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INFRA - RED DIVERGENCE ENFORCES A REARRANGED PERTURBATION EXPANSION

I. Scalar Electrodynamics

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ABSTRACT

We show that the ordinary perturbation expansion must, except in certain cases, be rearranged in order to uniquely carry out the infra-red exponentiation in a translation and gauge invariant way. Uniqueness of the exponent of order α follows from requiring exact order-by-order agreement with OPE before summation, and also requiring that exponentiation of all factorizable parts must be done before integration.

By this rearrangement the remaining infra-red regular part is defined in terms of correlations with respect to photon momenta in the integrands. This automatically excludes an exponentiation ambiguity of type $\exp \alpha (S) \rightarrow \exp \alpha (S + C')$ which is possible if the exponentiation is performed after integration.

We complete here the work on scalar electrodynamics initiated earlier by including radiative corrections, and also carry through the renormalization of this correlation rearrangement. All photons are treated identically. We believe that this makes our technique simpler and more advantageous than others.

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1. INTRODUCTION

By a simple rearrangement of the ordinary perturbation expansion (OPE) a new factorization technique for infra-red (IR) divergent field theories has been obtained ¹⁾. Here we complete the work on a two-particle scattering process in scalar electrodynamics, initiated in Ref. 1, by including radiative corrections for which renormalization is necessary.

We recall that, in conventional approaches, either the photon phase space is separated by a direct momentum cut-off into soft and hard contributions ²⁾, or a certain dependence on photon momenta is neglected ³⁾, in order to factorize and sum up the IR divergent part of OPE. Apart from breaking translation invariance and introducing a certain approximation error, this leads to a factorization ambiguity because it does not uniquely determine what to exponentiate and how to define the exact form and structure of the non-exponentiable IR regular part. These questions have been partially resolved by an improvement ⁴⁾ of the technique of Ref. 3), but a general formula for the IR regular part has yet to be found to exclude the above ambiguity.

For practical purposes, such as analytic fourth-order calculation of the electron form factors ⁵⁾, this ambiguity has been resolved by defining the exponential of Ref. 3) to be the physically significant one. The remaining IR regular part was then found by a manipulation "by hand" of the evaluated numbers F_n of each order α^n derived from OPE, e.g., for the Dirac form factor ⁵⁾ⁿ

$$\gamma_\mu \{ 1 + \alpha F_1 + \alpha^2 F_2 \}_{OPE} = \quad (1.1a)$$

$$= \gamma_\mu e^{\alpha S} \{ 1 + \alpha H_1 + \alpha^2 H_2 \} = \quad (1.1b)$$

$$= \gamma_\mu \{ 1 + \alpha (S + H_1) + \alpha^2 (\frac{1}{2} S^2 + H_2 + S H_1) \} \quad (1.1c)$$

As is seen by comparison of (1.1b) with (1.1c) and a term by term identification with (1.1a), the expansion (1.1b) is a rearrangement of OPE. It should be emphasized here that we can always make this rearranged perturbation expansion IR-regular translation invariantly, whereas except for certain cases this is not the case for OPE.

Such an exceptional case is the derivation of $(g-2)$, where a corresponding exponentiation and rearrangement can be done for the Pauli form factor ⁵⁾. The quantity $(g-2)$ is defined for zero momentum transfer for which $S=0$ and in which case there is no difference between OPE and this rearrangement.

Therefore, in the general case such a rearrangement of OPE and an exact definition of the physical S must be done to obtain a unique gauge and translation invariant IR regular result. If not, because of truncation of the IR regular part, we would be left with an indeterminacy of the type

$$\exp \alpha(S) \rightarrow \exp \alpha(S+c') \quad (1.2)$$

which, by a suitable choice of c' , could make (1.1b) and radiative corrections in general arbitrary small or big. Notice also that the IR regular one-photon effect H_1 describes part of the full two-photon effect F_2 . The exact relation is obvious from the definition of the corresponding IR regular part

$$H_2 = F_2 - \frac{1}{2} S^2 - S H_1 \quad (1.3)$$

We present here a formalism which exhibits the underlying form and structure of this rearrangement of OPE in terms of Feynman diagram integrals. This technique, which is applied before integration, "automatically" excludes an ambiguity of the type (1.2), which is obviously there if we make a formal manipulation of the integrated OPE (1.1), provided that we exponentiate before integration. Further, contrary to Refs. 2) and 4), all photons are here treated identically, which considerably simplifies all calculations.

In the case of scalar electrodynamics our rearrangement follows naturally from a generalization of Low's theorem ⁶⁾, first to all orders in the coupling constant ⁷⁾ and then for arbitrary "hard" photons ¹⁾, virtual as well as real. Such a generalization is only possible in cases where we know the model explicitly.

This leads to a rearrangement of OPE such that all terms are regrouped with respect to the "order of correlation" of photon momenta. The first term is totally factorizable, the second is factorizable up to a single correlation effect, the third up to a pair correlation effect, and so on. The totally factorizable parts, which contain all IR divergences, agree for the graphs in Fig. 1 with the exponential factor of Ref. 3) and for the generalized ladders (Fig. 2) with the refined field theoretical eikonal amplitude ^{8), 9)}.

Power counting for small photon momenta k and k^2 shows that all correlation effects are IR regular ¹⁰⁾. They are at least of order e^{2c+2} , where c is the order of correlation and e is the renormalized coupling constant, so they decrease with increasing c provided that c is not too strong.

We can thus calculate exactly to any desired finite order in α at the same time as we solve the IR problem in a translation and gauge invariant way to infinite order in α .

Throughout this paper we analyze single and pair correlation effects explicitly. The work is organized as follows. In Section 2 we rewrite all graphs in Fig. 1 on a summable form by means of our rearrangement of OPE. By use of the Ward identity and with a fictitious photon mass μ , for the same class of diagrams, we perform an on-shell renormalization in Section 3. The mass μ is kept only in the intermediate steps before emission corrections are added. In Section 4, still for the same class of diagrams, we generalize this to an arbitrary off-shell renormalization procedure. In Section 5, by means of the same rearrangement, we include all graphs in Fig. 2 in a summed up form. In Section 6 we combine the results of previous sections and obtain a summed up result for all graphs in Fig. 3. We then include graphs to first order in the quartic self coupling (Fig. 4) and then also the effects of scalar loops.

In Section 7 we give a summary of the work, emphasizing the important points of the results, discuss the summation of correlation effects, and present an explicit result for summed up single and pair correlation effects.

2. THE REARRANGEMENT OF OPE

The theory is defined by the Lagrangian density

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (\mathcal{D}^\mu \varphi)^* (\mathcal{D}_\mu \varphi) - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \\ & - m_0^2 (\varphi^* \varphi) - \lambda_0 (\varphi^* \varphi)^2; \quad m_0^2 > 0 \end{aligned} \quad (2.1)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2.2a)$$

$$\mathcal{D}_\mu = \partial_\mu - ie_0 A_\mu \quad (2.2b)$$

We will first perform an on-mass-shell renormalization and therefore give a small mass μ to the photon

$$\Delta_{\mu\nu}(k) = \frac{-i}{k^2 - \mu^2} \left(g_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right) \quad (2.3)$$

However, we could as well drop μ and carry out the renormalization off the mass shell.

As in the previous paper ¹⁾, we start with the tree diagram in Fig. 5 from which all types of graphs of OPE, except those including quartic self couplings, can be constructed. In this Section we derive the corrections due to the graphs in Fig. 1.

