Ref.TH.2178-CERN

PROTON AND ELECTRON INELASTIC SCATTERING
AND THE SHELL MODEL FOR ^{12}C

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ABSTRACT

The Glauber model is applied to the inelastic scattering of protons, and the electromagnetic transition form factor is derived from the same formula. The shell model of $^{12}\text{C}^*(2^+, 3^-)$ is improved in a new calculation. The spin effects for the proton scattering at small momentum transfers are discussed; they are found to explain an earlier discrepancy in the description of the 3^- excitation. The optical limit for the distortion factor is considered and the results compared with the exact Glauber formula. Our approximate version of the Glauber theory produces results that are much closer to the exact Glauber theory than those of the optical model.

Ref.TH.2178-CERN

22 June 1976

1. INTRODUCTION

The idea of using high-energy proton scattering data to learn about nuclear structure is based on the assumption that at high energy we have at our disposal a microscopic theory of nuclear reaction, the Glauber theory¹⁾. This theory has proved successful in several cases of proton scattering on nuclei between 1 and 20 GeV/c^{2,3)}; sometimes the agreement was not quantitative, but the qualitative features of the data at small momentum transfers have always been reproduced.

It seems worth while at this point to try to get even a quantitative agreement of the theory with experiments, and to achieve it with a parallel but independent improvement of the microscopic model used.

This has the twofold effect of increasing our confidence in the theory and improving the nuclear wave function. We shall use the harmonic oscillator wave function, since the marriage between this wave function and Glauber theory is a particularly happy one. The reason is that, on the one side, the Glauber model assumes that all elementary scatterings inside the nucleus are not correlated in space and time, and on the other, the shell model is an independent particle model, therefore the scattering matrix is completely factorized, enormously simplifying the problem.

As we shall see in Section 2, there is a formal but gratifying advantage in the use of Glauber theory: that is, we do not need to make the assumption that the excitation is produced in one step, i.e. during one interaction of the incident proton with a target nucleon, as is forcefully done in DWIA (distorted wave impulse approximation)⁴⁾. However, if the initial state is a closed-shell state and the final state is a superposition of one-particle one-hole states, the one-step mechanism turns out to be numerically dominant, but for more complex states, with two-particle two-hole components, the effect of multistep would become important.

In Section 3 we study the problem of the introduction of spin in the formalism: we reach some qualitative conclusions and give an approximate formula for calculating spin effects, once the nucleon-nucleon amplitude becomes known.

In Section 4, the procedure for calculating the shell model wave function of the ^{12}C excited state is outlined.

In Section 5 we discuss the numerical calculations for the 3^- and 2^+ states of ^{12}C , comparing the different approximations.

2. INDEPENDENT PARTICLE MODEL AND GLAUBER THEORY

The well-known formula¹⁾ for elastic and inelastic scattering of protons on nuclei at high energies is

$$F_{if}(\vec{q}_T) = \frac{ik}{2\pi} \int e^{i\vec{q}_T \cdot \vec{b}} \langle \psi_f | 1 - \prod_{\ell=1}^A (1 - \Gamma_{\ell}(\vec{b} - \vec{s}_{\ell})) | \psi_i \rangle d^2b \quad (2.1)$$

where $\vec{q}_T$ is the transverse momentum transfer, $\vec{b}$ the impact parameter, k the laboratory momentum of the incident proton and Γ_{ℓ} is the profile function of the ℓ^{th} nucleon proton scattering [see Eq. (3.2)]. This formula is valid whenever the excitation energy of the nucleus is small compared with the incident proton energy.

If the elementary profile function Γ_{ℓ} does not depend on the direction of $\vec{b} - \vec{s}_{\ell}$, i.e. it is spin independent, then the only angular dependence of the amplitude in Eq. (2.1) comes from the final and initial states, which behaves as spherical tensors under rotation.

To extract this dependence explicitly, we take the direction of $\vec{b}$ fixed, in the z - x plane, z being the incident beam direction: then we transform the result, performing a rotation around the z -axis of an angle $\phi_b = \hat{q}\hat{b}$:

$$\begin{aligned} & \langle J_f M_f | R^{-1}(\Omega) \prod_{\ell=1}^A (1 - \Gamma_{\ell}) R(\Omega) | J_i M_i \rangle = \\ & = \sum_{M M'} D_{M_f M}^{J_f}(\Omega) \langle J_f M | \prod_{\ell=1}^A (1 - \Gamma_{\ell}) | J_i M' \rangle D_{M' M_i}^{J_i}(\Omega) = \\ & = e^{-i(M_f - M_i)\phi_b} \langle J_f M_f | \prod_{\ell=1}^A (1 - \Gamma_{\ell}) | J_i M_i \rangle. \end{aligned} \quad (2.2)$$

The resulting phase factor can be integrated and the result is

$$F_{J_i M_i, J_f M_f}(q) = ik e^{i(M_f - M_i)\frac{\pi}{2}} \int_0^{\infty} J_{M_f - M_i}(qb) \langle J_f M_f | \prod_{\ell=1}^A (1 - \Gamma_{\ell}) | J_i M_i \rangle d^2b. \quad (2.3)$$

Since the profile operator is symmetric for interchange between any pair of nucleons, and the initial state is taken as fully antisymmetrized, we do not need to antisymmetrize the final state because only the antisymmetric part will survive in the scalar product⁵⁾.

