mu^b B_\nu^c. \quad (87)$$

The field equations are

$$(i \not{\nabla} - m) \psi_i(x) = g_0 B_a \frac{\lambda^a}{2} \psi_i(x), \quad (88)$$

$$\partial_\nu G^{a\mu\nu}(x) = g_0 \bar{\psi} \frac{\lambda^a}{2} \gamma^\mu \psi(x) + g_0 f^{abc} B_\nu^b G^{c\mu\nu}(x). \quad (89)$$

We will begin by constructing soliton equations for "mesons", i.e., colour singlet $q\bar{q}$ bound states. In analogy with Section 7 we shall temporarily ignore spin labels : these will be taken into account in the same way as it was done for the Abelian gauge model in Section 8. Let us consider the following set of states :

$$Q^a |q_i\rangle = \frac{\lambda_{ij}^a}{2} |q_j\rangle, \quad Q^a |\bar{q}_i\rangle = \frac{\lambda_{ji}^a}{2} |\bar{q}_j\rangle, \quad Q^a |q\bar{q}\rangle = 0, \quad (90)$$

$$Q^a = \int dx [\bar{\psi} \gamma^0 \frac{\lambda^a}{2} \psi + f^{abc} B_\nu^b G^{c0\nu}] . \quad (91)$$

Here $|q\bar{q}\rangle$ represents the meson bound state that we are seeking while $|q_i\rangle$ and $|\bar{q}_i\rangle$ are constituent states transforming as 3 and $\bar{3}$ under global $SU(3)$. Non-vanishing matrix elements may be reduced with the help of the Wigner-Eckart theorem for $SU(3)$:

$$\langle \bar{q}_i | \psi_j(x) | q\bar{q} \rangle = \frac{1}{\sqrt{3}} \delta_{ij} u(x) , \quad (92)$$

$$\langle q_i | B_\mu^a(x) | q_j \rangle = \frac{1}{2} \lambda_{ij}^a B_\mu(x) . \quad (93)$$

We approximate expectation values of currents and other operator products in a way similar to Eqs. (53)-(55). A notable difference is the occurrence of terms non-linear in the gauge field such as

$$f^{abc} \langle q_i | \{ B_\mu^b(x), B_\nu^c(x) \} | q_j \rangle \quad (94)$$

which, however, according to our saturation approximation, vanish :

$$\frac{1}{4} B_\mu(x) B_\nu(x) f^{abc} \{ \lambda^b, \lambda^c \} = 0 . \quad (95)$$

Thus we obtain

$$\frac{1}{\sqrt{3}} \delta_{ij} (i\not{D} - m) u(x) = \frac{1}{\sqrt{3}} \frac{1}{4} \sum_a (\lambda^a)_{ij}^2 g_0 \not{B} u(x) , \quad (96)$$

$$\frac{1}{2} \lambda_{ij}^a \square B_\mu = -\frac{1}{2} \lambda_{ij}^a g_0 \bar{u} \gamma_\mu u , \quad (97)$$

which imply the following (reduced) soliton equations (neglecting spin effects) :

$$(i\not{D} - m) u(x) = \frac{4}{3} g_0 \not{B}(x) u(x) , \quad (98)$$

$$\square B_\mu = -g_0 \bar{u} \gamma_\mu u(x) . \quad (99)$$

Apart from relative normalization of currents, these equations coincide with corresponding equations (62)-(63) derived in the Abelian gauge model.

Taking spin into account, as in Section 8, we arrive at the "meson" equations :

$$\begin{aligned} \text{spin 0} \\ \text{"mesons"} \end{aligned} \left\{ \begin{aligned} (i \not{\partial} - m) u_\uparrow(x) &= \frac{4}{3} g_0 (B_0 \gamma^0 - 3 B_3 \gamma^3) u_\uparrow(x) & (100) \\ \square B_\mu(x) &= -g_0 \bar{u}_\uparrow \gamma_\mu u_\uparrow(x) , & (101) \\ \frac{1}{\sqrt{3}} \delta_{ij} u_\uparrow(x) &= \sqrt{2} \langle \bar{q}_i, \frac{1}{2}, -\frac{1}{2} | \psi_j(x) | q\bar{q}, 0, 0 \rangle & (102) \end{aligned} \right.$$