We apply the factorization machinery developed in Ref. 1) to the graph in Fig. 5. All photon momenta are denoted k_ℓ and all external momenta of scalar particles p_i . By symmetrization over all vertices for photons attached to the i^{th} prong, as worked out in the Appendix, and by means of the generalized Low theorem ⁷⁾, the corresponding part of the N^{th} order contribution to the amplitude in Fig. 5 can be written in the form

$$M_{\mu_1 \dots \mu_N}^{(i)} \sim \left\{ \prod_{\ell=1}^N \int_{\mu_\ell}^{\wedge i} (\theta_\ell k_\ell) + \sum_{s \neq t} \prod_{\mu_s \mu_t}^{\wedge i} (\theta_s k_s, \theta_t k_t) \cdot \right. \\ \left. \cdot \prod_{\ell \neq st}^N \int_{\mu_\ell}^{\wedge i} (\theta_\ell k_\ell) + \text{higher correlations} \right\} \Delta(Q) V_0 \quad (2.4)$$

The quantities in Eq. (2.4) are defined in the following way. The contribution from single photon exchange in Fig. 5, $\Delta(Q) V_0$, is given by

$$V_0(p_i) = (p_a + p_{a'})^\mu \left(g_{\mu\nu} - (1-\xi) \frac{q_\mu q_\nu}{q^2} \right) (p_b + p_{b'})^\nu \quad (2.5)$$

and the recoil in $\Delta(q)$ from quanta attached to the i^{th} prong

$$Q = q - \sum_{\ell=1}^N \theta_\ell k_\ell ; \quad q = p_a - p_{a'} = p_{b'} - p_b \quad (2.6)$$

is conveniently included in the Fourier transformed propagator

$$\begin{aligned}\Delta(Q) &= \int d^4x \Delta_F(x) e^{iQ \cdot x} = \\ &= \int d^4x \Delta_F(x) e^{iqx} \prod_{\ell=1}^N e^{-i\theta_\ell k_\ell \cdot x}\end{aligned}\quad (2.7)$$

$$\varepsilon_i = \begin{cases} \begin{pmatrix} +1 \\ -1 \end{pmatrix} \end{cases} \text{ for } \begin{pmatrix} \text{out} \\ \text{in} \end{pmatrix} \text{ going } p_i \quad (2.8)$$

$$\theta_\ell = \begin{cases} \begin{pmatrix} +1 \\ -1 \end{pmatrix} \end{cases} \text{ for } \begin{pmatrix} \text{emitted} \\ \text{absorbed} \end{pmatrix} k_\ell \quad (2.9)$$

$$\hat{f}_{\mu_1}^i = f_{\mu_1}^i + D_{\mu_1}^i \quad (2.10)$$

$$f_{\mu_1}^i = ie \frac{2p_{\mu_1}^i + \varepsilon_i \theta_\ell k_{\ell\mu_1}}{2\varepsilon_i \theta_\ell p_i \cdot k_\ell + k_\ell^2} \quad (2.11)$$

$$D_{\mu_1} = f_{\mu_1}^i \hat{k}_1^i - ie \frac{\partial}{\partial p_{\mu_1}^i} \quad (2.12)$$

$$\hat{k}_1^i = \varepsilon_i \theta_\ell k_\ell \cdot \frac{\partial}{\partial p_i} \quad (2.13)$$

A general definition of the correlations is given in the Appendix.

Before we give the pair correlation tensor, notice that the first term in (2.12) correctly describes the recoil in the vertices $V_0(p_i)$ due to the attached momenta k_ℓ to the i^{th} prong p_i , and the second term in (2.12) reproduces exactly the effect of sea-gull terms where the exchanged photon in Fig. 5 is one of the involved quanta.

The other sea-gull terms are taken care of by a $g_{\mu\nu}$ term in the pair correlation tensor $\chi_{\mu\nu}$ and by higher order correlations

$$\hat{X}_{\mu_s \mu_t}^i = X_{\mu_s \mu_t}^i e^{\hat{k}_s^i + \hat{k}_t^i} = X_{\mu_s \mu_t}^i (1 + \hat{k}_s^i + \hat{k}_t^i) \quad (2.14)$$

$$X_{\mu_s \mu_t}^i = \left\{ f_{\mu_s}^i i e 2 \varepsilon^i \theta_s k_{s \mu_t} + f_{\mu_t}^i i e 2 \varepsilon^i \theta_t k_{t \mu_s} - f_{\mu_s}^i f_{\mu_t}^i 2 \theta_s \theta_t k_s \cdot k_t - (i e)^2 2 g_{\mu_s \mu_t} \right\} \frac{(y_{st}^i)^{-1}}{1 + X_{st}^i} \quad (2.15)$$

$$y_{st}^i = 2 \varepsilon^i p_i (\theta_s k_s + \theta_t k_t) + k_s^2 + k_t^2 \quad (2.16)$$

$$X_{st}^i = 2 k_s \cdot k_t (y_{st}^i)^{-1} \quad (2.17)$$

In Ref. 1), where all generalized ladders were considered, we showed in detail how to find (2.4) to (2.17) and that the currents (2.11), (2.12) and (2.15) are conserved.

Here we generalize and include all graphs in Fig. 1. A straightforward m^{th} order ansatz is found by connecting all "free" photon legs ($N = 2m$) in Fig. 5 pair-wise on, for example, the $A = \{aa'\}$ side (Fig. 6). After some combinatorical exercises and integration over all subspaces ¹¹⁾, the m^{th} order form factor can be written in the form

$$-i M_m(s, t) = \frac{e^2}{m!} \int d^4 x e^{i q x} \Delta_F(x) \cdot \left\{ (i \hat{R}_0)^m + \binom{m}{2} i^2 \hat{P}_0^r (i \hat{R}_0)^{m-2} + \dots \right\} V_0 \quad (2.18)$$

or

$$-i M_m(s, t) = \frac{e^2}{m!} \int d^4 x e^{i q x} \Delta_F(x) \cdot \left\{ (i \hat{R}_0)^m + \binom{m}{1} i \hat{S}_0^r (i \hat{R}_0)^{m-1} + \binom{m}{2} i^2 \hat{P}_0^r (i \hat{R}_0)^{m-2} + \dots \right\} V_0 \quad (2.19)$$

Naturally the same thing can be done for the $B=\{bb'\}$ side which we therefore include in the definitions of $\hat{R}_0$, R_0 , $\hat{S}_0^r$ and $\hat{P}_0^r$

$$\hat{R}_0 = \frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \left\{ \hat{J}_\mu^A(k) \hat{J}^{A\mu}(-k) + (A \rightarrow B) \right\} \quad (2.20)$$

$$\begin{aligned} \hat{P}_0^r = & \frac{i^2}{2^2} \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \Delta_F(k_1) \Delta_F(k_2) \left\{ \hat{X}^{A\mu\nu}(k_1, k_2) \cdot \right. \\ & \cdot \hat{J}_\mu^A(-k_1) \hat{J}_\nu^A(-k_2) + \hat{J}_\mu^{A\mu}(k_1) \hat{J}^{A\nu}(k_2) \hat{X}_{\mu\nu}^A(-k_1, -k_2) + \\ & + \hat{X}^{A\mu\nu}(k_1, k_2) \hat{X}_{\mu\nu}^A(-k_1, -k_2) + (A \leftrightarrow B) + \\ & + \hat{J}_\mu^B(k_1) D_\nu^A(k_2) D^{B\mu}(-k_1) \hat{J}^{A\nu}(-k_2) + \\ & \left. + D_\mu^A(k_1) \hat{J}_\nu^B(k_2) \hat{J}^{A\mu}(-k_1) D^{B\nu}(-k_2) + (k_1 \rightarrow -k_1) \right\} \quad (2.21) \end{aligned}$$

