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EXPANDING DISC AS A DYNAMICAL VACUUM INSTABILITY
IN REGGEON FIELD THEORY

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ABSTRACT

We show that the phase transition, obtained previously in Reggeon field theory for $\alpha(0) > 1$, is a consequence of a new kind of vacuum instability. The action of a field operator on the vacuum generates a disturbance that expands in space with a definite velocity towards a degenerate ground state. We show how the resulting J plane singularities can be interpreted as a sum over a highly degenerate spectrum with a non-vanishing energy gap. The resulting picture of the phase transition stems from the pseudo-hermiticity of the Reggeon field theory and is basically different from the spontaneous symmetry breaking.

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1. INTRODUCTION

In a preceding paper ¹⁾ a solution for the Reggeon field theory ²⁾ (RFT) for $\alpha(0) \equiv 1 + \mu > 1$ was attempted. The method was to introduce a lattice in transverse (impact parameter $\vec{b}$) space and to consider at each lattice site only the two lower lying states -- nearly degenerate for large μ -- obtained in solving the single site dynamics ³⁾. In this way a critical behaviour was obtained for a certain value $\mu = \mu_c$. For $\mu < \mu_c$ the only ground state is the (perturbative) vacuum -- call it ϕ_0 -- while for $\mu > \mu_c$ another state -- ψ_0 -- was found, degenerate with the vacuum and connected to it by field operators. This peculiar form of the phase transition in which the non-diagonal matrix elements play the role of the order parameter, is essentially due to the pseudo-hermiticity of the Hamiltonian.

In this paper we will study the evolution in time (i.e., rapidity $Y \equiv \log s$) of a family of states which connect, in $\vec{b}$ space, the two ground states ϕ_0 and ψ_0 . Even if they differ from ϕ_0 only locally at a certain Y , they always evolve towards ψ_0 . Local field operators acting on the vacuum ϕ_0 generate therefore states ending in ψ_0 for $Y \rightarrow \infty$. This shows the dynamical instability of the vacuum.

This evolution will also allow us to show that the Green's functions and the T matrix expand as a disc in (impact parameter) space. At finite Y the correlation length is of order Y , thus preserving the cluster decomposition of the T matrix at finite Y . This suggests that, at finite Y , the correlation functions and consequently the S matrix are regular functions of μ . It implies also a smooth matching with perturbation theory outside the disc.

The parameters of the theory, μ , the triple Pomeron coupling λ , the Pomeron slope α' , and the interlattice distance a playing the role of a cut-off, enter our approximate Hamiltonian in two combinations: the splitting Δ between the lowest lying levels in the single site dynamics, and J the intersite coupling.

The qualifying points of this paper are the exact solution of the one-transverse dimensional case ($D_T = 1$) for $\Delta = 0$ and the possibility to control the perturbation expansion in Δ at all orders. These results clarify the process of formation and evolution of the disc as well as the nature of the critical phenomenon. The extension of the qualitative conclusions to higher dimensional spaces ($D_T > 1$) is then obtained.

Even if our two-level single site approximation is definitely different from the full RFT, it is hoped that the critical behaviour we found is common to both theories. We conjecture that they belong to

the same class of universality as far as the critical behaviour is concerned. Then the Green's functions of the two theories should coincide in the critical region $|\mu - \mu_c| \ll 1$, $\alpha' k^2 \sim 0(\mu - \mu_c)$.

The structure of the paper is the following. In Section 2 we summarize the results of Ref. 1). In Sections 3 and 5 we discuss the evolution of the disc in the case $\Delta = 0$, $D_T = 1$. Particular attention is needed to be given to the study of the spectrum of the Hamiltonian in the finite and infinite volume cases. In the first case we find the spectrum of the Ising model in perfect agreement with Ref. 1). For the infinite volume case we find, however, that the spectrum is not the limit of that of the finite volume case: a complex continuum, related to the expanding disc, suddenly appears when the infinite volume limit is performed. Nevertheless, the evolution of a localized state (e.g., the disc) is continuous in that limit. In Section 4 another class of states, leading to field transition matrix elements showing soliton behaviour, is discussed.

In Sections 6 and 7 we treat the $\Delta \neq 0$ case, using a perturbative method in Δ . At all orders, Δ does not change the structure of the spectrum -- in particular, it does not break the ground state degeneracy -- while the disc is modified by a decrease in the expansion velocity and a smoothening of the edges. In Section 8 the generalization of our results to $D_T > 1$ is briefly discussed; the physical picture remains the same. Finally, a classical approximation is presented in Appendix I and an alternative proof of the degeneracy of the two ground states is given in Appendix II.

2. SUMMARY OF PREVIOUS RESULTS

The method used in Ref. 1) was to consider RFT in a lattice in impact parameter $\vec{b}$ space (interlattice distance a and total number of sites N). The Hamiltonian of such a system is given by

$$H = \sum_i \frac{\lambda}{a} q_i \left(\frac{\mu}{\lambda} a - p_i - q_i \right) p_i + \frac{\alpha'}{a^2} \sum_{(i,j)} q_i p_j \quad (2.1)$$

where (i,j) are next neighbours, α' is the Pomeron slope and λ the triple Pomeron coupling. p_i and q_i are related to the Gribov fields ²⁾ averaged over a cell in $\vec{b}$ space, and the vacuum Φ_0 is defined by $p_i \Phi_0 = 0$. The method of Ref. 1) was to solve first the single site dynamics [i.e., $\alpha' = 0$ in (2.1)] and then introduce the intersite coupling. The procedure was greatly simplified by the observation ³⁾ that the single site dynamics showed, for large μ , two closely lying lowest states with energy

gap $2\Delta \propto \mu e^{-\mu^2 a^2 / \lambda^2}$. In the approximation in which only these two levels are considered for each site, it was possible ¹⁾ to introduce a spin formalism and write the p, q fields and the Hamiltonian as

$$p = \frac{\mu a}{\lambda} \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{n}_p) \equiv \frac{\mu a}{\lambda} P, \quad q = \frac{\mu a}{\lambda} \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{n}_q) \equiv \frac{\mu a}{\lambda} Q, \quad (2.2)$$

$$H = \Delta \sum_i (1 - \sigma_z^i) + 2 \sum_{i,j} J_{ij} Q^i P^j, \quad J \equiv \sum_j J_{ij} = \alpha' \left(\frac{\mu}{\lambda}\right)^2 \quad (2.3)$$

where $J_{ij} \propto \alpha' (\mu/\lambda)^2$ for next neighbours and zero otherwise. The vectors $\vec{n}_p$ and $\vec{n}_q$ are given by

$$\vec{n}_{p,q} = (\pm 1, -i, -1) \quad (2.4)$$

so that $\vec{n}_p^2 = \vec{n}_q^2 = 1, \vec{n}_p \cdot \vec{n}_q = 1$.