With j - j coupling and in the $1p$ - $1h$ approximation the final state is (neglecting isospin for the moment)

$$\langle \tilde{\psi}_f | \vec{r}_1 \dots \vec{r}_A \rangle = (A!)^{\frac{1}{2}} \sum_{m_p} \begin{bmatrix} p & h & J_f \\ m_p & -m_h & M_f \end{bmatrix} \phi_{p, m_p}(\vec{r}_1) \prod_{\substack{g, m_g \neq h, m_h \\ c \neq 1}} \phi_{g, m_g}(\vec{r}_c) \quad (2.4)$$

where h and p are the total angular momenta of the hole and the particle, the square brackets is a C.G. coefficient, and ϕ are single-particle wave functions. The tild in $\tilde{\psi}_f$ denotes that the state is not antisymmetric. The fully antisymmetric initial state is

$$\langle \vec{r}_1 \dots \vec{r}_A | \psi_i \rangle = (A!)^{-\frac{1}{2}} \sum_{g, m_g} (-1)^{g-m_g} \phi_{g, m_g}(\vec{r}_1) \det \{ \phi_{g', m_{g'}}(\vec{r}_c) \}_{\substack{g', m_{g'} \neq g, m_g \\ c \neq 1}} \quad (2.5)$$

Since ϕ_{p, m_p} is orthogonal to all the ϕ_{g, m_g} in the initial state, we have for the profile function:

$$\begin{aligned} \langle \tilde{\psi}_f | \prod_{c=1}^A (1 - \tau_c) | \psi_i \rangle &= - \langle \tilde{\psi}_f | \Gamma_1 \prod_{c=2}^A (1 - \tau_c) | \psi_i \rangle = \\ &= - \sum_{g, m_g} (-1)^{g-m_g} \sum_{m_p} \begin{bmatrix} p & h & J_f \\ m_p & -m_h & M_f \end{bmatrix} \langle p, m_p | \Gamma_1 | g, m_g \rangle D_{g, m_g}^{h, m_h} \end{aligned} \quad (2.6)$$

where

$$D_{g, m_g}^{h, m_h} = \det \{ \langle g'', m_{g''} | 1 - \tau | g', m_{g'} \rangle \}_{\substack{g'', m_{g''} \neq h, m_h \\ g', m_{g'} \neq g, m_g}} \quad .$$

Starting from this formula one can calculate the inelastic scattering amplitude⁵⁾ integrating over b in (2.3). This is further described in Appendix B. However, it is worth doing some angular momentum algebra, which simplifies the final formula and clarifies the connections with the DWIA.

The starting point is the assumption that both the elementary amplitude Γ_1 and the distortion operator $\hat{D} = \prod_{\ell=2}^A (1 - \Gamma_\ell)$ are rotationally invariant. Expand Γ_1 in spherical harmonics and apply the Wigner-Eckart theorem⁶⁾:

$$\begin{aligned}
 \langle \tilde{\psi}_f | \prod_{e=2}^A (1 - \pi_e) | \psi_i \rangle &= \\
 &= \frac{2}{ik} \sum_{\lambda \mu} i^\lambda \int e^{-i \vec{q}' \cdot \vec{r}} Y_{\lambda \mu}^*(\hat{q}') f(q') \begin{bmatrix} J_i & \lambda & J_f \\ M_i & \mu & M_f \end{bmatrix} \langle \tilde{\psi}_f | \hat{O}_\lambda \hat{D} | \psi_i \rangle d^2 q' \quad (2.7)
 \end{aligned}$$

where $\hat{O}_{\lambda \mu} = Y_{\lambda \mu}(\hat{r}_1) j_\lambda(q' r_1)$.

The reduced matrix element is related to the original matrix element by the relation

$$\langle \tilde{\psi}_f | \hat{O}_\lambda \hat{D} | \psi_i \rangle = \frac{J_i^2}{J_f^2} \sum_{\mu' M'} \begin{bmatrix} J_i & \lambda & J_f \\ M_i & \mu' & M' \end{bmatrix} \langle \tilde{\psi}_f | \hat{O}_{\lambda \mu'} \hat{D} | \psi_i \rangle \quad (2.8)$$

where $\hat{J} = \sqrt{2J+1}$.

Substituting this expression in (2.7) and expanding the matrix element in the same way as (2.6), we obtain a standard sum of products of three 3j symbols⁶⁾. For the useful case $J_i = 0$, the resulting formula reads:

$$\begin{aligned}
 \langle \tilde{\psi}_f | \prod \hat{D} | \psi_i \rangle &= \frac{2}{ik} i^{J_f} \int e^{-i \vec{q}' \cdot \vec{r}} f(q') \times \\
 &\times Y_{J_f M_f}^*(\hat{q}') \frac{\hat{P}}{\hat{h}^2 \hat{J}_f} \langle P | \hat{O}_{J_f} | h \rangle \sum_{m_h} D_{h, m_h}^{h, m_h} d^2 q' \quad (2.9)
 \end{aligned}$$

where

$$\langle P | \hat{O}_\lambda | h \rangle = (-1)^{h+\lambda-\frac{1}{2}} \frac{\hat{\lambda} \hat{h}}{(4\pi)^{\frac{1}{2}}} \begin{bmatrix} h & P & \lambda \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \langle R_P | j_\lambda | R_h \rangle$$

and

$$\langle R_P | j_\lambda | R_h \rangle = \int_0^\infty R_P(r) R_h(r) j_\lambda(q'r) r^2 dr.$$

This is our main result: it clearly shows the separation between the transition density and the distortion factor, typical of the DWIA⁷⁾, and it can be reduced to that form if we assume that

$$\sum_{m_h} D_{h, m_h}^{h, m_h} = \hat{h}^2 \langle D \rangle = \hat{h}^2 \exp[-(A-1) \int d^3r \Gamma(\vec{r}-\vec{s}) \rho_0(r)]. \quad (2.10)$$

This approximation may be too crude, because if the final state ψ_f contains configuration mixing,

$$|\psi_f\rangle = \sum x_{ph} |ph^{-1}; J, M_f\rangle \quad (2.11)$$

then different holes correspond to different distortion factors.