The total currents are defined by

$$\hat{J}_\mu^A = \sum_{i \in A} \hat{f}_\mu^i; \quad D_\mu^A = \sum_{i \in A} D_\mu^i; \quad \hat{X}_{\mu\nu}^A = \sum_{i \in A} \hat{X}_{\mu\nu}^i \quad (A \rightarrow B) \quad (2.22)$$

and the definitions (2.5) to (2.17). Alternatively, we could write

$$\hat{R}_0 = R_0 + \hat{S}_0^r \quad (2.23)$$

where R_0 and $\hat{S}_0^r$ are given by

$$R_0 = \frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \left\{ J_\mu^A(k) J^{A\mu}(-k) + (A \rightarrow B) \right\} \quad (2.24)$$

$$\begin{aligned} \hat{S}_0^r = & \frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \left\{ D_\mu^A(k) J^{A\mu}(-k) + J_\mu^A(k) D^{A\mu}(-k) \right. \\ & \left. + D_\mu^A(k) D^{A\mu}(-k) + (A \rightarrow B) \right\} \quad (2.25) \end{aligned}$$

It is easy to check that the two forms (2.18) and (2.19) are equivalent. Due to the factorized form, we can now easily sum over m . From (2.18) we obtain

$$M(s,t) = ie^2 \int d^4x \Delta_F(x) e^{iqx} \left\{ 1 + \frac{i^2}{2!} \hat{P}_0^r + \dots \right\} e^{i\hat{K}_0} V_0 \quad (2.26)$$

and from (2.19) we find

$$M(s,t) = ie^2 \int d^4x \Delta_F(x) e^{iqx} \left\{ 1 + \frac{i}{1!} \hat{S}_0^r + \frac{i^2}{2!} \hat{P}_0^r + \dots \right\} e^{i\hat{K}_0} V_0 \quad (2.27)$$

The non-diagonal terms in (2.24) of form $f_\mu^a(k) f^{a'}(-k)^\mu$, corresponding ones in (2.21) and higher order correlations, contribute to the vertex, e.g., an arbitrary number of photons emitted from the a prong $f_\mu^a(+k)$ and then absorbed by the a' prong $f_\mu^{a'}(-k)$ (Fig. 6). This can easily be found directly from OPE by symmetrization of an m^{th} order vertex graph.

The remaining terms of self-energy type are diagonal in $i = f_\mu^i(k) f_\mu^i(-k)$, $i = a, a'$, i.e., the photon is emitted and absorbed by the same external leg. The problem is that these terms are not easily derived from higher orders in OPE. Instead we here fix their contribution in (2.24) and in the correlations by requiring conservation of all currents on the mass shell and then making the expression symmetric with respect to external particle indices.

Such an ansatz (2.24) is properly normalized if we restrict ourselves to the "soft" approximation, which means that the corresponding on-shell form factor in (2.26) is unity in the limit $p_a = p_{a'}$ and the obtained "soft" self energies equals those of OPE¹²⁾.

However, this is not true for arbitrary "hard" virtual contributions. To correct what is missing in this generalization we must look deeper into the effects of renormalization. We do this in the next Section by a detailed comparison with the ordinary form of the perturbation expansion, but first we give a graphical interpretation of the D_μ operators. The two last terms in (2.21) give graphs of the type given in Fig. 7, and the two first terms in (2.25) give graphs of the type in Fig. 8. The last term in (2.25) gives zero due to the form of V_0 , but is kept here because of symmetry. From the three first terms in (2.21) we get graphs illustrated in Figs. 9a-9c.

3. RENORMALIZATION

With a small photon mass $\mu > 0$ we first perform an on-mass shell renormalization, which we generalize in the next Section to allow for an arbitrary off-shell subtraction point.

As explained in the Introduction, an exact fourth-order calculation embedded in a coherent cloud of photons results if two arbitrary photons are included without approximation as compared to OPE. It is important to note that these two photons could be arbitrary two out of all photons, i.e., the symmetrization is carried out for all photons. The problem is partially solved in the foregoing Section, and the remaining question is thus solved by considering the full effect of one extra photon, except for the one exchanged photon in Fig. 1. At the end of this Section we then construct the renormalized pair correlation effects, which correspond to exact sixth order in OPE and of course the IR problem solved to all orders.

Starting from OPE, the second order self-energy correction (Figs. 10a to 10d) is given by

$$\frac{i}{p_i^2 - m_i^2} i \sum (p_i) V_0 \Delta(q)$$

$$\sum (p_i) = \frac{(ie)^2}{2!} \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \left\{ \frac{(2p_i - k)_\mu G_{\xi'}^{\mu\nu}(2p_i - k)_\nu}{(p_i - k)^2 - m_i^2} + \right. \quad (3.1)$$

$$\left. + \frac{(2p_i + k)_\mu G_{\xi'}^{\mu\nu}(2p_i + k)_\nu}{(p_i + k)^2 - m_i^2} - 2g_{\mu\nu} G_{\xi'}^{\mu\nu} \right\} \quad (3.2)$$

where $G_{\xi'}^{\mu\nu}$ is defined by

$$G_{\xi'}^{\mu\nu} = g_{\mu\nu} - \xi' \frac{k_\mu k_\nu}{k^2} \quad (3.3)$$

and $\Delta_F(k)$ is the regularized Feynman propagator.

In conventional approaches no difference is made between the two first terms in (3.2), which is of course correct since they contribute equal numbers. However, it is convenient to retain this form in order to obtain the summable form. In the transverse gauge we obtain

$$\begin{aligned} \Sigma(p_i) &= \frac{(ie)^2}{2!} \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \left\{ (2k^2 + 2\Delta m_i^2) \cdot \right. \\ &\quad \cdot \frac{2p_{i\mu}}{-2p_i \cdot k + k^2 + \Delta m_i^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{2p_{i\nu}}{2p_i \cdot k + k^2 + \Delta m_i^2} - 2g_{\mu\nu} G_{S=1}^{\mu\nu} \end{aligned} \quad (3.4)$$

where we have used the notation $\Delta m_i^2 = p_i^2 - m_i^2$.

We further introduce V_{ij} and S_{ij} for vertex and self-energy corrections, respectively.

$$V_{ij} = i \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) f_\mu^i(k) f^{j\mu}(-k) \quad (3.5)$$

$$S_{ii} = V_{ii} ; V = V_{aa'} + V_{bb'} ; S = \sum_i S_{ii} \quad (3.6)$$

and for the rest of (3.4) we define a quantity

$$\Pi(p_i) = \frac{i}{2!} \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) 2k^2 f_\mu^i(-k) f^{i\mu}(k) \quad (3.7)$$

where the currents are defined by (2.11) and taken off-shell. From (3.4) we then find the (gauge-dependent) result

$$\Sigma(p_i) = -\delta m_i^2 - i \Delta m_i^2 \left(\frac{\partial \Pi}{\partial p_i^2} \Big|_{\Delta m_i^2=0} + S_{ii} \right) + \mathcal{O}((\Delta m_i^2)^2) \quad (3.8)$$

where δm_i^2 is the mass renormalization due to the first term in the expansion

$$\Pi(p_i) = \Pi_{\Delta m_i^2=0} + \frac{\Delta m_i^2}{1!} \frac{\partial \Pi}{\partial p_i^2} \Big|_{\Delta m_i^2=0} + \dots \quad (3.9)$$

and also due to the sea-gull type loops (Figs. 10b and 10d).

