Ref.TH.2172-CERN

ANGULAR MOMENTUM, MAGNETIC MONOPOLES AND GAUGE THEORIES

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A B S T R A C T

When an exact gauge symmetry group H is embedded in a larger group G which is broken back to H spontaneously, there are situations in which the angular momentum operator acquires a new term, $\underline{t}$, where t_1, t_2, t_3 generate an angular momentum subalgebra of G . Thus internal and external symmetries are coupled. It is shown that the radial component of $\underline{t}$ is a linear combination of the generators of H . When H consists of a colour group and an electromagnetic $U(1)$ generated by the electric charge operator Q which is a colour singlet, then the coefficient of Q in this decomposition is minus the magnetic charge occurring in the situation. For general H the structure of the decomposition of the radial component of $\underline{t}$ into generators of H completely determines the topological quantum numbers of the solution considered. The result provides a useful new tool for the model building of monopoles.

Ref.TH.2172-CERN

31 May 1976

1. - INTRODUCTION

Recently new insights into magnetic monopoles for an exact gauge symmetry group H have been found by embedding H into a larger gauge group G which is broken back to H by the vacuum ^{1),2)}. One of the many intriguing features has been the coupling of external, space-time symmetries to internal (G) symmetries. This coupling arose because the solutions considered were invariant under the action of $\underline{J} + \underline{t}$ where $\underline{J}$ is the conventional angular momentum operator and $\underline{t}$ a set of three angular momenta selected from the generators of G ²⁾. The physical significance of this invariance is that the monopole solutions so considered are spherically symmetric and therefore spinless ³⁾: after rotation with the normal $\underline{J}$ the manifest invariance is restored by a gauge rotation with $\underline{t}$. Thus the total effective angular momentum is $\underline{J} + \underline{t}$ ³⁾. The structure of $\underline{t}$ has to do with the asymptotic behaviour of the Higgs' field, that is the topological quantum number or magnetic charge. It is our purpose to clarify this relationship.

First we show that the radial component of $\underline{t}$ is a linear combination of the generators of H (with the particular linear combination depending on the solution considered). A rotation through 4π generated by this radial component defines a closed path in the group H whose homotopy class is shown to be the "topological quantum number" in its most general form ¹⁾. This means that the topological quantum numbers of the spherically symmetric solutions are completely determined by the decomposition of the radial component of angular momentum. This is our main result.

We illustrate this general idea by considering H to be locally like $U(1)_{EM} \times K_{\text{colour}}$ (i.e., it has the same Lie algebra but not necessarily the same global structure). This is probably the situation in nature where $K = SU(3)$. Then we find that the coefficient of Q , the electric charge operator, is minus g where g is the magnetic charge of the spherically symmetric static monopole considered :

$$\hat{\underline{t}} \cdot \underline{t} = -gQ + \sum \beta_\alpha k_\alpha \quad (1.1)$$

The last term is the linear combination of colour generators.

This can be regarded as a generalisation of the old result (for $K=1$) ⁴⁾ that for a particle of electric charge q moving in the Coulomb magnetic field of a magnetic monopole charge g the radial component of the total angular momentum is $-gq$.

Since $\hat{f} \cdot \hat{t}$ is an angular momentum component, rotation through 4π with it must give unity and so it follows from (1.1) that

$$e^{4\pi i g Q} = k \text{ (an element of } K) \quad (1.2)$$

This is the generalized quantization condition, proved generally and discussed in a previous paper ⁵⁾, now rederived for spherically symmetric situations.

Condition (1.2) was a necessary condition for monopoles. It gives the range of values g may have but it does not guarantee that we can construct a model with a given value of g . Experience with model building indicates that given G , spherically symmetric monopoles can be constructed only for certain values of g ^{1),2)}. Other values of g which are additive combinations of these special values can be realized by assembling the constituent monopoles in a way which, inevitably, is no longer spherically symmetric.

The significance of Eq. (1.1) is manifold. Read one way it tell us immediately, in terms of the structure of G , what values of g spherically symmetric monopoles may have and so will be an important tool in model building. Given the Lagrangian describing G broken spontaneously to H , H is of course essentially determined (up to an isomorphism). But there may be many possible (non-isomorphic) choices of R , the rotation group generated by the three angular momenta $\hat{t}$ and this choice is at our disposal. For a given choice, we just read off

$$g = - \frac{\text{Tr} (Q \hat{t} \cdot \hat{t})}{\text{Tr} (Q^2)} \quad (1.3)$$

It seems surprising that g depends on so little.