The expression (2.9) is also useful for deriving the relation between the electromagnetic form factor and the hadron inelastic scattering amplitude. If the $\hat{D}$ operator is taken equal to unity, which means that $\langle D \rangle = 1$ in Eq. (2.10), then the profile function (2.6) becomes

$$\langle \tilde{\psi}_f | \Pi | \psi_i \rangle = \frac{1}{2\pi i k} \int d^2q' e^{-i\vec{q}' \cdot \vec{r}} f(\vec{q}') \langle \tilde{\psi}_f | e^{i\vec{q}' \cdot \vec{r}} | \psi_i \rangle. \quad (2.12)$$

Performing the integration over $\vec{b}$ in Eq. (2.1), we obtain a δ -function, which finally yields

$$F_m(\vec{q}) = f(\vec{q}) \langle \tilde{\psi}_f | e^{i\vec{q} \cdot \vec{r}} | \psi_i \rangle = A f(\vec{q}) \langle \psi_f | e^{i\vec{q} \cdot \vec{r}} | \psi_i \rangle. \quad (2.13)$$

This formula establishes the connection between the hadron and electron elastic scattering, because the last matrix element is by definition the electromagnetic form factor⁸⁾. The details of the proof are shown in Appendix A.

3. SPIN FORMALISM

The introduction of spin in the formalism spoils the simplicity of the previous section. Even the relation between the inelastic amplitude and the inelastic e.m. form factor is lost. However, it is worth while to develop the formalism with spin, to get some qualitative understanding of spin effects.

For this purpose we make the Fourier-Bessel transform of the NN amplitude in the laboratory system⁹⁾

$$\begin{aligned} \hat{f}_i(q_T) = & \alpha(q_T) + \beta(q_T)(\vec{\sigma}_0 \cdot \hat{n})(\vec{\sigma}_i \cdot \hat{n}) + i\gamma(q_T)(\vec{\sigma}_0 \cdot \hat{n} + \vec{\sigma}_i \cdot \hat{n}) + \\ & + \delta(q_T)(\vec{\sigma}_0 \cdot \hat{q})(\vec{\sigma}_i \cdot \hat{q}) + \epsilon(q_T)(\vec{\sigma}_0 \cdot \hat{e})(\vec{\sigma}_i \cdot \hat{e}) \end{aligned} \quad (3.1)$$

where $\vec{\sigma}_0, \vec{\sigma}_i$ refer, respectively, to the projectile and target nucleon i , $\hat{e}$ is the incident beam direction, and $\hat{n} = \hat{e} \times \hat{q}$. The result is

$$\begin{aligned} \hat{\Gamma}_i(b) = & \frac{1}{2\pi i k} \int f_i(q_T) e^{-i\vec{q}_T \cdot \vec{b}} d^2 q_T = \\ = & \Gamma_\alpha(b) + \Gamma_\beta(b)(\vec{\sigma}_0 \cdot \hat{n}')(\vec{\sigma}_i \cdot \hat{n}') + i\Gamma_\gamma(b)(\vec{\sigma}_0 \cdot \hat{n}' + \vec{\sigma}_i \cdot \hat{n}') + \\ & + \Gamma_\delta(b)(\vec{\sigma}_0 \cdot \hat{q})(\vec{\sigma}_i \cdot \hat{q}) + \Gamma_\epsilon(b)(\vec{\sigma}_0 \cdot \hat{e})(\vec{\sigma}_i \cdot \hat{e}) \end{aligned} \quad (3.2)$$

where $\hat{n}' = \hat{e} \times \hat{b}$. Taking $\hat{e}$ as the spin quantization axis, we can represent Γ_i in the following fashion¹⁰⁾:

$$\begin{aligned} \hat{\Gamma}_i(b) = & \Gamma_\alpha(b) + \sum_{\mu=\pm 1} \Gamma_\mu(b)(\sigma_\mu^0 + \mu_\mu^i) + \\ & + \sum_{\nu, \nu'=\pm 1} \Gamma_{\nu\nu'}(b) \sigma_\nu^0 \sigma_{\nu'}^i + \Gamma_\epsilon(b) \sigma_0^0 \sigma_0^i \end{aligned} \quad (3.3)$$

where

$$\sigma_{\pm 1} = (\sigma_x \pm i\sigma_y)/\sqrt{2}, \quad \sigma_0 = \sigma_z$$

and

$$\Gamma_{\pm 1}(b) = \pm \Gamma_\gamma(b) e^{\mp i\phi}/\sqrt{2}$$

(3.4)

$$\Gamma_{\nu\nu'}(b) = \frac{1}{2} [(-1)^{1+(\nu-\nu')/2} \Gamma_\rho(b) + \Gamma_\sigma(b)] e^{-i(\nu+\nu')\phi}$$

and ϕ is the angle between $\hat{b}$ and a fixed axis. The argument of Γ is not $\vec{b}$; but $\vec{b} - \vec{s}$, therefore the expressions (3.4) become more complicated because we have two angles ϕ_b and ϕ_s ^{*}). This concerns the actual calculation; for the formal considerations we simply insert the expansion (3.3) into (2.6). The result is