In (3.8) we can now identify the non-correlated contribution in (2.20) $\frac{1}{2}S_{ii}$. The factor $\frac{1}{2}$ arises because we are dealing with a wave function correction instead of the full propagator correction. This is realized here in a non-perturbative way since the non-correlated part exponentiates and we are therefore given an exact square root relation between the external particle propagator correction and the corresponding wave function correction

$$\sqrt{e^{iS}} = 1 + \frac{i}{1!} \left(\frac{1}{2} S \right) + \dots \quad (3.10)$$

The contribution $\partial\pi/\partial p_1^2$ is obtained by use of a symmetrized form of the Ward identity (see Appendix)

$$\begin{aligned} \frac{\partial \Sigma^i}{\partial p_\mu^i} = i \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \left\{ f_\mu^i(k) f^{i\mu}(-k) + \right. \\ \left. + k^2 \frac{\partial}{\partial p_i^2} f_\mu^i(k) f^{i\mu}(-k) \right\} 2 p_\mu^i = -V_\mu(p_i, p_i) \end{aligned} \quad (3.11)$$

where V_μ includes the $p_1 = p_1^i$ limit of all three graphs in Figs. 11a to 11c, which can be read off directly from the integrand in (A.7). This can also be written in the form

$$V_\mu(p_i, p_i) = \hat{V}_{ii} 2 p_\mu^i \quad (3.12)$$

where $\hat{V}_{ij}$ is defined by [compare (3.5)]

$$\hat{V}_{ij} = i \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \hat{f}_\mu^i(k) \hat{f}^{j\mu}(-k) \quad (3.13)$$

and the currents $\hat{f}_\mu$ are defined by (2.10). Thus the expression (2.20) correctly describes all vertex contributions if we include the D_μ operators in the currents.

Then by use of the definition (3.7) and summation over all external particles, we obtain

$$\begin{aligned} -i \sum_{i \in A, B} \frac{\partial \pi}{\partial p_i^2} \Big|_{\Delta m_i^2=0} = \frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \cdot \\ \cdot \left\{ \hat{J}_\mu^A(k) \hat{J}^{A\mu}(-k) + \hat{J}_\mu^B(k) \hat{J}^{B\mu}(-k) \right\}^{q=0}_{\Delta m_i^2=0} - C \end{aligned} \quad (3.14)$$

where C is given by

$$C = \frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \sum_{i \in AB} \left\{ D_\mu^i(k) f^{i\mu}(-k) + \right. \\ \left. + f_\mu^i(k) D^{i\mu}(-k) + D_\mu^i(k) D^{i\mu}(-k) \right\} \Big|_{\substack{q=0 \\ \Delta m_i^2=0}} \quad (3.15)$$

Thus (3.14) is simply obtained by adding and subtracting the same quantity C to the self-energy parts, e.g., the first term in the integrand of (3.11) and then removing all self-energy contributions to the right-hand side.

With the definition (2.20) of $\hat{R}_0$ we can then write (3.14) in the form

$$i \sum_{i \in A, B} \frac{\partial \Pi}{\partial p_i^2} \Big|_{\Delta m_i^2=0} = - \hat{R}_0^{S.P.} + C \quad (3.16)$$

where we have introduced the notation S.P. for the subtraction point $\{\Delta m_i^2=0; q=0\}$. The quantity C in (3.15) and (3.16) is then added to S_{ii} in (3.8) which gives a self-energy of the form [compare (3.6)]

$$\hat{S}_{ii} = i \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \hat{f}_\mu^i(k) \hat{f}^{i\mu}(-k) \quad (3.17)$$

which is exactly what we have in (2.20).

By means of (3.5), (3.8), (3.16), and (3.17) we are then led to the renormalized $\hat{R}$ defined by

$$\hat{R} = \hat{R}_0 - \hat{R}_0^{S.P.} \quad (3.18)$$

corresponding to renormalized quantities

$$R = R_0 - R_0^{S.P.}; \quad \hat{S}^r = \hat{S}_0^r - \hat{S}_0^{r S.P.} \quad (3.19)$$

which obey an equation of the same form as (2.23)

$$\hat{R} = R + \hat{S}^r \quad (3.20)$$

It is an easy task to control that the subtraction (3.18) applied to the exponential part of (2.26) normalizes the corresponding form factors to unity in the subtraction point. This is, of course, a direct consequence of the Ward identity

$$\Gamma_{\mu}^{(2,1)} \big|_{S.P.} = ie 2p_{\mu} \quad (3.21)$$

or

$$\frac{\partial \Gamma_{\mu}^{(2)}}{\partial p_{\mu}} \big|_{S.P.} = ie \quad (3.22)$$

where $\Gamma^{(n,m)}$ are the one-particle irreducible (1PI) vertices for n scalars and m photons.

Because of overlapping divergences a similar direct study of the full two-photon correction is not so easily done. However, the pure vertex part of the unrenormalized ansatz (2.21) agrees exactly with the corresponding part of OPE. Therefore, we cannot change the ansatz of the self-energy parts without breaking gauge invariance. The only possible change is thus an over-all subtraction at S.P.

$$\hat{P}^r = \hat{P}_0^r - \hat{P}_0^r \big|_{S.P.} \quad (3.23)$$

in order to fulfill the normalization condition (3.21). Thus the part we lost in the previous section, in generalizing the pure vertex type corrections in order to include also self-energies of higher order, was exactly the subtraction quantity. By the replacements (3.18) to (3.23) in (2.26) and (2.27) we then obtain the corresponding renormalized summed up amplitudes.

4. OFF-MASS SHELL RENORMALIZATION

The form (2.4) can also be obtained with $\Delta m_i^2 \neq 0$, starting from OPE with N attached quanta to the i^{th} prong (p_i) and summing over all permutations. However, the definitions of the currents have to be modified slightly:

$$\tilde{f}_{\mu}^i = \frac{2p_{\mu}^i + \varepsilon_i \cdot \theta_{\ell} k_{\ell} \mu_{\ell}}{2\varepsilon_i \cdot \theta_{\ell} p_i \cdot k_{\ell} + k_{\ell}^2 + \Delta m_i^2} \quad (4.1)$$

$$\tilde{D}_{\mu_\ell}^i = \tilde{f}_{\mu_\ell}^i \hat{k}_\ell^i - ie \frac{\partial}{\partial p_{\mu_\ell}^i} \quad (4.2)$$

where $\hat{k}_\ell$ is again defined by (2.13).

The new pair correlation tensor is

$$\begin{aligned} \tilde{X}_{\mu_s \mu_t}^i = & \left\{ \tilde{f}_{\mu_s}^i ie \frac{2 \varepsilon_i \theta_s k_{s \mu_t}}{\tilde{y}_{st}^i} + \tilde{f}_{\mu_t}^i ie \frac{2 \varepsilon_i \theta_t k_{t \mu_s}}{\tilde{y}_{st}^i} \right. \\ & \left. - \tilde{f}_{\mu_s}^i \tilde{f}_{\mu_t}^i \tilde{X}_{st}^i - \frac{g_{\mu_s \mu_t}}{\tilde{y}_{st}^i} \right\} \frac{1}{1 + \tilde{X}_{st}^i} \end{aligned} \quad (4.3)$$

where

$$\tilde{y}_{st}^i = 2 \varepsilon_i p_i (\theta_s k_s + \theta_t k_t) + k_s^2 + k_t^2 + 2 \Delta m_i^2 \quad (4.4)$$

$$\tilde{X}_{st}^i = (2 k_s \cdot k_t - \Delta m_i^2) (\tilde{y}_{st}^i)^{-1} \quad (4.5)$$

As in the on-shell case, we have

$$\hat{\tilde{X}}_{\mu_s \mu_t}^i = \tilde{X}_{\mu_s \mu_t}^i e^{\hat{k}_s^i + \hat{k}_t^i} = \tilde{X}_{\mu_s \mu_t}^i (1 + \hat{k}_s^i + \hat{k}_t^i) \quad (4.6)$$

and corresponding total currents

$$\hat{\tilde{J}}_\mu^Q = \sum_{i \in Q} \hat{\tilde{f}}_\mu^i ; \quad \hat{\tilde{f}}_\mu^i = \tilde{f}_\mu^i + \tilde{D}_\mu^i \quad (4.7)$$

$$\tilde{J}_\mu^Q = \sum_{i \in Q} \tilde{f}_\mu^i ; \quad \tilde{D}_\mu^Q = \sum_{i \in Q} \tilde{D}_\mu^i \quad (4.8)$$

$$\hat{\tilde{X}}_{\mu\nu}^Q = \sum_{i \in Q} \hat{\tilde{X}}_{\mu\nu}^i ; \quad Q = A \text{ or } B \quad (4.9)$$

Consequently, the $\Delta m_1^2 = 0$ limit will be denoted simply by dropping the tilde ($\sim$). With these definitions we easily obtain the corresponding quantities $\hat{R}_0$, $\hat{P}_0^r$, $\hat{R}_0$ and $\hat{S}_0^r$ by insertion of (4.1) to (4.9) into (2.20) to (2.25). The summed up amplitudes then follow by a corresponding replacement in (2.26) and (2.27).