and

$$\begin{aligned} \text{spin 1} \\ \text{"mesons"} \end{aligned} \left\{ \begin{aligned} (i \not{\partial} - m) u_\uparrow(x) &= \frac{4}{3} g_0 (B_0 \gamma^0 + B_3 \gamma^3) u_\uparrow(x) & (103) \\ \square B_\mu(x) &= -g_0 \bar{u}_\uparrow \gamma_\mu u_\uparrow(x) , & (104) \\ \frac{1}{\sqrt{3}} \delta_{ij} u_\uparrow(x) &= \langle \bar{q}_i, \frac{1}{2}, \frac{1}{2} | \psi_j(x) | q\bar{q}, 1, 1 \rangle . & (105) \end{aligned} \right.$$

These equations must be solved with the normalization condition given in Eq. (81). The energy of stationary solutions is :

$$\mathcal{E}_m = 2 \mathcal{E}_{Kn} + \frac{4}{3} \mathcal{E}_{el} + \frac{4}{3} \mathcal{E}_{mg} \begin{matrix} \nearrow 3 \text{ spin 0} \\ \searrow (-1) \text{ spin 1} \end{matrix} \quad (106)$$

with $\mathcal{E}_{Kn}$, $\mathcal{E}_{el}$ and $\mathcal{E}_{mg}$ defined as in the Abelian case by Eqs. (83)-(85).

Next we outline the construction of soliton equations for "baryons", i.e., colour singlet qqq bound states. As before, constituent states, $|(qq)_i\rangle$, are obtained by removing one quark from the bound state, $|qqq\rangle$. The left-over system consists of two quarks in an anti-symmetric combination transforming as a $\bar{3}$ under $SU(3)$:

$$Q^a |(qq)_i\rangle = \frac{\lambda^a_{ji}}{2} |(qq)_j\rangle, \quad Q^a |qqq\rangle = 0. \quad (107)$$

We assume the constituent states $|(qq)_i\rangle$ to carry spin 1 : this would correspond for example to $|qqq\rangle$ being the Δ resonance in a totally symmetric state with respect to "flavour". To describe the nucleon we would have to include spin zero constituent states also. This we shall not bother to do in the present paper. With the help of the Wigner-Eckart theorem we then reduce out colour and spin indices, as before, and obtain the following equations for spin $\frac{3}{2}$ baryons :

$$\left. \begin{array}{l} \text{spin } 3/2 \\ \text{"baryons"} \end{array} \right\} \begin{cases} (i \not{\partial} - m) u_\uparrow(x) = \frac{4}{3} g_0 (B_0 \gamma^0 + B_3 \gamma^3) u_\uparrow(x), & (108) \\ \square B_\mu(x) = -g_0 \bar{u}_\uparrow \gamma_\mu u_\uparrow(x), & (109) \\ \delta_{ij} u_\uparrow(x) = \langle (qq)_i, j=1, m=1 | \psi_j(x) | qqq, j=\frac{3}{2}, m=\frac{3}{2} \rangle. & (110) \end{cases}$$

Note that these equations are similar to the ones for spin 1 mesons, Eqs. (103)-(105). The energy of a (stationary) solution differs, however, by a factor $\frac{3}{2}$:

$$\mathcal{E}_B = 3 \mathcal{E}_{Kn} + 2 \mathcal{E}_{el} - 2 \mathcal{E}_{mg} = \frac{3}{2} \mathcal{E}_M (\text{spin } 1), \quad (111)$$

with $\mathcal{E}_{Kn}$, $\mathcal{E}_{el}$ and $\mathcal{E}_{mg}$ again defined by Eqs. (83)-(85) and $\mathcal{E}_M$ defined by Eq. (106).

In conclusion, we remind the reader of the two main ingredients of our derivation :

- 1) the constituent method, i.e., a saturation prescription for the operator field equations ;
- 2) the choice of constituent states, in particular of their transformation properties under internal and external symmetry groups.

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FIGURE CAPTION

Example of soliton solution for weak coupling.