It is useful to introduce the single site states

$$\chi_\sigma^i = \begin{pmatrix} 1 \\ -\sigma \end{pmatrix} \quad (2.5)$$

on which P and Q operate as follows:

$$\begin{aligned} P \chi_\sigma &= \chi_\sigma - (1 - \sigma) \chi_0, \quad P \chi_0 = 0, \\ Q \chi_\sigma &= \frac{1 - \sigma}{\sigma} (\chi_0 - \chi_\sigma), \quad Q \chi_1 = 0. \end{aligned} \quad (2.6)$$

In Ref. 1) it was found that the collective state

$$\phi_0 \equiv \prod_i \chi_0^i, \quad P^i \phi_0 = 0 \quad (2.7)$$

(perturbative vacuum) is a zero energy ground state for any value of Δ and J in (2.3), and that for $x = \Delta/J$ less than a certain critical value x_c there exists a second degenerate zero energy state

$$\Psi_0 \simeq \phi_0 - \phi_\sigma, \quad \phi_\sigma \equiv \prod_i \chi_\sigma^i, \quad \sigma = 1 - \frac{\Delta}{J} \equiv 1 - x. \quad (2.8)$$

This expression for Ψ_0 was obtained by a variational calculation. Actually the expression (2.8) is exact only for $\Delta=0$, but for $\Delta \neq 0$ we shall find that $\Psi_0 - \Phi_0$ has small components which are not direct products of single site states as in (2.8). These components are of short range and thus do not spoil the cluster property.

An important role in this theory is played by the $p \rightleftharpoons q^+$ symmetry of the Hamiltonian (2.1). Indeed, we can define an operator $M=M^+$ such that

$$M p M^{-1} = q^+ , \quad M q M^{-1} = p^+ ; \quad M H M^{-1} = H^+ , \quad (2.9)$$

which shows the "pseudo-hermiticity" of the Hamiltonian. In the representations (2.2) and (2.3)

$$M = M^{-1} = \prod_i \sigma_2^i . \quad (2.10)$$

Due to Eq. (2.9) M acts as a metric operator so that bra and ket states are connected by

$$\bar{\Phi}_\sigma \equiv \Phi_\sigma^+ M . \quad (2.11)$$

From (2.6) and (2.2) we find

$$\begin{aligned} (\bar{\Phi}_0 p_i \Phi_0) &= (\bar{\Phi}_0 q_i \Phi_0) = 0 , \\ (\bar{\Phi}_0 p_i \Phi_\sigma) &= (\bar{\Phi}_\sigma q_i \Phi_0) = \frac{\mu a}{\lambda} \sigma \end{aligned} \quad (2.12)$$

In Ref. 1) the asymptotic form of the S matrix was also derived from the definition

$$S(\vec{B}, \gamma) = (\bar{\Phi}_0 e^{-a \sum_i f_i p_i} e^{-H Y} e^{-a \sum_j g_j q_j} \Phi_0) \quad (2.13)$$

where g_i and f_i are the external sources (couplings of the Pomeron to the external particles) centered around $\vec{b}_i=0$ and $\vec{b}_i=\vec{B}$, respectively (Fig. 1). For $x > x_c$ (i.e., $\mu < \mu_c$), since Φ_0 is the only zero energy ground state, it was found that the T matrix vanishes asymptotically with a leading "Regge pole" behaviour. For $x < x_c$ (i.e., $\mu > \mu_c$) there are instead two ground states Φ_0 and Ψ_0 so that it was found

$$\begin{aligned}
 S(\vec{B}, Y) &\xrightarrow[Y \rightarrow \infty]{\substack{\vec{B} \text{ fixed}}} 1 - (\bar{\Phi}_0 e^{-a \sum f_i p_i} \Psi_0) (\bar{\Psi}_0 e^{-a \sum g_j q_j} \Phi_0) = \\
 &\xrightarrow{a \rightarrow 0} 1 - \left(1 - \exp - \sigma \frac{\mu}{\lambda} \int f d^2 b\right) \left(1 - \exp - \sigma \frac{\mu}{\lambda} \int g d^2 b\right) \quad (2.14) \\
 &\equiv 1 - T_f T_g \quad (\mu > \mu_c) .
 \end{aligned}$$

The approach to this asymptotic limit is of course controlled by the excited states. These come both from collective modes of the approximate Hamiltonian in Eq. (2.3) and from the inclusion of higher one-site excited levels. These latter contributions were shown in Ref. 1) to be not negligible outside the impact parameter region

$$\vec{B}^2 < \mu Y (R^2 + 4 \alpha' Y) , \quad (2.15)$$

where R is the radius of the sources (external particles). The collective excitations of (2.3) were also found in Ref. 1), with frequencies

$$\frac{1}{4} \omega^2(\vec{k}) = \begin{cases} [\Delta - J(\vec{k})]^2 & , \quad (x > x_c) \\ [J - x J(\vec{k})]^2 & , \quad (x < x_c) , \end{cases} \quad (2.16)$$

$$J(\vec{k}) \equiv \sum_e J_{em} e^{i \vec{k} \cdot (\vec{b}_e - \vec{b}_m)} ,$$

which have an energy gap $\omega(0) = 2|J - \Delta|$ vanishing only for $x = x_c$ (i.e., $\mu = \mu_c$).

In the following Sections we shall further investigate the collective modes and their contribution to the S matrix. In doing so we shall recognize an interesting phenomenon which gives rise to an expanding disc in $\vec{b}$ space. Moreover, we shall discuss the apparent inconsistency between the spectrum arising from an expanding disc (imaginary continuum with zero gap at $k=0$) and that of Eq. (2.16) which has a finite gap. Roughly speaking, this originates from a divergence over the contributions of type (2.16) when B is larger than Y .

3. EXPANDING BOX STATES FOR $\Delta = 0$

We shall start analyzing the case $\Delta = 0$ and $D_t = 1$, for which most of our results can be given in an analytic form. Indeed for $\Delta = 0$, $H\Phi_1 = 0$ and therefore Φ_0 and Φ_1 are degenerate ground states.