Clearly the currents in (4.7) to (4.9) are not conserved off the mass shell. The result agrees with OPE as in the on-shell case.

In order to find the contribution from

$$\tilde{\Pi}(p_i) = \frac{i}{2!} \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) 2k^2 \tilde{f}_\mu^{i,+}(k) \tilde{f}^{i,\mu}(-k) \quad (4.10)$$

we must make an off-shell expansion

$$\tilde{\Pi}(p_i) = \tilde{\Pi}_{p_i^2 = m_i^{*2}} + \frac{p_i^2 - m_i^{*2}}{1!} \frac{\partial \tilde{\Pi}}{\partial p_i^2} \Big|_{p_i^2 = m_i^{*2}} + \dots \quad (4.11)$$

By using the term proportional to Δm_1^2 in (A.8) in this case we obtain

$$\begin{aligned} i \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \Delta m_i^2 \frac{\partial}{\partial p_i^2} \tilde{f}_\mu^{i,+}(k) \tilde{f}^{i,\mu}(-k) &\approx \\ \approx S_{ii}(p_i^2 = m_i^{*2}) - S_{ii}(p_i^2 = m_i^2) \end{aligned} \quad (4.12)$$

This shows that in sufficiently small steps, we can reach an arbitrary off-shell subtraction point $S.P. = \{p_i^2 = m_i^{*2}\}$; in analogy with (3.16), we obtain

$$i \sum_{i \in A, B} \frac{\partial \Pi}{\partial p_i^2} \Big|_{p_i^2 = m_i^{*2}} = -R_0^{S.P.} + C \quad (4.13)$$

Therefore, again we are led to the same simple renormalization prescription (3.18) to (3.23)

$$\begin{aligned} \hat{R} &= \hat{R}_0 - \hat{R}_0^{S.P.}, \quad R = \tilde{R}_0 - \tilde{R}_0^{S.P.} \\ \hat{S}^r &= \hat{S}_0^r - \hat{S}_0^{r, S.P.}, \quad \hat{P}^r = \hat{P}_0^r - \hat{P}_0^{r, S.P.} \end{aligned} \quad (4.14)$$

however, now with an arbitrary off-shell subtraction point S.P. On the left-hand sides of (4.14) we have taken here the on-shell limit which we should not do if we were interested in the off-shell Green's functions.

In fact this could be realized more or less directly by observing that the contribution from (4.11), except for the first term on the right-hand side, is exactly cancelled by subtraction at any point which fulfills

$$\Gamma_{\mu}^{(2,1)}_{S.P.} = ie(p+p')_{\mu} \quad (4.15)$$

The above calculations are therefore not necessary, but are carried through here to explain the connection with the case of on-shell renormalization.

5. RENORMALIZATION OF THE GENERALIZED LADDERS

We treat here the pure exchange graph (Fig. 2), i.e., the generalized ladders without radiative corrections. The starting point will therefore be the result of Ref. 1). With $(n+1)$ exchanged photons and by use of the correlation rearrangement (2.4) after some combinatorics, the unrenormalized amplitude can be written in the form ¹⁾

$$\begin{aligned} -iM_{n+1}(s, t) &= \frac{e^2}{(n+1)!} \int d^4x e^{iqx} \Delta_F(x) \cdot \\ &\cdot \left\{ (iU_0)^n + \binom{n}{1} i \hat{S}_0^e (iU_0)^{n-1} + \binom{n}{2} i^2 \hat{P}_0^e (iU_0)^{n-2} + \dots \right\} V_0 = \\ &= \frac{e^2}{(n+1)!} \int d^4x e^{iqx} \Delta_F(x) \cdot \left\{ 1 + \right. \\ &+ \left. \frac{i \hat{S}_0^e}{1!} \frac{\delta}{\delta(iU_0)} + \frac{i^2 \hat{P}_0^e}{2!} \frac{\delta^2}{\delta(iU_0)^2} + \dots \right\} (iU_0)^n V_0 \quad (5.1) \end{aligned}$$

where

$$U_0 = i \int \frac{d^4k}{(2\pi)^4} \Delta_F(k) e^{-ikx} J_{\mu}^A(k) J^B_{\mu}(-k) \quad (5.2)$$

$$\hat{S}_0^e = i \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) e^{-ik \cdot x} \cdot \left\{ D_\mu^A(k) J^{B\mu}(-k) + J_\mu^A(k) D^{B\mu}(-k) + D_\mu^A(k) D^{B\mu}(-k) \right\} \quad (5.3)$$

and

$$\begin{aligned} \hat{P}_0^e &= i^2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \Delta_F(k_1) \Delta_F(k_2) e^{-i(k_1+k_2) \cdot x} \\ &\cdot \left\{ J_{\mu_1}^A(k_1) D_{\mu_2}^A(k_2) D^{B\mu_1}(-k_1) J^{B\mu_2}(-k_2) + D_{\mu_1}^A(k_1) J_{\mu_2}^A(k_2) \cdot \right. \\ &\cdot J^{B\mu_1}(-k_1) D^{B\mu_2}(-k_2) + \hat{X}_{\mu_1 \mu_2}^A(k_1, k_2) \hat{J}^{B\mu_1}(-k_1) \hat{J}^{B\mu_2}(-k_2) + \hat{J}_{\mu_1}^A(k_1) \hat{J}_{\mu_2}^A(k_2) \\ &\cdot \left. \hat{X}^{B\mu_1 \mu_2}(-k_1, -k_2) + \hat{X}_{\mu_1 \mu_2}^A(k_1, k_2) \hat{X}^{B\mu_1 \mu_2}(-k_1, -k_2) \right\} \quad (5.4) \end{aligned}$$

All notations and definitions of currents, etc., are given by (2.5) to (2.17). Because of the form of V_0 (2.5), the operator $\hat{S}_0^e$ can be included in the fully factorizable part [compare (2.23)]

$$\hat{U}_0 = U_0 + \hat{S}_0^e \quad (5.5)$$

$$\hat{U}_0 = i \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) e^{-ik \cdot x} \hat{J}_\mu^A(k) \hat{J}^{A\mu}(-k) \quad (5.6)$$

Compared with OPE, the virtue of this rearrangement is that it leads to the simple expression

$$\begin{aligned} M(s, t) &= i e^2 \int d^4 x e^{iq \cdot x} \Delta_F(x) \cdot \\ &\cdot \left\{ 1 + \frac{i \hat{S}_0^e}{1!} \frac{\delta}{\delta(i\hat{U}_0)} + \frac{i^2 \hat{P}_0^e}{2!} \frac{\delta^2}{\delta(i\hat{U}_0)^2} + \dots \right\} \frac{e^{i\hat{U}_0} - 1}{i\hat{U}_0} V_0 \quad (5.7) \end{aligned}$$

or

$$M(s, t) = i c^2 \int d^4 x e^{i q x} \Delta_F(x) \cdot \left\{ 1 + \frac{i^2 \hat{p}_0^e}{2!} \frac{\delta^2}{\delta (i \hat{u}_0)^2} e^{+\dots} \right\} \frac{e^{i \hat{u}_0 - 1}}{i \hat{u}_0} V_0 \quad (5.8)$$

The graphical meaning of the three terms in (5.3) is illustrated in Figs. 12a to 12c, and an example of the graphs corresponding to the two first terms of (5.4) is shown in Fig. 13. For a more detailed graphical interpretation of all terms, we refer to Ref. 1).