$$\begin{aligned} \langle \tilde{\psi}_f | \Gamma, \frac{A}{\ell=2} (1 - \Gamma_e) | \psi_i \rangle &= \sum_{g, m_g} (-1)^{g-m_g} \sum_{m_p} \begin{bmatrix} p & h & J_f \\ m_p & -m_h & M_f \end{bmatrix} \times \\ &\times \left\{ \langle p, m_p | \Gamma_\alpha | g, m_g \rangle + \sum_{\mu=\pm 1} (\sigma_\mu^0 \langle p, m_p | \Gamma_\mu | g, m_g \rangle + \right. \\ &+ \langle p, m_p | \Gamma_\mu \sigma_\mu^1 | g, m_g \rangle) + \sum_{\nu, \nu'=\pm 1} \sigma_\nu^0 \langle p, m_p | \sigma_{\nu'}^1 \Gamma_{\nu\nu'} | g, m_g \rangle + \\ &+ \sigma_0^0 \langle p, m_p | \sigma_0^1 \Gamma_e | g, m_g \rangle \left. \right\} D_{g, m_g}^{h, m_g}. \end{aligned} \quad (3.5)$$

In principle, any matrix element in the matrix $D_{gm_g}^{hm_g}$ is as complicated as the transition matrix element in (3.5). But it is reasonable to keep in the distortion factor only terms linear in the spin amplitudes, at least at small momentum transfers, that is:

$$D = D_\alpha + \sum_{\mu=\pm 1} (\sigma_\mu^0 D_\mu + D_\mu^1) + \sum_{\nu, \nu'=\pm 1} \sigma_\nu^0 D_{\nu\nu'} + \sigma_0^0 D_e \quad (3.6)$$

and

$$\begin{aligned} D_\alpha &= \det \{ \langle i | 1 - \Gamma_\alpha | j \rangle \}, \quad i = (g'', m_g'') \neq (h, m_h), \quad j = (g', m_g') \neq (g, m_g) \\ D_\mu &= - \sum_{e, m} \det \{ \langle i | 1 - \Gamma_\alpha | j \rangle; \langle e | \Gamma_\mu | m \rangle \} \\ D_\mu^1 &= - \sum_{e, m} \det \{ \langle i | 1 - \Gamma_\alpha | j \rangle; \langle e | \Gamma_\mu \sigma_\mu^1 | m \rangle \} \\ D_{\nu\nu'} &= - \sum_{e, m} \det \{ \langle i | 1 - \Gamma_\alpha | j \rangle; \langle e | \sigma_{\nu'}^1 \Gamma_{\nu\nu'} | m \rangle \} \\ D_e &= - \sum_{e, m} \det \{ \langle i | 1 - \Gamma_\alpha | j \rangle; \langle e | \sigma_0^1 \Gamma_e | m \rangle \} \end{aligned} \quad (3.7)$$

*) For the second definition of (3.4), this means only that for mixed terms the exponential is of the type $e^{-i\nu\phi_s} e^{-i\nu'\phi_b}$.

where we mean that we substitute the matrix element after the semicolon in the (ℓ, m) element of the matrices, except for D_α .

Substituting (3.6) in (3.5) we find that the scattering operator is of the following form:

$$\begin{aligned} \hat{O} &= S + \sum_{\mu=-1}^1 V_\mu \sigma_\mu^0 + \sum_{\mu, \mu' \neq 0, 0} T_{\mu\mu'} \sigma_\mu^0 \sigma_{\mu'}^0 = \\ &= \sum_{g, m_g} (-1)^{g-m_g} \sum_{m_p} \begin{bmatrix} p & h & J_f \\ m_p & -m_n & M_f \end{bmatrix} \times \\ &\times \left(S^{pg} + \sum_{\mu=-1}^1 V_\mu^{pg} \sigma_\mu^0 + \sum_{\mu, \mu' \neq 0, 0} T_{\mu\mu'}^{pg} \sigma_\mu^0 \sigma_{\mu'}^0 \right) \end{aligned} \quad (3.8)$$

where

$$S^{pg} = \langle p, m_p | \Gamma_\alpha | g, m_g \rangle D_\alpha' + \langle p, m_p | \sigma_0 \Gamma_\alpha | g, m_g \rangle D_E$$

$$V_\mu^{pg} = \langle p, m_p | T_\mu' | g, m_g \rangle D_\alpha + \langle p, m_p | \Gamma_\alpha | g, m_g \rangle D_\mu'$$

$$T_{0\mu}^{pg} = \langle p, m_p | \sigma_0 T_\mu' | g, m_g \rangle D_\mu' \quad (3.9)$$

$$T_{\mu 0}^{pg} = \langle p, m_p | T_\mu' | g, m_g \rangle D_E$$

$$T_{\mu\mu'}^{pg} = \langle p, m_p | T_\mu' | g, m_g \rangle D_{\mu'}'$$

and

$$\begin{aligned} D_{\alpha}^{\prime} &= D_{\alpha} + \sum_{\mu=\pm 1} D_{\mu}^{\prime} \\ D_{\mu}^{\prime} &= D_{\mu} + \sum_{\nu=\pm 1} D_{\mu\nu} \\ T_{\mu}^{\prime} &= T_{\mu} + \sum_{\nu=\pm 1} T_{\mu\nu} . \end{aligned}$$