Let us define the box states (see Fig. 2)

$$\xi^{l,m} = \prod_{i < l} \chi_0^i \prod_{i=l}^m \chi_1^i \prod_{i > m} \chi_0^i, \quad (3.1)$$

$$(\bar{\Phi}_0 \xi^{l,m}) = 1, \quad \lim_{m-l \rightarrow \infty} \xi^{l,m} = \Phi_1$$

Noting that [see Eq. (2.6)]

$$P\chi_0 = Q\chi_1 = 0, \quad P\chi_1 = \chi_1, \quad Q\chi_0 = \chi_0 - \chi_1, \quad (3.2)$$

we find

$$Q^i \xi^{l,m} = 0 \quad \text{for } (l \leq i \leq m),$$

$$P^i \xi^{l,m} = \begin{cases} \xi^{l,m} & (l \leq i \leq m) \\ 0 & \text{otherwise} \end{cases}. \quad (3.3)$$

In order to study the time evolution of the box states, we have to apply the Hamiltonian (2.3) which for $\Delta = 0$ reduces to

$$H_0 = J \sum_{(ij)} P^i Q^j \quad (3.4)$$

with ij next neighbours. From (3.3), noticing that the only terms in H_0 contributing are $(ij) = (m, m+1)$ and $(l-1, l)$, we have

$$-\frac{d}{dy} \xi^{l,m} \equiv H_0 \xi^{l,m} = J(2\xi^{l,m} - \xi^{l-1,m} - \xi^{l,m+1}). \quad (3.5)$$

This finite difference equation becomes, in the continuum limit, a differential equation for

$$\xi(b_l, b_m, y) \equiv e^{-H_0 y} \xi^{l,m}, \quad (b_i = ia), \quad (3.6)$$

i.e.,

$$-\frac{d}{dy} \xi(b, b', y) = Ja \left(\frac{\partial}{\partial b} - \frac{\partial}{\partial b'} \right) \xi(b, b', y) . \quad (3.7)$$

The solution of this equation is

$$\xi(b, b', y) = \xi(b - Ja y, b' + Ja y, 0) \quad (3.8)$$

which implies that the box state ξ expands with velocity Ja . This fact gives rise to the expanding disc behaviour of all the Green's functions.

Indeed the two-point function is

$$\begin{aligned} G(b_i, b_j, Y) &\equiv (\bar{\Phi}_0 p_i e^{-H_0 Y} q_j \Phi_0) = -\left(\frac{\mu}{\lambda}\right)^2 (\bar{\Phi}_0 p^i \xi(b_j, b_j, Y)) \\ &= -\left(\frac{\mu}{\lambda}\right)^2 (\bar{\Phi}_0 p^i \xi(b_i - Ja Y, b_i + Ja Y, 0)) \\ &= -\left(\frac{\mu}{\lambda}\right)^2 \Theta(Ja Y - |b_j - b_i|) , \end{aligned} \quad (3.9)$$

whose Fourier transform

$$\tilde{G}(E, \vec{k}) \equiv \int dy d\vec{b} G(\vec{b}, 0, y) e^{-i\vec{b} \cdot \vec{k}} e^{E y} \quad (3.10)$$

is

$$\tilde{G}(E, k) = -\left(\frac{\mu}{\lambda}\right)^2 / (E^2 + J^2 a^2 k^2) \quad (D_t = 1) \quad (3.11)$$

Analogously the expression for the S matrix (for point-like sources) is

$$\begin{aligned} S(b_i, b_j, Y) &= (\bar{\Phi}_0 e^{-f \frac{\mu}{\lambda} p^i} e^{-H_0 Y} e^{-g \frac{\mu}{\lambda} q^j} \Phi_0) \\ &= (\bar{\Phi}_0 (1 - T_f p^i) e^{-H_0 Y} (1 - T_g q^j) \Phi_0) = 1 - T_f T_g \Theta(Ja Y - |b_j - b_i|) , \end{aligned} \quad (3.12)$$

in agreement with the result of Ref. 1) [see Eq. (2.14)] for $Y \rightarrow \infty$, and $|b_i - b_j|$ fixed.

We must remark that the velocity of expansion of this disc is $dB/dY = Ja$, which is vanishingly small for physical values of the parameters [fixed $J = \alpha'(\mu/\lambda)^2$ and $a \rightarrow 0$]. Therefore this is not the disc of Eq. (2.15) which expands with velocity $2\sqrt{\alpha'\mu}$. This fact is not surprising because (2.15) was obtained by estimating the contribution of the higher single site excited levels neglected here. Indeed it was conjectured in Ref. 1) that the collective excitations (phonons) coming from the higher levels should be the ones responsible for the disc expansion of Eq. (2.15). Nevertheless, by the universality argument of Section 1, we find it instructive to discuss the expansion mechanism we just found. We shall therefore conventionally consider the continuum limit for Ja fixed.

The preceding analysis shows that there is a dynamical instability of the vacuum ϕ_0 . In fact, once this state is locally excited (e.g., by applying q_i on ϕ_0) the local "disturbance" propagates over all sites ^{*}) through the box states $\xi^{\ell,m}$ towards the degenerate ground state ϕ_1 . This is not the usual vacuum instability occurring in (Hermitean) spontaneously broken theories. Indeed ϕ_1 is not lower lying than ϕ_0 , and moreover the expectation values of the fields p and q satisfy the symmetry (2.9).

4. STEP STATES AND SOLITON SOLUTIONS ($\Delta = 0$)

Before entering the discussion of the spectrum of H_0 , it will prove convenient to study the evolution of other states which are similar to the box states, but with different boundary conditions.

Let us, in fact, define the step states (see Figs. 3a and b)

$$\begin{aligned} R^e &= \prod_{i \leq e} \chi_1^i \prod_{i > e} \chi_0^i \\ L^e &= \prod_{i < e} \chi_0^i \prod_{i \geq e} \chi_1^i \end{aligned} \quad (4.1)$$

and the hole states (see Fig. 3c)

$$\eta^{\ell,m} = \prod_{i \leq \ell} \chi_1^i \prod_{\ell < i < m} \chi_0^i \prod_{i \geq m} \chi_1^i \quad (4.2)$$

The time evolution of these states can be studied much in the same way as we did for the ξ states. Indeed

^{*}) A classical picture of the disc which has in common with us the local modification of the vacuum has been recently proposed by Czechowski (CERN TH. 2179).

$$\begin{aligned} H_0 R^\ell &= J (R^\ell - R^{\ell+1}) , \\ H_0 L^\ell &= J (L^\ell - L^{\ell-1}) , \\ H_0 \eta^{\ell,m} &= J (2 \eta^{\ell,m} - \eta^{\ell+1,m} - \eta^{\ell,m-1}) , \quad (\eta^{\ell,\ell} = \Phi_1) , \end{aligned} \quad (4.3)$$

and the time evolution of these states is completely analogous to the one of the box states. In the continuum limit we have

$$\begin{aligned} R^\ell(y) &= R(b_\ell, y) = R(b_\ell + J a y, 0) , \\ L^\ell(y) &= L(b_\ell, y) = L(b_\ell - J a y, 0) , \end{aligned} \quad (4.4)$$

and similarly for $\eta^{\ell,m}(y)$. Recalling that $R(b,0)$ [$L(b,0)$] are step states having χ_0 on the right (left), we clearly have a rightwards (leftwards) expansion (see Figs. 3a and b). This again shows the instability of the vacuum Φ_0 towards the companion state Φ_1 .