Contrary to ordinary spinor QED, the generalized ladders in scalar electrodynamics are divergent from fourth order on and must be renormalized. We therefore go over to off-shell quantities (5.2) to (5.6), as in the previous Section.

This is the point at which the quartic self-coupling in (2.1)

$$\lambda_0 (\varphi^* \varphi)^2 \quad (5.9)$$

must be considered. We begin with the definition of the renormalized coupling

$$\lambda = \Gamma^{(4,0)} / S.P. \quad (5.10)$$

where $\Gamma^{(4,0)}$ is the 1PI scalar four vertex. The contribution to $\Gamma^{(4,0)}$ can then be split into two parts

$$\Gamma^{(4,0)} = \Gamma_{ex}^{(4,0)} + \Gamma_{con}^{(4,0)} \quad (5.11)$$

where Γ_{ex} stands for the pure exchange part (Fig. 2) and Γ_{con} contains, at least, one contact vertex (Fig. 4) due to the term (5.9).

The normalization condition (5.10) then simply expresses that all renormalized contributions to Γ_{ex} , i.e., from generalized ladders from fourth order on, vanish at the off-shell subtraction point S.P.

The renormalized amplitude up to fourth order is given by

$$M = ie^2 \int d^4x e^{iqx} \Delta_F(x) \left\{ 1 + \frac{i}{2!} (\tilde{U}_0 + \hat{S}_0^e) - \frac{i}{2!} (\tilde{U}_0 + \hat{S}_0^e)_{S.P.} + \dots \right\} V_0 \quad (5.12)$$

which leads to the definitions of the renormalized $\tilde{U}$ and $\hat{S}^e$

$$U = \tilde{U}_0 - \tilde{U}_0^{S.P.} \quad (5.13)$$

$$\hat{S}^e = \hat{S}_0^e - \hat{S}_0^{e S.P.} \quad (5.14)$$

or

$$\hat{U} = \hat{U}_0 - \hat{U}_0^{S.P.} \quad (5.15)$$

Again it should be said that if we were interested in the off-shell Green's function, we should not take the on-shell limit on the left-hand sides of (5.13) to (5.15). It is easy to check that the subtraction (5.15) applied to the eikonal-like part of (5.8) normalizes this according to the condition (5.10). The same prescription clearly also yields the renormalized pair correlations (and all higher correlations),

$$\hat{P}^e = \hat{P}_0^e - \hat{P}_0^{e S.P.} \quad (5.16)$$

which leads to the summed up renormalized amplitude

$$\begin{aligned} M &= ie^2 \int d^4x e^{iqx} \Delta_F(x) \left\{ 1 + \frac{i \hat{S}^e}{1!} \frac{\delta}{\delta(iu)} + \frac{i^2 \hat{P}^e}{2!} \frac{\delta^2}{\delta(iu)^2} + \dots \right\} \frac{e^{iu} - 1}{iu} V_0 = \\ &= ie^2 \int d^4x e^{iqx} \Delta_F(x) \int_0^1 d\theta \left\{ 1 + \frac{i}{1!} \hat{S}^e(x) \theta + \frac{i^2}{2!} \hat{P}^e(x) \theta^2 + \dots \right\} e^{i\theta u(x)} V_0 \end{aligned} \quad (5.17)$$

In the last line we have made use of

$$\frac{e^{iu} - 1}{iu} = \int_0^1 d\theta e^{i\theta u} \quad (5.18)$$

We are now in a position to construct the full amplitude.

6. THE FULL AMPLITUDE

We combine here the higher order vertex and self-energy results (2.19), renormalized by means of (3.18) to (3.23) or in the off-shell case by (4.14), with that for exchanges (5.1) renormalized by use of (5.13) to (5.16). With $(n+1)$ exchanged photons (Fig. 2) and m other virtual photons (Fig. 1) of the above type, we obtain the combined renormalized amplitude

$$\begin{aligned} -i M_{n+1, m}(s, t) &= \frac{e^2}{(n+1)! m!} \int d^4x e^{iqx} \Delta_F(x) \cdot \\ &\cdot \left\{ (iu)^n (iR)^m + \binom{n}{1} i \hat{S}^e (iu)^{n-1} (iR)^m + \binom{m}{1} i \hat{S}^r \cdot \right. \\ &\cdot (iu)^n (iR)^{m-1} + \binom{n}{2} i^2 \hat{P}^e (iu)^{n-2} (iR)^m + n \cdot m i^2 \hat{P}^{er} \cdot \\ &\cdot (iu)^{n-1} (iR)^{m-1} + \binom{m}{2} i^2 \hat{P}^r (iu)^n (iR)^{m-2} + \dots \left. \right\} V_0 = \\ &= \frac{e^2}{(n+1)! m!} \int d^4x e^{iqx} \Delta_F(x) \left\{ 1 + \frac{i}{1!} \left(S^e(x) \frac{\delta}{\delta(iu)} + \right. \right. \\ &+ S^r \frac{\delta}{\delta(iR)} \left. \right) + \frac{i^2}{2!} \left(\hat{P}^e(x) \frac{\delta^2}{\delta(iu)^2} + 2 \hat{P}^{er}(x) \frac{\delta}{\delta(iu)} \frac{\delta}{\delta(iR)} + \right. \\ &+ \hat{P}^r \frac{\delta^2}{\delta(iR)^2} \left. \right) + \dots \left. \right\} (iu)^n (iR)^m V_0 \end{aligned} \quad (6.1)$$

In this combined amplitude there is one new term $\hat{P}^{er}(x)$ which accounts for correlations between exchanged photons (Fig. 2) and vertex or self-energy photons (Fig. 1). With the use of (2.4) and the topology required by the Lagrangian (2.1), its form is easily found to be

$$\begin{aligned} \hat{P}_0^{er} = & \frac{i^2}{2!} \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \Delta_F(k_1) \Delta_F(k_2) e^{-i k_2 \cdot x} \\ & \cdot \left\{ \hat{J}_{\mu_1 \mu_2}^A(k_1, k_2) \hat{J}_{(-k_1)}^{A\mu_1} \hat{J}_{(-k_2)}^{B\mu_2} + \hat{J}_{\mu_1 \mu_2}^B(-k_1, -k_2) \hat{J}_{(k_1)}^{B\mu_1} \hat{J}_{(k_2)}^{A\mu_2} \right. \\ & + \hat{J}_{\mu_1}^B(k_1) \hat{D}_{\mu_2}^A(k_2) \hat{D}_{(-k_1)}^{B\mu_1} \hat{J}_{(-k_2)}^{B\mu_2} + \\ & \left. + \hat{D}_{\mu_1}^A(k_1) \hat{J}_{\mu_2}^A(k_2) \hat{J}_{(-k_1)}^{A\mu_1} \hat{D}_{(-k_2)}^{B\mu_2} + (k_1 \rightarrow -k_1) \right\} \end{aligned} \quad (6.2)$$

The renormalized quantity $\hat{P}^{er}$ follows from the prescriptions in Sections 3 and 4, and is defined by

$$\hat{P}^{er} = \hat{P}_0^{er} - \hat{P}_0^{er}{}_{S.P.} \quad (6.3)$$

The graphical interpretations of the last two terms in (6.2) are illustrated in Figs. 14a and 14b. We then easily sum up (6.1) to the "full" amplitude

$$\begin{aligned} M(s, t) = & i e^2 \int d^4 x e^{i \not{x}} \Delta_F(x) \left\{ 1 + \frac{i}{1!} (\hat{S}^e \frac{\delta}{\delta(iu)} + \hat{S}^r) + \right. \\ & \left. + \frac{i^2}{2!} (\hat{P}^e \frac{\delta^2}{\delta(iu)^2} + 2 \hat{P}^{er} \frac{\delta}{\delta(iu)} + \hat{P}^r) + \dots \right\} \frac{e^{iu-1}}{iu} e^{iR} V_0 \end{aligned} \quad (6.4)$$

The result can be further generalized by replacing bare photon propagators in (6.4) by dressed propagators (Fig. 17). This does not disturb the factorization machinery. For example, including fourth order corrections in the photon propagator $\Delta_F^4(x)$, we are able to reproduce sixth order OPE exactly, and of course IR effects to all orders.

