Ref.TH.2181-CERN

SPIN AND ISOSPIN IN A GAUGE THEORY WITH MONOPOLES

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A B S T R A C T

The mechanism by which systems of spinless bosons with isospin and spinless magnetic monopoles acquire a spin in their ground states is investigated in the context of $SU(2)$ gauge models. Contrary to recent claims, it is demonstrated that isospin degrees of freedom are not converted into spin degrees of freedom. For a spin $\frac{1}{2}$ state, solutions are explicitly constructed, exhibiting the isospin degree of freedom.

Ref.TH.2181-CERN
9 June 1976

In classical electrodynamics, extended following Dirac to include magnetic monopoles ¹⁾, the minimal angular momentum of the system of a spinless monopole and a spinless charged particle is non-zero, because the electromagnetic field carries an angular momentum proportional to the product of the electric and magnetic charges ²⁾. This is mirrored in the first quantized theory by a spin degeneracy of the ground state ³⁾⁻⁵⁾. Recently, there have been attempts to generalize this phenomenon to non-Abelian gauge theories ^{6),7)} where monopoles appear as soliton solutions of the classical field equations ⁸⁾. In these theories, the "isospin" gauge group is spontaneously broken, and electromagnetic gauge invariance corresponds to a left-over unbroken $U(1)$ subgroup. It follows that electrically charged particles are necessarily introduced in isospin multiplets. Because of the spontaneous symmetry breakdown, one no longer expects an isospin degeneracy of the physical states, in particular of the relevant classical solutions. However, in a model theory, a degeneracy of classical solutions does exist, which, in a particular gauge, has the appearance of an isospin degeneracy. It has been shown that this degeneracy is in fact a spin degeneracy ⁷⁾. Does this mean that, in the field of a monopole of a non-Abelian gauge theory, isospin degrees of freedom are converted into spin degrees of freedom ?

This question was recently raised, and answered affirmatively, in Ref. 7). In the light of the results in Abelian electrodynamics alluded to above, this strikes us as a very puzzling conclusion, since a similar phenomenon occurs there, in spite of the absence of any isospin degrees of freedom. We shall actually see that the conclusion in question ⁷⁾ is false : in fact, spin is generated in a way that is completely analogous to the Abelian case, and the isospin degrees of freedom remain.

To see this, we will present a more explicit construction of the relevant classical solutions in a model theory. This will also be helpful in clarifying other apparently obscure aspects of the dynamics of charges and extended monopoles.

The model Lagrangian we want to consider is :

$$\begin{aligned}
 \mathcal{L} = & -\frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu} + \frac{1}{2} (D^\mu \phi_a)(D_\mu \phi_a) - V_1(\phi^2) \\
 & + (D^\mu U^\dagger)(D_\mu U) - V_2(U^\dagger U, \phi^2) ; \\
 F_a^{\mu\nu} = & \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e \varepsilon_{abc} A_b^\mu A_c^\nu ; \\
 D^\mu U = & \partial^\mu U - ie \frac{1}{2} \tau^a A_a^\mu U ;
 \end{aligned}
 \tag{1}$$

$$\Phi^2 = \phi_a \phi^a ;$$

$$V_1 = \lambda^2/2 [(\mu^2/\lambda^2) - \Phi^2]^2 ;$$

$$V_2 = m^2 U^+ U + g^2 (U^+ U)^2 - h^2 U^+ U [(\mu^2/\lambda^2) - \Phi^2] .$$

Setting $U=0$, this reduces to the model used in Ref. 8) to demonstrate the existence of monopoles as solitons. In the radial gauge, this solution is given by

$$\phi_a = \frac{r_a}{r} \phi(r) , \quad A_a^i = \epsilon_{iaj} r_j A(r) , \quad A_a^0 = 0 , \quad (2)$$

and in the so-called Abelian gauges, we have

$$\phi_a = 0 \quad (a=1,2) , \quad \phi_3 = \phi(r) , \quad A_3^i = A_D^i , \quad A_a^0 = 0 , \quad (3)$$

where $\vec{A}_D$ is the Dirac monopole potential, and we have not bothered to fix the remaining $U(1)$ gauge freedom. To exhibit some physical features of the theory, we begin by considering a related first quantized problem ⁶⁾

$$H U = (E^2 - m^2) U$$

with a Hamiltonian given by

$$H = [p_j - e A_j^a(x) \frac{1}{2} \tau^a]^2 - h^2 (\mu^2/\lambda^2 - \phi^2(r)) + g^2 u^+ u . \quad (4)$$

This is related to the original problem by regarding U as the wave function of a charged test particle or alternatively as a classical field interacting with external fields A and ϕ , identified with the monopole solution (2) or (3).