It is amusing to find the evolution of the transition matrix elements

$$\begin{aligned} \langle \bar{L}^i q_j(y) R^i \rangle &= \langle \bar{L}^i e^{Hy} q_j e^{-Hy} R^i \rangle \\ &= \langle \bar{L}(b_i + J a y) q_j R(b_i + J a y) \rangle = \frac{\mu}{\lambda} \Theta(b_j - b_i - J a y) , \end{aligned} \quad (4.5)$$

$$\langle \bar{L}^i p_j(y) R^i \rangle = \frac{\mu}{\lambda} \Theta(-b_j + b_i + J a y) . \quad (4.6)$$

These matrix elements represent therefore a soliton-like solution ^{*)} travelling towards the right (Fig. 4a). Due to the symmetry (2.9), we have also

$$\begin{aligned} \langle \bar{R}^i q_j(y) L^i \rangle &= \langle \bar{L}^i p_j(-y) R^i \rangle , \\ \langle \bar{R}^i p_j(y) L^i \rangle &= \langle \bar{L}^i q_j(-y) R^i \rangle , \end{aligned} \quad (4.7)$$

which then represent a leftwards travelling soliton (Fig. 4b).

*) A soliton solution for the classical fields has been recently obtained by Bronzan, Cardy and Sugar (we thank John Cardy for communicating to us their results).

We find here another striking difference with the soliton solutions of usual spontaneously broken theories. In these the soliton appears as a field expectation value on a state that interpolates the two vacua. Here instead it appears as the transition matrix element between two different states which interpolate the two vacua.

5. SPECTRUM ($\Delta = 0$)

The spectrum of H_0 can be read from (3.11) and is a continuum on the imaginary axis ($E = \pm ikJa$) as expected for an expanding disc. As mentioned in the Introduction we want to find the connection between this spectrum and the collective excitations discussed in Ref. 1) [cf. Eq. (2.16)]. Let us first notice that the spectrum in Eq. (2.16) degenerates for $\Delta = 0$ into a point spectrum at $E = 2J$ with very high multiplicity. This is not striking if one remembers ¹⁾ the analogy for $\Delta = 0$ with the pure Ising model in which -- for periodic boundary conditions -- the first excited level is at $E = 2J$ and is $2N$ fold degenerate (only one spin opposite to all others).

In order to understand the relation between the two spectra, we start by keeping the number of the lattice sites N finite and considering different boundary conditions. Let us represent the Hamiltonian H_0 [Eq. (3.4)] in the subspaces generated by the states R, L, ξ or η depending on the four respective boundary conditions. For instance, in the subspace of R states, i.e., with boundary condition $\chi_\alpha^0 = \chi_1^0, \chi_\alpha^N = \chi_0^N$, call it R , H_0 as given by (4.3) can be represented by

$$H_0 = J \begin{bmatrix} 1 & -1 & & 0 \\ & 1 & -1 & \\ & & 1 & -1 \\ 0 & & & \ddots & -1 \\ & & & & 0 \end{bmatrix} \quad (5.1)$$

The spectrum of this finite dimensional triangular matrix is controlled by the diagonal elements. It is therefore either $E = 0$ (corresponding to $R^N = \xi_1$) or $E = J$ with $N-1$ fold degeneracy. Even if this feature is common with the Ising model (with analogous boundary conditions), the structure of the resolvent operator is basically different. In fact, the matrix (5.1) is not diagonalizable and the resolvent therefore develops multiple poles at $E = J$. This can be verified on the exact evolution of the states R^i which, always for N finite, is obtained by integrating Eq. (4.3)

$$e^{-H_0 y} R^i = \sum_{j=i}^{N-1} \frac{(Jy)^{j-i}}{(j-i)!} e^{-Jy} R^j + \left[1 - e^{-Jy} \sum_{n=0}^{N-1-i} \frac{(Jy)^n}{n!} \right] R^N \quad (5.2)$$

where $R^N = \Phi_1$. From this equation one can easily notice the existence in the resolvent of multiple poles $(E-J)^{-n}$ up to $n=N$. One can verify that, while the eigenvalue $E=J$ is $(N-1)$ fold degenerate, the only eigenfunction is $R^N - R^{N-1} = \Phi_1 - R^{N-1}$. Moreover, for $y \rightarrow \infty$, the evolution operator $e^{-H_0 y}$ (in this R subspace) becomes a projection over the state Φ_1 . This was essentially the property used in Ref. 1) for evaluating the asymptotic S matrix.

For $N \rightarrow \infty$ the spectrum changes abruptly, while the time evolution does not. This last fact is seen immediately from Eq. (5.2), where for $N \rightarrow \infty$ the last term cancels, thus giving

$$e^{-H_0 y} R^i = \sum_{n=0}^{\infty} \frac{(Jy)^n}{n!} e^{-Jy} R^{i+n}, \quad (N \rightarrow \infty), \quad (5.3)$$

as can also be directly seen from (4.3).

On the other hand, the approximate point spectrum ^{*}) acquires the whole continuum of points (Fig. 5)

$$E_k = J(1 - e^{ika}) = -iJa k + O(Ja^2) \quad (5.4)$$

The approximate eigenfunctions $\tilde{R}_n(k)$ are in fact given by the "Fourier transforms"

$$\tilde{R}_n(k) = \frac{1}{n} \sum_{\ell=1}^n e^{i\ell k a} R^\ell. \quad (5.5)$$

In fact from (4.3) we have

$$H_0 \tilde{R}_n(k) = E_k \tilde{R}_n(k) + \frac{J}{n} e^{ika} R^1 \xrightarrow{n \rightarrow \infty} E_k \tilde{R}_n(k). \quad (5.6)$$

^{*}) E belongs to the approximate point spectrum ⁴⁾ of H if there is a sequence of unit norm vectors $\{R_n\}$, such that $\|(H-E)R_n\| \rightarrow 0$ as $n \rightarrow \infty$.

We have therefore seen that the evolution operator $e^{-H_0 Y}$ in the subspace R at N finite is controlled by a point spectrum (the non-degenerate point $E=0$ corresponding to Φ_1 and the $N-1$ fold degenerate point $E=J$), while for $N=\infty$ it is controlled by a continuum lying on the complex circle (5.4). The abrupt change in the structure of the spectrum in the $N \rightarrow \infty$ limit is due to the fact that non-diagonal elements in a triangular finite matrix [as that of (5.1)] are irrelevant for the eigenvalues, while this is not the case for infinite matrices. In spite of this fact, the evolution operator does not abruptly change in the limit $N \rightarrow \infty$ [cf. (5.2) and (5.3)] and matrix elements of local operators are then continuous in this limit.