The unpolarized cross-section is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &\sim \frac{1}{2} \text{Tr} [\hat{\sigma}^+ \hat{\sigma}] = \\ &= \frac{1}{2} \text{Tr} \left[(S^* + \sum_{\mu} V_{\mu}^* \sigma_{\mu}^0 + \sum_{\mu\mu'} T_{\mu\mu'}^* \sigma_{\mu}^0 + \sigma_{\mu'}^0) \times \right. \\ &\quad \left. \times (S + \sum_{\nu} V_{\nu} \sigma_{\nu}^0 + \sum_{\nu\nu'} T_{\nu\nu'} \sigma_{\nu}^0 + \sigma_{\nu'}^0) \right] \\ &= |S|^2 + \sum_{\mu} |V_{\mu}|^2 + 4 \text{Re} [S^* (T_{11} + T_{-11}) + \\ &\quad + \sum_{\mu=\pm 1} (T_{\mu 0}^* T_{\mu 0} + T_{0\mu}^* T_{0\mu} + T_{\mu 0}^* T_{0\mu})] + \\ &\quad + 4 (|T_{1-1}|^2 + |T_{-11}|^2) . \end{aligned} \quad (3.10)$$

From this expression and from (3.9) we see that the effect of spin on the elementary NN amplitude is mainly incoherent and therefore plays a role similar to that of a large real part. In the optical model calculation¹¹⁾ of spin effects, we take into account only the diagonal elements of (3.7) ($\ell = m$); this might not always be reasonable because non-diagonal elements can be larger than the diagonal ones.

4. EXCITED STATES OF THE ^{12}C NUCLEUS

Numerical results of the present investigation were obtained with the particle-hole (p-h) nuclear model. To fix the amount of configuration mixing, we convert the many-particle wave function of Eq. (2.4) into a particle-hole one. The procedure induces an additional phase factor¹²⁾ and we end up with the basic p-h ket of the form

$$|ph J_f M_f T_f M_T\rangle = \sum_{\substack{p m_p \\ \tau_p}} \sum_{\substack{h m_h \\ \tau_h}} (-1)^{h-m_h+\frac{1}{2}-\tau_h} \begin{bmatrix} p & h & J_f \\ m_p & -m_h & M_f \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & T_f \\ \tau_p & -\tau_h & M_T \end{bmatrix} a_{p m_p \tau_p}^\dagger a_{h m_h \tau_h} |0\rangle \quad (4.1)$$

where $a^\dagger(a)$ is the fermion creation (annihilation) operator. The Hamiltonian matrix with Tabakin's non-local realistic potential¹³⁾ was diagonalized within the space of $1\hbar\omega$ (negative parity) and $0\hbar\omega + 2\hbar\omega$ (positive parity) excitation of the type (4.1). We have used a standard set of the single-particle energies [cf. Gillet et al.¹⁴⁾]. Harmonic oscillator radial wave function were defined to be positive near the origin and $\vec{\ell} + \vec{s} \rightarrow \vec{j}$ is the order of coupling.

Our results in Table 1 are compared with those by Gillet et al.¹⁴⁾. The 3^- excited state (two components) differs little from that of Gillet et al. On the other hand, the lowest 2^+ level (seven components) shows, with Tabakin's interaction, considerably more configuration mixing than the corresponding (Tamm-Dancoff) result of Ref. 14 obtained with a Gaussian potential. Also, the excitation energy (4.63 MeV) almost coincides with the experimental value (4.43 MeV).

The high degree of collectivity obtained gives us confidence in using such wave functions in the present paper with the aim of developing the basic formalism and extracting the information on the validity of the frequently used approximations (DWIA, spin- and isospin-dependence of the elementary amplitude). At the same time, the p-h approach allows for a natural extension to the np-nh excitations. The two-particle two-hole components should be both technically feasible and physically appealing, in the cases of ^{12}C and ^{16}O , to describe the two-step excitation process.

5. NUMERICAL CALCULATIONS

It is well known that the spin dependence of the nucleon-nucleon amplitude is not only necessary in principle, but also for phenomenological purposes. Bourrely et al.¹⁵⁾ have noted that the differential cross-section is under-evaluated, even in the forward direction, if one uses only the spin-independent part of the amplitude determined by the optical theorem and Coulomb interference. Thus

$$\frac{d\sigma}{dt} = \frac{\pi}{k^*{}^2} |f^*(q)|^2 (1+\delta) \quad (5.1)$$

where

$$f^*(q) = \frac{k^4 \sigma_{tot}}{4\pi} (i + \alpha) e^{-Bq^2}$$

is the usual parametrization of the nucleon-nucleon amplitude⁵⁾. The value for δ is 0.29 for proton-proton scattering; for neutron-proton it is 0.12 if we use the Bugg data extracted from deuteron scattering, or 0.34 if we take results from neutron beam experiments¹⁶⁾. By introducing the parameter δ we approximately take into account the square of the double spin-flip amplitudes which are different from zero in the forward direction. The simplest way to do this is to take an artificially large value of the real part of the nucleon-nucleon amplitude; we use $\frac{1}{2}(\alpha_{pp} + \alpha_{np}) = -0.5$, taking the Bugg value for the n-p total cross-section. This procedure is based on the hypothesis that the double spin-flip amplitudes are incoherent with the scalar amplitude even in the nuclear case, which is rigorously true if we neglect the distortion factor. We showed this in Section 3. With this procedure we raise the cross-section by 20%, which is needed to have theory and experiment agree, within the theoretical error bars. The error in theory comes from the error in $d\sigma/dt$, which is of the order of 10% in the forward direction and bigger for larger angles, but not big enough to account for the large-angle discrepancy.