It is then an easy task to find the summed up amplitude for the crossed channels (Figs. 15a and 15b) and the first order contact interaction (Fig. 16). We give here the form of the latter ($x=0$)

$$M(s, t)_{con} = \lambda \left\{ 1 + \frac{i}{1!} (\hat{S}^e(0) + \hat{S}^r) + \frac{i^2}{2!} (\hat{P}^e(0) + 2\hat{P}^{er}(0) + \hat{P}^r) + \mathcal{O}(\alpha^3 \lambda) \right\} e^{i\mathcal{U}(0)} e^{iR} \quad (6.5)$$

where all notations are the same as in (6.4). By addition of (6.4), (6.5) and cross-terms, our subtraction point independent result reads

$$\begin{aligned} M(s, t) = & \lambda e^2 \int d^4x e^{i q \cdot x} \Delta_F'(x) \int_0^1 d\theta e^{i\theta \mathcal{U}(x)} e^{iR} \cdot \\ & \cdot \left\{ 1 + \frac{i}{1!} (\hat{S}^e(x) \theta + \hat{S}^r) + \frac{i^2}{2!} (\hat{P}^e(x) \theta^2 + 2\hat{P}^{er}(x) \theta + \hat{P}^r) \right\} \\ & + (\text{cross-terms}) + \lambda \left\{ 1 + \frac{i}{1!} (\hat{S}^e(0) + \hat{S}^r) + \right. \\ & + \frac{i^2}{2!} (\hat{P}^e(0) + 2\hat{P}^{er}(0) + \hat{P}^r) \left. \right\} \cdot e^{i\mathcal{U}(0)} e^{iR} + \\ & + \mathcal{O}(\alpha^4, \lambda^2) \end{aligned} \quad (6.6)$$

It should be remembered, however, that the choice of renormalized λ is completely at our disposal. In passing we also notice that all operators, in the term proportional to λ in (6.6), can be dropped because they operate on the constant λ .

Higher order vacuum graphs (Fig. 18) are symmetrically included according to the rule illustrated in Fig. 18, where everything inside the square is treated as an effective vertex. Such a vertex contributes to the same order of correlation as the number of photons which cross the boundary of this square.

The inclusion of such graphs also forces us to generalize the Low theorem one step further. In this case the starting point should be the dressed photon propagator in place of the bare one in the single photon exchange graph. The leakage currents obtained in this case are then found to correspond to graphs of the type depicted in Fig. 18.

7. SUMMARY

We have shown that OPE must, except in certain cases, be rearranged into a correlation expansion in order to uniquely ¹³⁾ carry out the infrared exponentiation in a translation and gauge invariant way.

From the form of these correlations, we find that inclusion of single correlation effects enables us to make exact e^4 order calculations as compared to the full OPE. The exact e^6 order result is obtained by inclusion of pair correlation effects, and so on. In principle, this can be continued to any desired order, following the recursive definition of correlations in the Appendix and including the effects of scalar loops to a sufficiently high order. Clearly, we can also apply the same technique to real photon emission, but will not do so here.

As mentioned in the Introduction, we can expect a rapidly decreasing correlation expansion provided that the coupling constant e is not too strong. However, clearly also the effective numbers which multiply the various powers of e are important in this respect, as we can understand from an interesting example of exponentiation of IR divergences in dual models ¹⁴⁾. The IR divergences there are caused by dilatons which factorize completely because the correlations vanish with the phase space, irrespective of the value of the coupling constant.

In discussing the strength of the correlations, it should also be mentioned that we can expect non-leading logarithmic enhancement of α due to ultra-violet divergences.

To obtain an improvement compared to earlier results, it is not necessary to include pair correlation effects as we have done here. Already the inclusion of single correlations and, of course, the IR divergent factorizable part, which like in (2.26) and (5.8) can be compromised in an operatorial exponential form

$$M(s, t) = ie^2 \int d^4x e^{i q x} \Delta_F'(x) \int_0^1 d\theta \cdot e^{i \theta \hat{U}(x)} e^{i \hat{K}} V_0 + (\text{cross-terms}) + e^{i U(\omega)} e^{i R} 1 \quad (7.1)$$

is an improvement compared to the eikonal approximation of the generalized ladders ⁹⁾. Clearly also a corresponding improvement of the eikonal form factors is obtained, i.e., like (1.1b) in the case of QED ⁵⁾.

We close this discussion by giving a formula for the summed up pair effects ($\hat{P}$), double pair effects ($\hat{P}^2$), etc. In the special case of exchanges, we obtain

$$M(s, t) = ie^2 \int d^4x e^{iq \cdot x} \Delta_F(x) \int_0^1 d\theta e^{i\theta \hat{U}(x) + \frac{i\theta^2}{2} \hat{P}(x)} \quad (7.2)$$

which might be of interest in massless scalar electrodynamics ¹⁵⁾. This is because $\hat{P}$ contains the double sea-gull which is IR divergent for forward scattering. This causes an attractive force not unlike the case of superconductivity ¹⁶⁾. In Fig. 20 we have illustrated the graphical content of formula (7.2).

Further conclusions on this rearrangement are developed in the application to spinor QED ¹⁷⁾, after which we intend to apply this to the problem of Coulomb interference. As far as we can see that problem cannot be solved without this technique, i.e., via the generalized Low theorem. From the behaviour of electromagnetic form factors, we know that this IR structure, obtained in a "pointlike" theory, must be the IR limit also if we are dealing with extended sources. It is therefore conceivable that this is a possible way to introduce particles with structure or corresponding solutions into an IR divergent theory with local currents.