A completely parallel treatment of the spectrum can be given in subspaces generated by the states L , ξ or η . For simplicity, we only briefly discuss the ξ sector of box states which is the one directly relevant for the hadronic T matrix [see Eq. (3.9)]. For N finite we have the points $E=0$ (corresponding to both Φ_0 and Φ_1 in this case) and $E=2J$ [$2(N-1)$ fold degenerate], as in the Ising model with periodic boundary contributions. The time evolution of ξ states is given by (3.5) which for $N \rightarrow \infty$ gives

$$e^{-H_0 Y} \xi^{\ell, m} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(JY)^{i+j}}{i! j!} e^{-2JY} \xi^{\ell-i, m+j}, \quad (5.7)$$

which in the continuum limit reduces to Eq. (3.8).

Also in this case the spectrum at $N \rightarrow \infty$ is a continuum

$$E_{q,k} = J(2 - e^{ika} - e^{iqa}) \quad (5.8)$$

The approximate eigenfunctions are obtained from the sequence

$$\tilde{\xi}_n(q, k) = \frac{1}{n^2} \sum_{\ell=1}^n \sum_{m=1}^n \xi^{\ell, m} e^{i(\ell q - m k) a} \quad (5.9)$$

In fact from Eq. (3.5) we find

$$\begin{aligned} (H_0 - E_{q,k}) \tilde{\xi}_n(q, k) &= \frac{J}{n} \left[2\phi_2 \delta_{q,k} + \right. \\ &\left. + (1 + e^{i(q-k)a}) \frac{1}{n} \sum_{\ell} \xi^{\ell, \ell} e^{i(q-k)\ell a} \right] \xrightarrow{n \rightarrow \infty} 0 \end{aligned} \quad (5.10)$$

This shows, in particular, that the only exact eigenstates for finite N are $\tilde{\xi}_N(k, k+\pi/a)$, all corresponding to $E=2J$. The important point is again that the matrix elements of the evolution operator $e^{-H_0 Y}$ do not change with increasing N .

We shall now obtain the two-point Green's function defined in (3.9). By inserting the evolution of $\xi(b_i, b_j, Y)$ as given by (5.7) into (3.9), we find ($j > i$)

$$\begin{aligned} G(b_i, b_j, Y) &= -\frac{\mu}{\lambda} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{(JY)^{\ell+m}}{\ell! m!} e^{-2JY} (\bar{\phi}_0 | i \xi^{j-\ell, j+m}) \\ &= -\left(\frac{\mu}{\lambda}\right)^2 \sum_{\ell=0}^{\infty} \frac{(JY)^{\ell}}{\ell!} e^{-JY} \theta(i-j+\ell) = -\left(\frac{\mu}{\lambda}\right)^2 \left[1 - \sum_{\ell=0}^{j-i} \frac{(JY)^{\ell}}{\ell!} e^{-JY} \right] \end{aligned} \quad (5.11)$$

which shows the expanding disc structure in lattice space.

We recognize in the singularities of

$$G(b_i, b_j, E) = \int_0^{\infty} dY e^{EY} G(b_i, b_j, Y)$$

a pole at $E=0$ and the multiple poles at $E=J$, i.e., the structure of the spectrum of the Hamiltonian for finite N . For $N \rightarrow \infty$, by performing the Fourier transform with respect to $b_j - b_i = a(j-i)$, we find that the singularities of $\tilde{G}(k, E)$ are indeed given by the $N=\infty$ spectrum of Eq. (5.4) for $j > i$ or its conjugate for $i > j$. In the continuum limit $a \rightarrow 0$, Ja fixed [see Eq. (3.11)], we find again

$$\tilde{G}(k, E) = \frac{1}{E} \left(\frac{1}{E - iJa k} + \frac{1}{E + iJa k} \right) = \frac{1}{E^2 + J^2 a^2 k^2} \quad (5.12)$$

We have then explicitly seen how an expression sharing the singularities of the finite N Hamiltonian is Fourier transformed in an expression sharing the singularities of the $N=\infty$ Hamiltonian.

6. THE GROUND STATE ($\Delta \neq 0$)

We have, up to now, treated the $\Delta = 0$ case which is particularly simple since Φ_0 and Φ_1 are exact degenerate eigenstates. In Ref. 1) it was argued that the degeneracy of the ground state is not lifted for $\Delta \neq 0$ as long as $x < x_c$ ($x = \Delta/J$). While Φ_0 remains an exact zero energy eigenvalue of (2.3), the other degenerate state was constructed by a variational method which gives the expression (2.8) and a critical value $x_c = 1$.

The variational result of Ref. 1) can be further supported by a perturbative calculation. Indeed we shall prove, perturbatively in x around $x = 0$, that the degeneracy of the two ground states is not lifted at any order in x and that the second eigenstate is Φ_0 [see Eq. (2.8)], apart from short-range correlations.

Let us decompose the Hamiltonian of Eq. (2.3) as

$$\begin{aligned} H &= H_0 + H_I, \\ H_I &= 2Jx \sum_i \frac{1}{2}(1 - \sigma_3^i) \end{aligned} \quad (6.1)$$

We shall again use the basis constructed by products of local states χ_0^i , χ_1^i [Eq. (2.5)]. Let us, in fact, introduce the collective states of the type

$$\gamma^{l_1 \dots l_n} = \prod_{i \in \{l_j\}} \chi_0^i \prod_{i \notin \{l_j\}} \chi_1^i \quad (6.2)$$

which can be interpreted as a state Φ_1 with n holes located at sites $l_1, l_2, \dots, l_n$.

Due to

$$\frac{1}{2}(1 - \sigma_3) \chi_0 = 0, \quad \frac{1}{2}(1 - \sigma_3) \chi_1 = \chi_1 - \chi_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \equiv \chi_-, \quad (6.3)$$

we have

$$H_I \Phi_1 = 2Jx \sum_{\ell} (\Phi_1 - \gamma^{\ell}) \quad (6.4)$$

$$H_I \gamma^{l_1 \dots l_n} = 2Jx \sum_{m \notin \{l_j\}} (\gamma^{l_1 \dots l_n} - \gamma^{l_1 \dots l_n m}) \quad (6.5)$$

Let us now discuss how $\gamma^{l_1 \dots l_n}$ evolves under H_0 . With a calculation similar to that of Sections 3 and 4, we can show that the action of H_0 on γ shrinks the holes so that the state evolves towards Φ_1

$$e^{-Y H_0} \gamma^{l_1 \dots l_n} \rightarrow \Phi_1 + O(e^{-2JY}) \quad (6.6)$$

Therefore

$$e^{-H_0 Y} (\gamma^{l_1 \dots l_n} - \gamma^{l_1 \dots l_n m}) = O(e^{-2JY}) \quad (6.7)$$

For example,

$$e^{-H_0 Y} (\Phi_1 - \gamma^l) = e^{-2JY} (\Phi_1 - \gamma^l) \quad (6.8)$$

We see that the action of H_I [see Eqs. (6.4) and (6.5)] gives states which are asymptotically annihilated by $e^{-Y H_0}$. It is therefore reasonable to expect that $e^{-Y(H_0 + H_I)} \Phi_1$ goes to a finite limit for $Y \rightarrow \infty$. This finite limit would be an eigenstate of the full Hamiltonian (6.1) with zero energy, and therefore degenerate with the vacuum Φ_0 .