In Fig. 1a we show the results for the 3^- (9.64 MeV) excited state of ^{12}C and in Fig. 1b for the 2^+ (4.44 MeV) state. The experimental results are those of Bertin et al.¹⁷⁾. The wave functions are those of Gillet and Vinh Mau¹⁴⁾, and formula (2.6) is used for the solid-line curves. This calculation -- "exact Glauber" -- is explained in Appendix B. The harmonic oscillator parameter is $\alpha^2 = 0.401 \text{ F}^{-2}$ ⁵⁾, and B in (5.1) is 0.106 F^2 . The dotted-line curves are obtained from formula (2.9). The difference is very small, noticeable only for higher momentum transfers. Thus it is justified to treat the distortion operator as rotationally invariant.

These cross-sections are the squares of the amplitudes (2.3) summed over M_f . For 3^- the amplitudes vanish for $M_f = 0$ and ± 2 for 2^- when $M_f = \pm 1$.

In Fig. 2 we show the square of the electromagnetic form factor F_{em}^2 for the transition from the ground state to the 3^- state in (a) and to the 2^+ state in (b). These results are obtained with the same computer program as for proton scattering, without distortion and multiplying the differential cross-section by a q-dependent factor:

$$F_{em}^2(q) = \frac{\exp(Bq^2)}{A^2 k^+{}^2 |\sigma_{tot}(i+a)/4\pi|^2} \frac{d\sigma}{d\Omega}. \quad (5.2)$$

The experimental results are those of Cranell¹⁸⁾ and Nakad et al.¹⁹⁾. The results disagree surprisingly with experiment for the 3^- state around $q^2 = 4 F^{-2}$, where the agreement with proton scattering is very good.

It is interesting to check if the optical limit of the distortion factor is good for ^{12}C . This approximation leads to (2.11), where

$$\langle D \rangle = \exp \left[-(A-1) \int d^3r \, \Gamma(\vec{r}-\vec{z}) \rho_0(r) \right] \quad (5.3)$$

where

$$\Gamma(\vec{r}-\vec{z}) = \Gamma(0) \exp \left[-(\vec{r}-\vec{z})^2/4B \right], \quad \Gamma(0) = f(0)/(2ikB)$$

$$\rho_0(r) = \frac{1}{3} (R\sqrt{\pi})^{-3} \left(1 + \frac{4}{3} \frac{r^2}{R^2} \right) \exp(-r^2/R^2), \quad R = 1.58 F.$$

After integration we obtain

$$\langle D \rangle = \exp \left\{ - \frac{(A-1)f(0)}{18ikB} R'^2 \left(5 + 4 \frac{b^2}{R^2} \frac{1}{\chi^2} + 4 \frac{R'^2}{R^2} \right) e^{-\frac{b^2}{R^2}} \right\} \quad (5.4)$$

where

$$\frac{1}{R'^2} = \frac{1}{R^2} + \frac{1}{4B}, \quad \chi = 1 + \frac{4B}{R^2}$$

The results are shown in Fig. 3. We see that the optical limit (which means, among other things, that we neglect the non-diagonal elements in D) is rather good for ^{12}C , but worse than our approximation (2.9).

The shell-model calculations of Gillet and Vinh Mau¹⁴⁾ were repeated, as explained in Section 4, with a realistic nuclear interaction. The resulting wave functions yield the results shown in Fig. 4 for the proton inelastic cross-sections. For the 3^- state there is no significant change of cross-sections at low-momentum transfers, but the agreement in normalization for the 2^+ state is much better. Let us stress that the excitation energy of the 2^+ level is much closer to the experimental value than it was previously¹⁴⁾. From Table 1 one can see the corresponding change in the configuration mixing.

At $q^2 = 5 F^{-2}$ a minimum in the cross-section for the 3^- state has been predicted³⁾, in disagreement with experiment. This was discussed by Saudinos and

Wilkin³⁾ in considerable detail. In our calculation this oscillation for the 3^- level is damped, and the cross-section curves have the same slope (in a log plot) as the experimental points at higher momentum transfers, but the normalization is no longer right. This is because the real part of the nucleon-nucleon amplitude is large in order to simulate the effect of the spin-flip amplitudes. For smaller values of the parameter α , the old unsatisfactory results are reproduced.

It is worth stressing that for higher momentum transfers, our results for 3^- are too small, whereas for 2^+ they are too large.

The electromagnetic form factors obtained with our improved wave functions are also plotted in Fig. 2. The fit for the 2^+ case is improved, but the disturbing situation for 3^- remains. For this state, the transition charge radius given by the shell-model wave functions is about 25% too small.

In Fig. 5 we show the prediction for the density matrix elements:

$$\rho_{M_f M_f'} = F_{00, J_f M_f}^* F_{00, J_f M_f'} / \sum_{M_f} |F_{00, J_f M_f}|^2 \quad (5.5)$$

where F is the amplitude (2.3). This may be compared with an analysis of the angular distribution of decay photons^{20,21)}. It is noticeable that the excited states are almost completely aligned along the incident beam, since $2\rho_{22}$ and $2\rho_{33}$ are close to 1. This could provide an interesting test of the deformation of the neutron distribution, as proposed by Kofoed-Hansen and Jacob²²⁾. Similar results were obtained by Manaenkov²³⁾, where configuration mixing was not taken into account.