Read conversely (1.1) indicates how the topological quantum numbers determine the structure of $\hat{t}$.

We should like to emphasize that the decomposition of radial angular momentum into generators of H has a meaning independently of G . Now the concept of spherical symmetry to be used depends very much on G . We therefore conjecture that the decomposition of radial angular momentum into generators of H has a fundamental significance for any solution whether or not it is spherically symmetric.

Section 2 establishes the framework, shows that the radial component of $\underline{t}$ can be linearly decomposed into generators of H and that the coefficient of Q is $-g$ if $H = "U(1) \times K"$. Section 3 illustrates how Eq. (1.1) works in the known examples of monopole solutions, and how it can be used to get information about new solutions. Section 4 returns to the general theory and shows that the structure of the radial component of $\underline{t}$ determines the topological quantum numbers for general H .

2. - THE DECOMPOSITION OF RADIAL ANGULAR MOMENTUM

As in previous papers ^{2),5)} we consider the Yang-Mills gauge theory of a compact semi-simple group G , and suppose that the Lagrangian is such that, when the Higgs' field ψ is in its ground state, its little group is H . H is then the resultant exact gauge symmetry and is not yet assumed to be $"U(1) \times K"$. Because of the assumed structure of G we can choose a basis for its generators $\underline{T}$ such that the structure constants are totally anti-symmetric and the metric is the unit matrix.

At large distances from the centre of the monopole the Higgs' field is in its ground state : so

$$D(X)\psi = 0 \Leftrightarrow X = \sum x_\alpha h_\alpha \quad (2.1)$$

i.e., X is a linear combination of the generators h_α of H if the representative of X appropriate to ψ , $D(X)$, annihilates ψ (ψ may be in any representation of G , even reducible). Further the covariant derivative of ψ vanishes :

$$\mathcal{D}^\mu \psi \equiv \partial^\mu \psi - ie D(\underline{W}^\mu \cdot \underline{I})\psi = 0 \quad (2.2)$$

where $\underline{W}^\mu$ is the gauge potential, a vector in the self-adjoint representation of G . All fields are assumed to be non-singular at large distances (i.e., string free).

Now we introduce the extra assumptions that the solution is everywhere static :

$$\frac{\partial \psi}{\partial t} = 0 \quad (2.3)$$

and spherically symmetric. We formulate this as :

$$(-i(\underline{r} \wedge \nabla) + D(\underline{t}))\psi = 0 \quad (2.4)$$

$[\underline{r}, \underline{t}]$ denote three-vectors, $\underline{T}$, etc. (dim. G) vectors], where $\underline{t} = (t_1, t_2, t_3)$ are three generators of G satisfying

$$[t_i, t_j] = i\epsilon_{ijk} t_k \quad (2.5)$$

thereby generating an $SO(3)$ or $SU(2)$ subgroup of G which we shall call R . As explained before (2.4) implies spherical symmetry and we shall assume (without proof), the converse : if a solution is spherically symmetric then there exists a non-singular gauge such that (2.4) is satisfied for some choice of R .

The analogous conditions to (2.3) and (2.4) are also assumed for the gauge potentials $\underline{W}^\mu$. In the analogue of (2.4) there is an extra spin term acting on the space indices of $\underline{W}^\mu$ ($\mu = 1, 2, 3$).

The t_i need not constitute an irreducible representation of (2.5) but the $D(t_i)$ must have integral eigenvalues since $-i\underline{r} \wedge \nabla$ does and their sum has zero eigenvalues.

Since the orbital angular momentum $-i\underline{r} \wedge \nabla$ has zero radial component (2.4) implies

$$D(\hat{\underline{r}} \cdot \underline{t})\psi = 0$$

So we have our first important deduction : when ψ is in its ground state (2.1) implies that $\hat{\underline{r}} \cdot \underline{t}$ is a linear combination of the generators h_α of H,

$$\hat{\underline{r}} \cdot \underline{t} = \sum \gamma_\alpha h_\alpha \quad (2.6)$$

Actually if our monopole solution is to have any angular dependence at large distances $\hat{\underline{r}} \cdot \underline{t}$ must be the unique common generator of H and R.