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THE REARRANGEMENT

We start here from the on-shell amplitude depicted in Fig. 5 with two photons attached to the i^{th} prong. In exact form the corresponding part of the amplitude reads

$$M_{\mu_1 \mu_2} = (ie)^2 \sum_{\text{perm}} \frac{(2p_i + \varepsilon_i \cdot \theta, k_1)_{\mu_1} (2p_i + \varepsilon_i \cdot 2\theta, k_1 + \varepsilon_i \cdot \theta_2 k_2)_{\mu_2}}{y_1^i y_{12}^i} \cdot \left(1 - \frac{x_{12}^i}{1 + x_{12}^i}\right)$$

$$y_{12}^i = 2\varepsilon_i \cdot p_i (\theta, k_1 + \theta_2 k_2) + k_1^2 + k_2^2$$

$$y_1^i = 2\varepsilon_i \cdot \theta, p_i \cdot k_1 + k_1^2; \quad x_{12}^i = 2k_1 \cdot k_2 (y_{12}^i)^{-1}$$

(A.1)

Notations are the same as in Section 2. The last factor in (A.1) is invariant under permutations and the rest can be split into two parts

$$M_{\mu_1 \mu_2}^{(1)} = (ie)^2 \sum_{\text{perm}} \frac{(2p_i + \varepsilon_i \cdot \theta, k_1)_{\mu_1} (2p_i + \varepsilon_i \cdot \theta_2 k_2)_{\mu_2}}{y_1^i y_{12}^i}$$

(A.2)

$$M_{\mu_1 \mu_2}^{(2)} = (ie)^2 \sum_{\text{perm}} \frac{(2p_i + \varepsilon_i \cdot \theta, k_1)_{\mu_1} (2\varepsilon_i \cdot \theta, k_1)_{\mu_2}}{y_1^i y_{12}^i}$$

(A.3)

Summation over permutations in (A.2) gives the product $f_{\mu_1}^i f_{\mu_2}^i$ where f_{μ}^i is defined by (2.11). After summation over permutations, multiplication by the above factor $(1 + x_{12}^i)^{-1}$ and addition of sea-gull terms, the amplitude reads

$$M_{\mu_1 \mu_2}^{i \text{ exact}} = f_{\mu_1}^i f_{\mu_2}^i + (f_{\mu_1}^i i\epsilon \frac{2\varepsilon_i k_{1\mu_2}}{y_{12}^i} + f_{\mu_2}^i i\epsilon \frac{2\varepsilon_i k_{2\mu_1}}{y_{12}^i} - f_{\mu_1}^i f_{\mu_2}^i \chi_{12}^i - (i\epsilon)^2 \frac{g_{\mu_1 \mu_2}}{y_{12}^i}) \frac{1}{1 + \chi_{12}^i} = f_{\mu_1}^i f_{\mu_2}^i + \chi_{\mu_1 \mu_2}^i \quad (\text{A.4})$$

which defines the pair correlation tensor. Repeating this for three photons gives

$$M_{\mu_1 \mu_2 \mu_3}^{i \text{ exact}} = \prod_{l=1}^3 f_{\mu_l}^i + \sum_{\substack{S \subset \{1,2,3\} \\ |S|=2}} \chi_{\mu_S \mu_{S^c}}^i \prod_{l \in S^c} f_{\mu_l}^i + T_{\mu_1 \mu_2 \mu_3}^i \quad (\text{A.5})$$

which defines the triple correlation tensor. Similarly, the correlation tensor of rank four is uniquely defined by exact fourth order amplitude and the lower order definitions. This defines, in a recursive way, the general N^{th} order correlation tensor. However, here we just write out the formula explicit for the lowest orders, i.e., including pair correlations

$$M_{\mu_1 \dots \mu_N}^i = \left\{ \prod_{l=1}^N f_{\mu_l}^i + \sum_{\substack{S \subset \{1, \dots, N\} \\ |S|=2}} \chi_{\mu_S \mu_{S^c}}^i \prod_{l \in S^c} f_{\mu_l}^i + \text{higher order correlations} \right\} \quad (\text{A.6})$$

For the off-shell situation $\Delta m_1^2 = p_1^2 - m_1^2 \neq 0$ we obtain a result of the same form, depending on the off-shell currents given in Section 4. By use of the generalized Low theorem ⁷⁾, it is then an easy task to incorporate the recoil from the vertex at the single photon exchange (Fig. 5), which gives the formula (2.4).

The Ward identity

Starting from the symmetrized self-energy expression (3.2), the Ward identity reads

$$\begin{aligned} \frac{\partial \Sigma}{\partial p_\mu^i} = & \frac{(ic)^2}{2!} \int \frac{d^4 k}{(2\pi)^4} \Delta_F(k) \left\{ \frac{4(2p_\mu^i - k_\mu)}{-2p_i \cdot k + k^2 + \Delta m_i^2} + \right. \\ & + \frac{4(2p_\mu^i + k_\mu)}{2p_i \cdot k + k^2 + \Delta m_i^2} - \frac{(2p_i - k)^2}{(-2p_i \cdot k + k^2 + \Delta m_i^2)^2} (2p_\mu^i - 2k_\mu) \\ & \left. - \frac{(2p_i + k)^2}{(2p_i \cdot k + k^2 + \Delta m_i^2)^2} (2p_\mu^i + 2k_\mu) \right\} = V_\mu(p_i, p_i) \end{aligned} \quad (A.7)$$

where V_μ is defined by (3.12).

The vertex defined by (A.7) is everywhere contracted with some other vertex via $G_\xi^{\mu\nu}$ defined by (3.3). For simplicity then, we can choose the transverse gauge and drop all terms linear in k_μ , as we did in (3.4).

By the use of

$$\begin{aligned} & (f(+k))^2 + (f(-k))^2 + 2 f_\mu(+k) f^\mu(-k) = (f_\mu(+k) + f_\mu(-k))^2 \\ & \cdot G_{g=1}^{\mu\nu} (f_\nu(+k) + f_\nu(-k)) = 2(k^2 + \Delta m^2) \cdot \frac{2p_\mu G_{g=1}^{\mu\nu}}{2p \cdot k + k^2 + \Delta m^2} \\ & \cdot \frac{2p_\nu}{-2p \cdot k + k^2 + \Delta m^2} \cdot \left(\frac{1}{2p \cdot k + k^2 + \Delta m^2} + \frac{1}{-2p \cdot k + k^2 + \Delta m^2} \right) = \\ & = -2(k^2 + \Delta m^2) \frac{\partial}{\partial p^2} \left(f_\mu(+k) G_{g=1}^{\mu\nu} f_\nu(-k) \right) + \frac{2(k^2 + \Delta m^2) \cdot 4}{2p \cdot k + k^2 + \Delta m^2} \\ & \cdot \frac{1}{-2p \cdot k + k^2 + \Delta m^2} = -2(k^2 + \Delta m^2) \frac{\partial}{\partial p^2} f_\mu(k) f^\mu(-k) + \\ & + 4 \left(\frac{1}{2p \cdot k + k^2 + \Delta m^2} + \frac{1}{-2p \cdot k + k^2 + \Delta m^2} \right) \end{aligned} \quad (A.8)$$

which we insert in (A.7), we obtain (3.11).

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FIGURE CAPTIONS

- Fig. 1 : All vertex and self-energy corrections without scalar loops.
- Fig. 2 : The generalized ladders.
- Fig. 3 : All graphs without scalar loops.
- Fig. 4 : Inclusion of first order quartic self coupling.
- Fig. 5 : Tree graph of arbitrary high order.
- Fig. 6 : Example of vertex and self-energy corrections without scalar loops.
- Fig. 7 : Example of pair correlation effect from the two last terms in (2.21).
- Fig. 8 : Single correlation effects from the two first terms in (2.25).
- Fig. 9 : Pair correlation effects from the three first terms in (2.21).
- Fig. 10 : Self-energy corrections.
- Fig. 11 : Sea-gull effects in the self-energy corrections.
- Fig. 12 : Single correlation effects among exchanged quanta.
- Fig. 13 : Pair correlation effects among exchanged quanta.
- Fig. 14 : Pair correlation effects among exchanged and self-energy quanta.
- Fig. 15a : Example of s-t crossed graph.
- Fig. 15b : Example of s-u crossed graph.
- Fig. 16 : First order contact interaction.
- Fig. 17 : Dressed photon propagator.
- Fig. 18 : Inclusion of scalar loops.
- Fig. 19 : Double sea-gull exchange.
- Fig. 20 : Iterated double sea-gull exchanges.

FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6

FIG. 7

FIG. 8

(a)

(b)

(c)

FIG. 9

(a)

(b)

(c)

(d)

FIG.10

FIG.11

FIG.12

FIG.13

FIG.14

(a)

(b)

FIG.15

FIG.16

FIG.17

FIG.18

FIG.19

FIG.20