The Hamiltonian (4) commutes with an angular momentum operator, which differs from the naïve one by the inclusion of the angular momentum of the gauge field. Its explicit form is gauge dependent, and was given in the Abelian case in Ref. 4) :

$$\vec{J} = \vec{r} \times \left(\frac{1}{r} \frac{\partial}{\partial r} - e \vec{A}^3 \frac{1}{r} \tau^3 \right) + \frac{1}{2} \tau^3 \frac{\vec{r}}{r}. \quad (5)$$

In the radial gauge, it is ^{6),7),9)} :

$$\vec{J} = \vec{r} \times \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{2} \tau^3. \quad (6)$$

We now make a further simplification, ignoring - for the moment - terms non-linear in U and the effect of the extension of the monopole. Their effect will be included later. We will use two different gauges : the radial gauge, in which calculations are simpler, and an Abelian gauge, where it is easier to make the connection with electrodynamics and to interpret the results.

Let us begin with the Abelian gauge, where the upper and lower components of the isospinor decouple. Each of them separately obeys a Schrödinger equation which has been extensively discussed in the literature ³⁾⁻⁵⁾. The lowest energy solutions, in the gauge where the Dirac string points in the positive z direction, are :

$$u_+^+ = \begin{pmatrix} \cos \theta/2 e^{i\varphi} \\ 0 \end{pmatrix} 2 u_+(r), \quad u_-^+ = \begin{pmatrix} 0 \\ -\sin \theta/2 \end{pmatrix} 2 u_-(r), \quad (7)$$

$$u_+^- = \begin{pmatrix} \sin \theta/2 \\ 0 \end{pmatrix} 2 u_+(r), \quad u_-^- = \begin{pmatrix} 0 \\ \cos \theta/2 e^{-i\varphi} \end{pmatrix} 2 u_-(r),$$

$[u_+(r) = u_-(r)]$ is the solution of a radial equation.

These solutions correspond to total angular momentum $\frac{1}{2}$, and the upper label refers to the spin component in the z direction. The lower label refers to the electric charge, which in this gauge coincides with third component of isospin.

Next we solve the Schrödinger equation in the radial gauge, where the Hamiltonian takes the form

$$H = -\frac{1}{2r} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{2r} \left\{ (\vec{L} + \frac{1}{2} \vec{\tau})^2 - \frac{1}{4} \right\} + V(r) \quad (8)$$

with $\vec{L} = \vec{r} \times \frac{1}{r} \frac{\partial}{\partial \vec{r}}$. $\vec{J} = \vec{L} + \frac{1}{2} \vec{\tau}$ is the spin operator in this gauge, as noted earlier, and $C = \frac{1}{2} (\vec{r}/r) \cdot \vec{\tau}$ is the charge. These can again be diagonalized simultaneously, and the lowest energy solutions are

$$\begin{aligned} V_+^+ &= \begin{pmatrix} 1 + \cos \Theta \\ \sin \Theta e^{i\varphi} \end{pmatrix} u_+(r), & V_-^+ &= \begin{pmatrix} 1 - \cos \Theta \\ -\sin \Theta e^{i\varphi} \end{pmatrix} u_-(r), \\ V_+^- &= \begin{pmatrix} \sin \Theta e^{-i\varphi} \\ 1 - \cos \Theta \end{pmatrix} u_+(r), & V_-^- &= \begin{pmatrix} -\sin \Theta e^{-i\varphi} \\ 1 + \cos \Theta \end{pmatrix} u_-(r), \end{aligned} \quad (9)$$

where u_+ , u_- and the labels have the same meaning as before. The gauge transformation bringing us from the radial to the Abelian gauge is given by

$$R = \begin{pmatrix} \cos \Theta/2 e^{i\varphi} & \sin \Theta/2 \\ -\sin \Theta/2 & \cos \Theta/2 e^{-i\varphi} \end{pmatrix}. \quad (10)$$