We shall now analyze (2.2) regarded as an equation for the gauge potential $\underline{W}^\mu$ in terms of a given Higgs' field ψ . The remarkable fact is that the spherical symmetry condition (2.4) enables us to find the general solution to (2.2) (in a suitable gauge) :

$$\underline{W}^\mu \cdot \underline{I} = (\underline{r} \wedge \underline{t})^\mu / er^2 + \sum \omega_\alpha^\mu h_\alpha \quad (2.7)$$

where the ω_α^μ are arbitrary functions.

The first step is to choose a gauge in which

$$\underline{r} \wedge \underline{W}_\mu = 0$$

Then at large distances (2.2) implies that

$$\underline{r} \cdot \nabla \psi = 0 \quad (2.8)$$

This is achieved by

$$\psi(\underline{x}) \rightarrow \psi'(\underline{r}) = T \exp\left(ie \int_{\underline{r}}^{\underline{r}_0} d\underline{x}_\mu \underline{W}^\mu \cdot \underline{I}\right) \psi(\underline{x})$$

where the contour of integration runs along the radius vector from $\underline{r}$ to the point $\underline{r}_0$ on the unit sphere. Owing to the assumed spherical symmetry of the gauge potential $\underline{W}^\mu$ and the form of the integration contour, the spherical symmetry property of ψ , in the form (2.4) is preserved in the new gauge.

Taking the vector product of (2.4) with $\underline{r}$ yields

$$\underline{r} \wedge (\underline{r} \wedge \nabla) \psi \equiv (\underline{r}(\underline{r} \cdot \nabla) - r^2 \nabla) \psi = -ic D(\underline{r} \wedge \underline{t}) \psi$$

But by (2.8) we see that (2.7) is indeed the general solution to (2.2).

Now let us specialize to the case that $H = "U(1)_{EM} \times K_{\text{colour}}"$. The inverted commas mean that the direct product is to be understood as referring to the local structure near the identity and not necessarily to the global structure [there may be points of intersection of the $U(1)$ and K other than the identity ⁵⁾]. Thus the generators of H , the h_α , now consist of the electric charge

$$Q = e \underline{\phi} \cdot \underline{I} ; \quad \phi^2 = 1 \quad (2.9)$$

which is a colour singlet, i.e. commutes with the colour generators

$$k_\alpha = \underline{k}_\alpha \cdot \underline{I} \quad \alpha=1,2,\dots \dim K. \quad (2.10)$$

which themselves close under commutation and generate the "colour group" K . Without loss of generality we can take

$$\underline{\phi} \cdot \underline{k}_\alpha = 0 \quad \alpha=1,2,\dots \dim K. \quad (2.11)$$

as was shown in Ref. 5). Now (2.6) means that $\hat{f} \cdot \underline{t}$ must be a linear combination of Q and the k_α as expressed by Eq. (1.1). We now proceed to show that the coefficient of Q is indeed minus the magnetic charge.

In the $H = "U(1) \times K"$ situation it was shown ⁵⁾ that if $\underline{w}_0^\mu$ is a particular solution of (2.2) then the gauge field tensor

$$\underline{G}^{\mu\nu} = \underline{G}_0^{\mu\nu} + (\partial^\mu A^\nu - \partial^\nu A^\mu) \underline{\phi} + \sum B_\alpha^{\mu\nu} \underline{k}_\alpha$$

Here $\underline{G}_0^{\mu\nu}$ is constructed solely out of $\underline{w}_0^\mu$ and determines the magnetic charge $\tilde{g}$ since the remaining terms do not contribute by (2.11) and Stokes theorem :

$$\tilde{g} \equiv \frac{1}{4\pi} \int_S d\sigma_{\mu\nu}^* \underline{\phi} \cdot \underline{G}^{\mu\nu} = \frac{1}{4\pi} \int_S d\sigma_{\mu\nu}^* \underline{\phi} \cdot \underline{G}_0^{\mu\nu} \quad (2.12)$$