To formalize this argument, let us write

$$e^{H_0 Y} e^{-(H_0 + H_I)Y} = T \left(\exp - \int_0^Y dy H_I(y) \right) \quad (6.9)$$

where

$$H_I(y) = e^{H_0 y} H_I e^{-H_0 y} \quad (6.10)$$

Expanding Eq. (6.9) we have

$$\begin{aligned}
 e^{-HY} \phi_1 &= \sum_n (-)^n \int_0^Y dy_n \int_0^{y_n} dy_{n-1} \cdots dy_1 e^{-H_0 Y} H_I(y_n) \cdots H_I(y_1) \bar{\Phi}_1 \\
 &= \sum_n x^n \varphi_n(Y) .
 \end{aligned}
 \tag{6.11}$$

From Eqs. (6.5) and (6.7) we have

$$e^{-H_0 Y} H_I(y_n) \cdots H_I(y_1) \phi_1 = O\left(e^{-2J(Y-y_1)}\right) \quad (Y \geq y_n \geq y_{n-1} \geq \cdots y_1) \tag{6.12}$$

so that in (6.11) we can safely send Y to infinity giving $\varphi_n(Y) = \varphi_n + O(e^{-2JY})$. The eigenvalue of H therefore vanishes and the corresponding eigenvector is given by

$$\bar{\Phi}(x) = \sum_n x^n \varphi_n . \tag{6.13}$$

This perturbative argument does not provide a radius of convergence in x which should be related to x_c . In Appendix II we give an alternative algebraic proof of this result.

We now want to argue that

$$\bar{\Phi}(x) = \bar{\Phi}_{\sigma(x)} + \text{short-range corrections} \tag{6.14}$$

where $\bar{\Phi}_0$ is defined in (2.8). At second order in x , for instance, for $D_t = 1$, we have (cf. Appendix II)

$$\bar{\Phi}(x) = \bar{\Phi}_{\sigma(x)} + x^2 \sum_{\ell} \chi_{-}^{\ell} \chi_{-}^{\ell+1} \prod_{i \neq \ell, \ell+1} \chi_{-}^i \tag{6.15}$$

where χ_{-}^i is defined in (6.3) and

$$\sigma(x) = 1 - x - x^2 + O(x^3) . \tag{6.16}$$

The corrections to $\bar{\Phi}_0$ in (6.15) are, in fact, of short range and it is simple to see that this will extend to higher orders.

Equation (6.16) for $\sigma(x)$ suggests that $x_c < 1$ [$\sigma(x_c) = 0$] where 1 is the value obtained by variational methods. This tends to coincide with the usual situation in second order phase transitions ¹⁾.

7. THE EXPANDING BOX STATE ($\Delta \neq 0$)

Having understood the nature of the two vacua for $x < x_c$, we pass now to study the time evolution of the box states in Eq. (3.1) which are relevant for the behaviour of the correlation function and the S matrix [see Eq. (3.12)]. We start by studying the case of x small by perturbative methods. This will, however, allow us to reach a qualitative picture of the time evolution of these states for any x and to understand the nature of the phase transition.

Let us define for $x \neq 0$ the box state

$$\xi^{l,m} = \prod_{i < l} \chi_0^i \prod_{i=l}^m \chi_\sigma^i \prod_{i > m} \chi_0^i \quad (7.1)$$

in analogy with (3.1), where $\sigma = 1 - x$. We introduce now also the single holed box states $\xi_k^{l,m}$ which are obtained from (7.1) by substituting at the site k the state χ_σ^k with χ_0^k . Similarly, we can introduce n holed box states. It is convenient to introduce also the states

$$\beta_k^{l,m} = \xi_k^{l,m} - \xi^{l,m}. \quad (7.2)$$

In fact at the order $x = 1 - \sigma$ we have

$$H \xi^{l,m} = J \left(2 \xi^{l,m} - \xi^{l-1,m} - \xi^{l,m+1} \right) + Jx \beta_m^{l,m+1} + Jx \beta_l^{l-1,m} + O(x^2), \quad (7.3)$$

$$H \beta_k^{l,m} = J \left(4 \beta_k^{l,m} - \beta_k^{l-1,m} - \beta_k^{l,m+1} \right) + O(x). \quad (7.4)$$

The physical meaning of Eqs. (7.3) and (7.4) is clear: we find the box expansion of Section 3 [cf. Eq. (3.5)], but now holes are produced at the edges. The evolution of box states with one hole is such that the radius increases while the hole fills up at fixed position. The state $\beta_k^{l,m}$ in fact tends exponentially to zero as can be seen by integrating Eq. (7.4):

$$e^{-HY} \beta_k^{e, m} = e^{-2JY} \sum_{i, j} e^{-2JY} \frac{(JY)^{i+j}}{i! j!} \beta_k^{e-i, m+j} \quad (7.5)$$

In the large time limit, near the edges of the box, a dynamical equilibrium situation will be achieved among the creation of holes at the edges given by the second term in (7.3) proportional to x , and the natural death of holes given by Eq. (7.4). This will produce a smoothening of the edges of the expanding box.

This picture should not be modified by taking into account the neglected terms in (7.3) and (7.4) of higher order in x . One needs to introduce states of type β [Eq. (7.2)] with more than a single hole, which are generated by the further action of H_I and which will again evolve to zero under the H_0 time evolution.

From the time evolution of the box state we can obtain, as in Section 3, the resulting picture for the hadronic T matrix. For small x the T matrix is an expanding disc with an opacity controlled by $\sigma \simeq 1-x$ [see Eq. (2.14)] with edges that are smoothened by holes which have a limited penetration length. This length increases with x , while the velocity of the expansion of the disc decreases. We can therefore visualize the critical phenomenon at $x=x_0$ as the complete infiltration of holes up to the centre.

In Appendix I we produce a classical treatment which allows us to understand how this penetration of holes into the centre peripheralizes the process so as to give the transition to a Regge-like structure for $x > x_0$.

8. EXTENSION TO SEVERAL SPACE DIMENSIONS

Up to this point we have studied the one-dimensional case. The physically more interesting case of two dimensions is more complicated and an explicit solution cannot be found even for $x=0$. The physical picture is, however, the same: for $x < x_0$ there are two degenerate vacua (the proof of Section 6 holds for any D_t). Φ_0 in this case is also dynamically unstable, i.e., there are states which differ from Φ_0 only locally which evolve towards the other degenerate state.