The fourfold degeneracy we find arises from a twofold spin degeneracy $J=\frac{1}{2}$ and a degeneracy between positively and negatively charged states. It is clear that the isospin degree of freedom has not metamorphosed into a spin degree of freedom, as it is still present in addition to the spin. Rather, a spin of the ground state has appeared because of the peculiar angular momentum properties of a charged particle in the velocity dependent potential of a monopole. We note that, in the radial gauge, there is a spherically symmetric solution with a twofold spin degeneracy, $v_+^+ + v_-^+$ and $v_+^- + v_-^-$. Had we been looking exclusively for solutions with such symmetry, we would have found only two solutions, losing on the way the charge degree of freedom. Then it would have been tempting

$$\vec{J} = \vec{r} \times \left(\frac{1}{r} \frac{\partial}{\partial r} - e \vec{A}^3 \frac{1}{r} \right) + \frac{1}{2} \vec{r}^3 \frac{\partial}{\partial r} . \quad (5)$$

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to conclude erroneously that isospin had turned into spin ^{7),*)}, especially in view of the fact that this spin degeneracy can be parametrized as ⁷⁾

$$\psi = u(r) \exp(i \vec{\alpha} \cdot \frac{\vec{\tau}}{2}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (11)$$

However, this parametrization is only valid for the spherically symmetric solution. In general we have

$$\psi = u(r) \exp[i \vec{\alpha} \cdot (\vec{L} + \frac{1}{2} \vec{\tau})] \chi(\theta, \varphi), \quad (12)$$

where χ is an isospinor with an angular dependence.

We now proceed to include the effect of the extension of the monopole. In the radial gauge, the Hamiltonian becomes

$$H = p_r^2 + \frac{1}{2r^2} \left\{ \vec{L}^2 + W(r) \vec{L} \cdot \vec{\tau} + \frac{1}{2} W^2(r) \right\} + V(r) \quad (13)$$

where we introduced the notation $W(r) = -er^2 A(r)$. Since $\vec{J} = \vec{L} + \vec{\tau}$ still commutes with H , the separation of the angular variables is done in the same way as before. Because of the extension, however, we can no longer choose eigenstates of $(1/r) \vec{r} \cdot \vec{\tau}$. The solutions corresponding to (9) now become

$$\begin{aligned} V_1^+ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_1(r), & V_2^+ &= \begin{pmatrix} \cos \theta \\ \sin \theta e^{i\varphi} \end{pmatrix} u_2(r), \\ V_1^- &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(r), & V_2^- &= \begin{pmatrix} \sin \theta e^{-i\varphi} \\ -\cos \theta \end{pmatrix} u_2(r). \end{aligned} \quad (14)$$

The radial wave functions are now different and obey the equations ($K = 1 - W(r)$)

$$\begin{aligned} H u_1 &= \left(p_r^2 + \frac{1}{2r^2} (1-K)^2 \right) u_1 = E_1 u_1, \\ H u_2 &= \left(p_r^2 + \frac{1}{2r^2} (1+K)^2 \right) u_2 = E_2 u_2. \end{aligned} \quad (15)$$

*) That such a conclusion had to be wrong might have been guessed from other considerations. For instance, if one generalizes the Lagrangian (1) to include a term $u^\dagger \vec{\tau} u \vec{\phi}$, the isospin degeneracy between oppositely charged states is lifted, yet the spin degeneracy survives.

The degeneracy corresponding to a charge degeneracy in the case of a Dirac monopole is therefore removed. It is clear, however, that the degrees of freedom remain as before. The effect of the non-linearity is trivially included, by remarking that quantities like $(v_1)^\dagger v_1$ are independent of angles. Correspondingly, only the radial equations are modified.

Considering now the full classical field theory given by (1), we remark that the angular expansion displayed in detail above can be used to actually solve the classical field equations. We will not discuss here the meaning of these solutions for the quantum theory, but only attract attention to the degeneracy of classical solutions (12), which is the spin degeneracy.

We wish to thank J.S. Bell, D. Olive and B. Zumino for discussions.

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