S is a sphere centre O , large enough for (2.1) and (2.2) to be valid thereon. We can take $\underline{w}_0^\mu$ to be given by (2.7) with all $w_\alpha^\mu = 0$. Then

$$\begin{aligned} \underline{I} \cdot \underline{G}_0^{\mu\nu} &= - \frac{\epsilon_{\mu\nu\sigma}}{r^2} \hat{r}_\sigma \frac{\hat{r} \cdot \underline{t}}{e} \\ &= \frac{\epsilon_{\mu\nu\sigma}}{r^2} \hat{r}_\sigma \left(g \underline{\phi} \cdot \underline{I} - \frac{1}{e} \sum \beta_\alpha k_\alpha \right) \end{aligned}$$

by (1.1). Inserting this into (2.12) and integrating we find $\tilde{g}=g$ and have thereby established the result (1.1) with g interpreted as the magnetic charge.

3. - CHECKS AND ILLUSTRATIONS OF THE RADIAL ANGULAR MOMENTUM DECOMPOSITION

Before developing the general theory further we shall look at illustrative examples.

First consider $H=U(1)_{EM}$. The simplest possibility is $G=SU(2)$ (as considered by 't Hooft¹⁾). It follows that $R=G$. So by (1.1)

$$\hat{r} \cdot \underline{t} = -g Q$$

But the minimum eigenvalue of $\hat{r} \cdot \underline{t}$ is $\frac{1}{2}$, and so if the smallest eigenvalue of Q is e_0 we have $g = \frac{1}{2}e_0$, i.e., the spherically symmetric solution must have one Dirac unit of monopole charge, as indeed was found¹⁾. To get spherically symmetric monopoles with higher values of g we must consider groups G larger than $SU(2)$, with subgroups R whose angular momentum is larger than $\frac{1}{2}$.

Now we shall see how (1.1) is satisfied for each of the six spherically symmetric point monopole solutions presented in Ref. 2) when $G=SU(3)$, $H=U(2) = "U(1) \times SU(2)"$. In that situation the Higgs' field was $\frac{1}{2}\underline{\phi} \cdot \underline{\lambda} = Q/e$. It was always expanded as

$$Q(r) = \beta \underline{t} \cdot \hat{r} + B$$

The first term always satisfies (2.4) (for any group). B, the remainder was orthogonal to $\underline{t} \cdot \hat{\underline{t}}$ but does not affect the magnetic charge whatever it is, as we shall see.

Comparing with (1.1)

$$g = \text{Tr}(Q \underline{t} \cdot \hat{\underline{t}}) / \text{Tr}(Q^2) = \beta \text{Tr}((\underline{t} \cdot \hat{\underline{t}})^2) / \text{Tr}(Q^2)$$

There were two choices of R considered : if $\underline{t} = (\frac{1}{2}\lambda_1, \frac{1}{2}\lambda_2, \frac{1}{2}\lambda_3)$ then $\hat{\underline{t}} \cdot \underline{t} = \frac{1}{2}\hat{\underline{t}} \cdot \underline{\lambda} = \psi_1$ in the notation of Ref. 2), so $\text{Tr}((\hat{\underline{t}} \cdot \underline{t})^2) = \frac{1}{2}$. The other choice was $\underline{t} = (\lambda_7, -\lambda_5, \lambda_2)$ so $\hat{\underline{t}} \cdot \underline{t} = -\psi_2$ in the notation of Ref. 2), and $\text{Tr}((\hat{\underline{t}} \cdot \underline{t})^2) = 2$. Now β can be read off the first row of the table of Ref. 2) and g calculated from the above. $\tilde{g}$ the monopole charge is read off the fourth row and in each case coincides, as it should.

Now we shall illustrate the use of (1.1) by showing that if we embed $U(N) = "SU(N) \times U(1)"$ in $SU(N+1)$ we can only construct spherically symmetric monopoles with $|ge_0| \leq \frac{1}{2}$ where e_0 is the minimum possible electric charge.