In order to control this phenomenon, let us first consider the case $x=0$. In analogy with the $D_t=1$ case, we define the bubble state ξ_0 , where C is a connected region in the lattice space

$$\xi_c = \prod_{i \in c} \chi_i \prod_{i \notin c} \chi_o^i \quad (8.1)$$

Let us define as δC all the points next neighbours of C and external to it. We find that the action of the Hamiltonian for $x=0$ is

$$H_0 \xi_c = J \sum_{k \in \delta C} (\xi_c - \xi_{c+k}) n(k) \quad (8.2)$$

where $C+k$ is the region obtained adding to C the point k , and $n(k)$ is the number of points of C which are next neighbours of k . For $D_t=2$ the possible values for $n(k)$ are 1, 2, 3, and 4 as shown in the example of Fig. 6.

The action of the Hamiltonian H_0 on a bubble state has then an effect similar to the $D_t=1$ case: the bubble expands with increasing Y . The fact that $n(k)>1$, at a concave angle of the contour of C , clearly shows that the time evolution tends to eliminate the concavities and to fill possible holes. This effect is similar to the surface tension on real bubbles.

In order to be more quantitative, let us define a general superposition of bubble states

$$\xi(F) = \sum_c F(c) \xi_c \quad (8.3)$$

Applying H_0 to $\xi(F)$ we find the equation of motion for F

$$\frac{d}{dy} F(c, y) = -J \left[\sum_{k \in \delta C} n(k) F(c, y) - \sum_{c' \in \delta C} \sum_{k \in \delta c'} n(k) \delta(c'+k-c) F(c', y) \right] \quad (8.4)$$

where $\delta(c'+k-c)$ is defined to be one if $c'+k$ is equal to c and zero otherwise. Notice that

$$\sum_{k \in \delta C} n(k) = l(c) , \quad (8.5)$$

where $l(c)$ is the length of the contour of the surface C . Equation (8.4) can then be written also as

$$\dot{F}(C, y) = -J \ell(C) F(C, y) + J \sum_{C' \in \delta C} \sum_{k \in \delta C'} n(k) \delta(C' + k - C) F(C', y). \quad (8.6)$$

Equation (8.6) is very similar in form to the Pauli master equation: the second term in the right-hand side represents the increase in F due to the expansion of the surface, while the first one is the consequent local decrease of F . Equation (8.6) gives then an irreversible evolution which implies an indefinite expansion towards the ground state Φ_1 , i.e., C fills up the full space.

The function $F(C, y)$ may be indeed interpreted as a probability distribution; this interpretation is sustained by the fact that

$$\frac{d}{dy} \sum_C F(C, y) = 0 \quad (8.7)$$

(conservation of total probability)

$$F(C, y) \geq 0 \quad \text{for } y \geq 0 \quad \text{if } F(C, 0) \geq 0 \quad (8.8)$$

(positivity of probability)

Equation (8.7) is simply obtained from Eq. (8.4) by summing over C .

Equation (8.8) can be derived noticing that Eq. (8.6) implies, for positive $F(C, y)$

$$\dot{F}(C, y) \geq -J F(C, y) \ell(C) \quad (8.9)$$

and therefore the function $F(C, y)$ cannot change sign

$$(F(C, y) \geq F(C, 0) \exp[-\ell(C) J y]).$$

Altogether we understood that the bubble states expand with increasing Y and that the most likely shape is the one with few concavities. Consequently, the contour cannot be very rough so that $\ell(C) \propto A^{(D-1)/D}$, A being the area of the region C . In order to find the velocity of the expansion, let us restrict ourselves to these regions. Integrating Eq. (8.6) over all C with area A we find

$$\dot{F}(A, y) \simeq -J A^{1-\frac{1}{D}} F(A, y) + J (A-1)^{1-\frac{1}{D}} F(A-1, y). \quad (8.10)$$

Here $F(A,y)$ can be interpreted as the probability of finding a bubble of area A at time Y . The solution of (8.10) is a Poisson distribution centered around $A \sim (JY)^{D_t}$. This implies a linear rise with Y of the radius of the bubble ($R(C) \sim JaY$).

The influence of $\Delta \neq 0$ is analogous to that discussed in Section 7 for $D_t = 1$. We expect here a further effect in smoothening the edges of the bubble coming from $H_\perp$.

From the fact that the action of the local operator q_i on the vacuum Φ_0 produces a (small) bubble [see Eq. (3.2)], we can obtain from its evolution the Green's functions and the S matrix, as we have done in Section 7 with obviously similar results.

APPENDIX I - SEMI-CLASSICAL APPROXIMATION

A simple insight on the behaviour of the Green's functions for general transverse dimension D_t and $x \neq 0$ can be given in the mean field classical approximation.

Let us consider the following matrix element of the field $P^\ell(y)$

$$\begin{aligned} \sigma_\ell(y) &\equiv (\bar{\Phi}_0 P^\ell(y) \Phi(\{\sigma_i\})) , \\ \Phi(\{\sigma_i\}) &= \prod_i \chi_{\sigma_i}^i = \prod_i \begin{pmatrix} 1 \\ -\sigma_i \end{pmatrix} \end{aligned} \quad (A.1)$$

where $\Phi(\{\sigma_i\})$ is a functional state with $(\bar{\Phi}_0 \Phi(\{\sigma_i\})) = 1$. From Eq. (3.9) we have that for $\sigma_\ell(0) = (1-x) \delta_{\ell m}$, $\sigma_\ell(y)$ is the two-point Green's function $G(b_\ell, b_m, y) [-\lambda^2/\mu^2(1-x)]$.

Noting that [see Eq. (2.2)]

$$[P, Q] = P+Q-1, \quad \frac{1}{2} [P, (1-\sigma_3)] = P - \frac{1}{2} (1-\sigma_3), \quad (A.2)$$

the Heisenberg equation of motion for $P^\ell(y)$ gives

$$(\bar{\Phi}_0 [\dot{P}^\ell(y) + 2Jx P^\ell(y) + 2J \sum_m' P^m(y) (P^\ell(y) - 1)] \Phi(\{\sigma_i\})) = 0, \quad (A.3)$$

with m next neighbour of ℓ .

Making the mean field approximation (no correlations)

$$(\bar{\Phi}_0 P^\ell(y) P^m(y) \Phi(\{\sigma_i\})) \simeq \sigma_\ell(y) \sigma_m(y), \quad (\ell \neq m), \quad (A.4)$$

Eq. (A.3) reduces, in the continuum limit, to

$$\frac{d}{dy} \sigma(\vec{b}, y) = 2J \sigma(\vec{b}, y) [1-x - \sigma(\vec{b}, y)] + [1-\sigma(\vec{b}, y)] J a^2 \vec{\nabla}_b^2 \sigma(\vec{b}, y) \quad (A.5)$$

which is similar to the classical equation of RFT for a single field.