Finally, we calculate the cross-section for $\pi^- ^{12}\text{C} \rightarrow \pi^- ^{12}\text{C}^*(2^+)$; this requires only a change in the parameters in the elementary amplitude (5.1). The $I = 0$ part of the π^- nucleon total cross-section²⁴⁾ and the real part given by dispersion relations²⁵⁾ were used. The result is given in Fig. 6 and compared with the data of Koester et al.²⁰⁾ at 4.5 GeV/c. The agreement is fair except for a slight difference at the peak.

In conclusion we can say that with our present knowledge of hadron-nucleon amplitudes, we can fit the inelastic scattering data of hadron- ^{12}C reasonably well at small momentum transfers. At high momentum transfers the theory itself becomes questionable²⁶⁾ and perhaps more detailed nuclear information, such as the presence of two-particle two-hole components in the wave function, or deformations²⁷⁾, is needed to understand the data.

APPENDIX A

To show the relation between hadron and electron inelastic scattering, we calculate explicitly the two matrix elements of formula (2.13):

$$\begin{aligned}
 \langle \tilde{\psi}_3 | e^{i \vec{q}'_T \cdot \vec{r}_1} | \psi_i \rangle &= \sum_{g, m_g} (-1)^{g-m_g} \sum_{m_p} \begin{bmatrix} p & h & J_3 \\ m_p & -m_n & M_3 \end{bmatrix} \times \\
 &\times \int d^3 r_1 \dots d^3 r_A \phi_{p, m_p}^*(\vec{r}_1) \prod_{\substack{g', m'_g \neq h, m_h \\ e \neq 1}} \phi_{g', m'_g}(\vec{r}_e) e^{i \vec{q}'_T \cdot \vec{r}_1} \times \\
 &\times \phi_{g, m_g}(\vec{r}_1) \det \{ \phi_{g'', m''_g}(\vec{r}_e) \} .
 \end{aligned} \tag{A.1}$$

The integral is different from zero only if $g, m_g = h, m_h$, therefore

$$\begin{aligned}
 \langle \tilde{\psi}_3 | e^{i \vec{q}'_T \cdot \vec{r}_1} | \psi_i \rangle &= \sum_{m_p} \begin{bmatrix} p & h & J_3 \\ m_p & -m_n & M_3 \end{bmatrix} (-1)^{h-m_h} \times \\
 &\times \int d^3 r_1 \phi_{p, m_p}(\vec{r}_1) e^{i \vec{q}'_T \cdot \vec{r}_1} \phi_{h, m_h}(\vec{r}_1) .
 \end{aligned} \tag{A.2}$$

Doing the same for the form factor, one obtains

$$\begin{aligned}
 \langle \psi_3 | e^{i \vec{q}'_T \cdot \vec{r}_1} | \psi_i \rangle &= \frac{1}{A!} \sum_{g, m_g} (-1)^{g-m_g} \sum_{m_p} \begin{bmatrix} p & h & J_3 \\ m_p & -m_n & M_3 \end{bmatrix} \times \\
 &\times \sum_i (-1)^{i+1} \int d^3 r_1 \dots d^3 r_A \phi_{p, m_p}^*(\vec{r}_i) \det \{ \phi_{g', m'_g}^*(\vec{r}_e) \} \times \\
 &\times e^{i \vec{q}'_T \cdot \vec{r}_1} \phi_{g, m_g}(\vec{r}_1) \det \{ \phi_{g'', m''_g}(\vec{r}_e) \} .
 \end{aligned} \tag{A.3}$$

If $i \neq 1$ the integral is zero because of the orthogonality of ϕ_{p, m_p} to all single-particle functions ϕ_{g, m_g} . It is zero also if $g, m_g \neq h, m_h$, for similar reasons. Doing the integration of the minors, one obtains

$$\langle \psi_j | e^{i \vec{q}'_T \cdot \vec{r}_1} | \psi_i \rangle =$$

$$= \frac{1}{A} \sum_{m_p} \begin{bmatrix} p & h & J_j \\ m_p & -m_h & M_j \end{bmatrix} (-1)^{h-m_h} \int d^3 r_1 \phi_{p, m_p}^*(\vec{r}_1) e^{i \vec{q}'_T \cdot \vec{r}_1} \quad (A.4) \\ \times \phi_{h m_h}(\vec{r}_1).$$

Comparing this with (A.2), the result (2.13) follows.

APPENDIX B

The matrix element $\langle p, m_p | \Gamma(\vec{r} - \vec{s}) | g, m_g \rangle$ in (2.6) is a linear combination (with C.G. coefficients) of two (or one) integrals of the form

$$\int d^3r R_{n\ell}(r) R_{n'\ell'}(r) Y_{\ell m}^*(\hat{r}) Y_{\ell' m'}(\hat{r}) \Gamma(\vec{r} - \vec{s}) \quad (B.1)$$

where $\Gamma(\vec{b} - \vec{s}) = \Gamma(0) \exp \{ -(\vec{b} - \vec{s})^2 / 4B \}$ and $R_{n\ell}$ are harmonic oscillator radial wave functions²⁸⁾. These integrals are evaluated analytically. Writing

$$Y_{\ell m}^*(\theta, \varphi) Y_{\ell' m'}(\theta, \varphi) = e^{i(m'-m)\varphi} \sum_{k=0}^{\ell+\ell'} a_k(\ell m, \ell' m') \sin^k \theta \times \\ \times \cos^{\ell+\ell'-k} \theta$$

the integral (B.1) becomes, since $s = r \sin \theta$ and $z = r \cos \theta$:

$$\Gamma(0) e^{-b^2/4B} \sum_{k=0}^{\ell+\ell'} a_k(\ell m, \ell' m') \int_{-\infty}^{\infty} dz \int_0^{\infty} s ds e^{-s^2/4B} \times \\ \times s^k z^{\ell+\ell'-k} r^{-(\ell+\ell')} \int_0^{2\pi} d\varphi e^{i(m'-m)\varphi} e^{bs \cos(\varphi - \varphi_0)/2B}$$

where the integral over ϕ gives a Bessel function J . Since $R_{n\ell} R_{n'\ell'}$ is of the form $r^{\ell+\ell'} \exp(-\alpha^2 r^2)$ times a polynomial in r^2 , where α^2 is the harmonic oscillator parameter, the integration over z factors out. It is of the trivial form (p is an integer):

$$\int_{-\infty}^{\infty} dz e^{-\alpha^2 z^2} z^p.$$

The final integral over s is of the form

$$\int_0^{\infty} ds e^{-a^2 s^2} s^{r+1} J_q(-ib s / 2B) = \\ = \frac{1}{2} e^{b^2/(16B^2 a^2)} \left(\frac{-ib}{4B} \right)^q a^{-(r+q+2)} \times \\ \times \Gamma((r+q+2)/2) M((q-r)/2, q+1; -b^2/(16B^2 a^2)) / \Gamma(q+1) \quad (B.2)$$

where $a^2 = \alpha^2 + 1/4 B$. This formula is obtained from Ref. 29, p. 48b, with one Kummer transformation (Ref. 29, p. 505). The confluent hypergeometric functions M

in (B.2) all reduce to polynomials. The matrix elements (B.1) vanish when $m + m'$ is even if $\ell + \ell'$ is odd, and when $m + m'$ is odd if $\ell + \ell'$ is even.

The determinant D_{g, m_g}^{h, m_h} in (2.6) is obtained as the minor of the corresponding element in the matrix calculated by Guardiola⁵).

For each of the (2 or 7) configurations in the wave functions we have a sum like (2.6) which must be evaluated for $M_f = 1$ and 3 for the 3^- case, and for $M_f = 0$ and 2 for 2^+ . The largest number of non-vanishing terms we get in these sums is 12, in the case if the $(1p\ 3/2)^{-1} (1f\ 7/2)$ configuration for $M_f = 2$.

Since the matrix element in the integrand in (2.3) is a sum of products of exponentials in $-b^2$ and powers of b , the final integration can also be done analytically by a formula similar to (B.2). If very many configurations are included, or if one wishes to treat two-particle two-hole excitations, one should probably set up a computer program to do the C.G. and configuration mixing algebra and multiply out all the functions of b . We did the algebra by hand and the final integration in (2.3) numerically, without working out the explicit b -dependence of the integrand.

Table 1

Decomposition of the shell model wave function for ^{12}C

3^-	E	$1p_{3/2}$ $1d_{3/2}$	$1p_{3/2}$ $1d_{5/2}$
Gillet and Vinh Mau	13.4	-0.47	0.88
Ours	12.70	-0.627	0.778

2^+	E	$1p_{3/2}$ $1p_{1/2}$	$1p_{3/2}$ $1f_{5/2}$	$1p_{3/2}$ $1f_{7/2}$	$1p_{3/2}$ $2p_{1/2}$	$1p_{3/2}$ $2p_{3/2}$	1s $1d_{3/2}$	1s $1d_{5/2}$
Gillet and Vinh Mau	8.23	0.94	-0.05	0.23	0.07	-0.07	-0.13	0.19
Ours	4.63	0.765	-0.166	0.43	-0.005	-0.033	-0.258	0.367

The sign of the two last components of Vinh Mau's 2^+ wave function has been changed because they used a sign convention different from ours.

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Figure captions

- Fig. 1 : Cross-section of inelastic scattering of 1.04 GeV protons on ^{12}C . The solid line is the result of (2.3) with (2.6) as transition matrix element ("exact" Glauber theory) and the dashed line with approximate expression (2.9). Wave functions of 3^- (Fig. 1a) and 2^+ (Fig. 1b) nuclear states are taken from Gillet and Vinh Mau (Ref. 14).
- Fig. 2 : Square of the electromagnetic form factor calculated with Gillet's parameters (Ref. 14) (dashed line) and ours (see Table 1, solid line), compared with experimental data. (a) and (b) as in Fig. 1.
- Fig. 3 : Comparison between the theoretical predictions using expression (2.9) for the transition matrix element (dashed line) and using the optical limit of the distortion factor (2.10) (solid line).
- Fig. 4 : Test of the shell-model wave function. Cross-section of the inelastic scattering of 1 GeV protons calculated with the wave function corresponding to the Tabakin two-nucleon potential (see Table 1). Figures 4a and 4b correspond to the excitation of the 3^- and 2^+ states of ^{12}C , respectively.
- Fig. 5 : Density matrices as functions of the momentum transfer.
a) solid line: $2\rho_{33}$ and dashed line: $2 \operatorname{Re} \rho_{31}$;
b) solid line: $2\rho_{22}$ and dashed line: $2 \operatorname{Re} \rho_{20}$.
- Fig. 6 : Comparison of the theory with experiment for $\pi^- ^{12}\text{C} \rightarrow \pi^- ^{12}\text{C}^*(2^+)$.

Fig. 1a

Fig. 1b

Fig. 2a

Fig. 2b

Fig. 3a

Fig. 3b

Fig. 4a

Fig. 4b

Fig. 5a

Fig. 5b

Fig. 6