We must have

$$Q = e_0 \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & -N \end{pmatrix} = e_0 I - (N+1)e_0 \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0_1 \end{pmatrix}$$

in diagonal form. $\hat{\underline{t}} \cdot \underline{t}$ commutes with Q and so can be simultaneously diagonalized. Since they are angular momentum eigenvalues its diagonal elements must be an integer/2. Let $m/2$ be the bottom diagonal element and let it be part of an irreducible representation of dimension d . Then

$$2|m|/2 + 1 \leq d \leq N+1$$

Hence $|m| \leq N$. But

$$g = -\text{Tr}(\hat{\underline{t}} \cdot \underline{t} Q) / \text{Tr}(Q^2) = (N+1)e_0 \frac{1}{2} m / (e_0^2 (N+1)^2) \\ = m / (2e_0 N)$$

Hence $|ge_0| = |m| / (2N) \leq \frac{1}{2}$ as claimed.

4. - $\hat{f}, t$ AND TOPOLOGICAL QUANTUM NUMBERS : ANY H

Once R is given Eqs. (2.4) and (2.8) completely determine the Higgs' field $\psi(r)$ at large distances in terms of its value in one direction $\psi(r\hat{k})$, say. But the general definition of topological quantum number is as the homotopy class of the map, $\psi(\underline{r})$, from a large sphere (in space) to the vacuum manifold G/H (the permissible values ψ may take when it minimizes its self-interaction). Mathematically this is written

$$\pi_2(G/H)$$

The element of this group corresponding to a particular monopole solution must be determined by R and $\psi(r\hat{k})$. Now it is a mathematical theorem that ¹⁾

$$\pi_2(G/H) = \pi_1(H) \quad (4.1)$$

[if $\pi_1(G)=1$, i.e., G is simply connected as we shall suppose without loss of generality by considering its covering group]. The element of $\pi_1(H)$ is constructed by considering the path dependent phase factor ⁶⁾

$$U(\underline{r}, \Gamma) = \text{Texp} \left(ic \int_{\Gamma} \underline{W}^{\mu} \cdot \underline{I} d\alpha_{\mu} \right) \quad (4.2)$$

where Γ is a path starting and finishing at $\underline{r}$ and lying on a sphere S , centre o , sufficiently large that Eq. (2.2) is valid thereon. Then it follows that

$$U(\underline{r}, \Gamma) \psi(\underline{r}) = \psi(\underline{r}) \quad (4.3)$$

Hence $U(\underline{r}, \Gamma)$ lies in H . Now let Γ move over the sphere S starting and finishing as an infinitesimal loop. Then correspondingly $U(\underline{r}, \Gamma)$ traces a path in H starting and finishing at the identity. It is this closed path, which we call J , whose homotopy class defines the topological quantum number on the right-hand side of Eq. (4.1). Now $\underline{W}^{\mu}$ is given by (2.7). The arbitrary functions w_{α}^{μ} are irrelevant to the topological quantum number since they can be varied continuously and just cause a homotopy of J . So we may take the $w_{\alpha}^{\mu} = 0$. Then (4.2) can be evaluated. If, for example, Γ is a circle of longitude subtending angle Θ with the z axis,

$$U(\underline{r}, \underline{r}) = e^{i \underline{r} \cdot \underline{r} 2\pi(1-\cos\theta)} \quad (4.4)$$

This is indeed both an element of R (which it must be by the form of $\underline{W}_0$) and H as (4.3) implies. This result can be proved by a singular gauge transformation to the Dirac potential as is done, for example, by Arafune et al. ¹⁾. Now we see that as θ varies the path J traced by U in H is simply a rotation through 4π generated by $\underline{r} \cdot \underline{r}$ (since solid angle 4π is covered). In R this path can be distorted to a point, of course, but this is not necessarily the case in H . In fact it is the homotopy class of this path, $\exp i\omega \underline{r} \cdot \underline{r}$, $0 \leq \omega \leq 4\pi$, in H which is the topological quantum number, (4.1).

Now it is clear how fundamental is the decomposition (2.6) of $\underline{r} \cdot \underline{r}$ into generators of H : it completely determines the topological quantum number of the spherically symmetric solution under study.

If $H = "U(1) \times K"$ it is now obvious why $-g$ is the coefficient of Q ; it counts the number of times J winds round the $U(1)$. We hope to discuss other examples in more detail in later work.

Let us note that we have only really used the spherical symmetry (2.4), and time independence (2.3) for large values of r .

Finally, let us emphasize the point made in the Introduction : that the decomposition of radial angular momentum makes sense independently of G , or of any assumed spherical symmetry and is probably still of fundamental importance.

I would like to thank Edward Corrigan for many discussions concerning monopoles.

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