The corresponding equation for a $\vec{b}$ independent field $\sigma(y)$ admits two fixed points at $\sigma=0$ and $\sigma=1-x$, and we have two different regimes according to whether $x>1$ or $x<1$. In the first case $\sigma=0$ is stable and $\sigma=1-x$ unstable, while the opposite is true for $x>1$:

$$\lim_{y \rightarrow \infty} \sigma(y) = \begin{cases} 0 & (x > 1) \\ 1-x & (x < 1) \end{cases} \quad (A.6)$$

For $x>1$ a $\vec{b}$ dependent solution can be obtained by perturbing around $\sigma=0$. By linearizing Eq. (A.5) we have (e.g., for $D_t=2$),

$$\sigma(\vec{b}, y) = \int d^2 b' \sigma(\vec{b}', 0) \frac{1}{4Ja^2 y} \exp \left[2J(1-x)y - \frac{(\vec{b}-\vec{b}')^2}{4Ja^2 y} \right], \quad (A.7)$$

leading to a "Regge pole" for the T matrix with intercept $1+2J(1-x)$ smaller than one for $x>1$ and slope Ja^2 .

For $x<1$, $\sigma(\vec{b}, y)$ is not necessarily small and non-linear effects are important in (A.5). Using the same arguments as in Ref. 5), it can be argued that there exists a solution $\sigma(\vec{b}, y)$ which for $y \rightarrow \infty$ and b fixed tends to $(1-x)$, and which vanishes for $b \rightarrow \infty$ and y fixed in such a way as to match with the perturbative solution (A.7) for $\vec{b}^2 > 8J^2 a^2 (1-x)y^2$.

This picture is in qualitative agreement with the features discussed in the full quantum treatment: the T matrix is an expanding disc for x small. By increasing x , the opacity of the disc and its velocity of expansion are reduced (holes start to enter from the border of the disc). As x overcomes 1 the disc turns into a Regge pole as in (A.7) (the holes are penetrating into the centre of the disc and the process becomes peripheral).

APPENDIX II - ALTERNATIVE PROOF OF GROUND STATE DEGENERACY

In this Appendix we give an algebraic proof that, besides the state Φ_0 , there exists -- to any order in x -- a second eigenstate of H , $\Phi(x)$, with zero eigenvalue. For simplicity, we consider $D_t = 1$, but the generalization to any D_t is evident.

Let us define the p states $(\chi_- = \chi_1 - \chi_0)$

$$p_{l_1 \dots l_n} = \prod_{i \in \{l_n\}} \chi_- \prod_{i \notin \{l_n\}} \chi_+ \quad . \quad (B.1)$$

The action of H_0 is

$$\begin{aligned} H_0 p_l &= 2J p_l , \\ H_0 p_{l_1 l_2} &= 4J p_{l_1 l_2} \quad , \quad (l_1 \neq l_2 \pm 1) , \\ H_0 p_{l, l+1} &= 2J p_{l, l+1} + J p_l + J p_{l+1} \\ H_0 p_{l_1 \dots l_n} &= 2nJ p_{l_1 \dots l_n} , \\ & \quad (l_i \neq l_j \pm 1 \text{ for any } i, j) \quad . \end{aligned} \quad (B.2)$$

In general we have

$$H_0 p_{l_1 \dots l_n} = J c_0 p_{l_1 \dots l_n} + J \sum_{i=1}^n c_i p_{l_1 \dots \bar{l}_i \dots l_n} , \quad (B.3)$$

where $\bar{l}_i$ means that the label l_i is suppressed. c_i are positive weights depending on the neighbouring structure of $\{l_i\}$.

The action of H_I is, in general,

$$H_I p_{l_1 \dots l_n} = 2Jx \left[\sum_{l \notin \{l_i\}} p_{l_1 \dots l_n l} + n p_{l_1 \dots l_n} \right] . \quad (B.4)$$

We want to find out how the zero eigenvalue state Φ_1 of H_0 is modified by H_I . The important point to realize is that, up to order x^n , it is possible to restrict this search to the finite dimensional space generated, by applying to the state Φ_1 polynomials in H_0 and H_I which are of the order $m \leq n$ in H_I . For example, for $n=2$ a basis of this space is

$$\Phi_1, \alpha \equiv \sum_e p_e, \beta = \sum_{|l-m|>1} p_{lm}, \gamma = \sum_e p_{e,e+1}. \quad (B.5)$$

On this basis the operator H has the form

$$H = 2J \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 1+x & 0 & 1 \\ 0 & x & 2 & 0 \\ 0 & 2x & 0 & 1 \end{bmatrix} \quad (B.6)$$

A general feature for all n is that the first line of the corresponding matrix of H is zero. This is because H does not make transitions from a p state to Φ_1 . Consequently a zero eigenvalue is present for finite n ($\det H=0$).

From (B.6) we find the corresponding eigenfunction

$$\Phi(x) = \Phi_1 - x \sum_e p_e + x^2 \left[\sum_{l < m} p_{lm} + \sum_e (p_{e,e+1} - p_e) \right] + O(x^3), \quad (B.7)$$

which coincides with Eq. (6.15).

Of course, for infinite n we cannot prove in this way the existence of a zero eigenvalue of H in this space, and we have to study the radius of convergence of the series (B.7) which is probably related to x_c .

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* * *

FIGURE CAPTIONS

- Fig. 1 : Graphical representation of a contribution to the S matrix.
 $G^{n,m}$ are the Green's functions of Reggeon field theory.
- Fig. 2 : Pictorial representation of the box state $\xi^{\ell,m}$ [Eq. (3.1)].
The arrows indicate the time evolution of the state.
- Fig. 3 : Pictorial representation of the step states L^{ℓ} and R^{ℓ} and of the hole state $\eta^{\ell,m}$ [Eqs. (4.1) and (4.2)]. Arrows indicate the time evolution.
- Fig. 4 : Form and evolution of the transition matrix elements of the operator p_{ℓ} (dashed lines) and q_{ℓ} (solid lines).
(a) represents a rightwards moving soliton [Eqs. (4.5) and (4.6)], while (b) represents a leftwards moving one [Eq. (4.7)].
- Fig. 5 : Representation of $E_K = J(1 - e^{ika})$ which belongs to the approximate point spectrum of H_0 for $N = \infty$.
- Fig. 6 : Pictorial representation of a bubble state ξ_c for $D_t = 2$.
The region C is represented shaded; a, b, c, and d are examples of next neighbour points with $n(a) = 1$, $n(b) = 2$, $n(c) = 3$, and $n(d) = 4$.

FIG. 1

$\xi^{l,m}$

FIG. 2

L^l

(a)

R^l

(b)

$\eta^{l,m}$

(c)

FIG. 3

FIG.4

FIG.5

FIG.6