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ABOUT THE ZERO MODE PROBLEM IN THE
QUANTIZATION OF SOLITONS

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ABSTRACT

It is shown that the zero mode singularities cancel in the manifestly covariant approach to the quantization of solitons.

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During the last two years several ways to calculate the quantum corrections to properties of localized particle-like solutions of classical field equations were developed ^{1),2)}. The most natural and manifestly Lorentz covariant approach based on the functional integral faces an obstacle called the "zero mode problem". To show what it is, let us consider the two-dimensional model of a scalar self-interacting field $u(x,t)$ with the action functional

$$A[u] = \int \left(\frac{1}{2} (\dot{u}_t^2 - u_x^2) - V(u) \right) dx dt$$

It is the common wisdom of the functional integral method that the scattering amplitude is given by the integral

$$\int \exp \left\{ \frac{i}{\hbar} A[u] \right\} \prod_{x,t} du(x,t)$$

where we are to integrate over the field histories with the prescribed asymptotic behaviour

$$u(x,t) \rightarrow u_{\text{in}} \quad , \quad t \rightarrow -\infty$$

$$u(x,t) \rightarrow u_{\text{out}} \quad , \quad t \rightarrow +\infty$$

and u_{in} , u_{out} are the appropriate asymptotic functions [see, e.g., Ref. 3]. When classical solutions localized in space are present they are to be included in the list of these asymptotics. The stationary phase method can be described via Feynman diagrams in x - t space in terms of a propagator

$$G = H^{-1} \quad , \quad H = \square + V''(u) \big|_{u=u_d}$$

and of vertices

$$V_n = V^{[n]}(u) \big|_{u=u_d} \quad , \quad n=3,4,\dots$$

Here u_{cl} is a solution of the classical field equations, connecting the given asymptotic functions,

$$\square u_d + V'(u_d) = 0 \quad , \quad u_d \rightarrow u_{\text{in}} \quad , \quad t \rightarrow -\infty$$

$$u_d \rightarrow u_{\text{out}} \quad , \quad t \rightarrow +\infty$$

When u_{cl} is a localized soliton solution, depending on some parameters $\alpha(a,\dots)$, where a is a centre-of-mass and we do not specify other internal co-ordinates (if any); then the derivatives $u_\alpha = (\partial/\partial\alpha)u_{cl}$ satisfy the equation:

$$\square u_\alpha + V''(u_d) u_\alpha = 0 \quad (1)$$

and vanish rapidly when $|x| \rightarrow \infty$. This fact is considered as an indication that the propagator $(H+i\epsilon)^{-1}$ has a pole when $\epsilon \rightarrow 0$ so that G is not defined and the Feynman rules are to be modified. To deal with this problem, several methods were developed ^{1),2)}, most of them based on the introduction of collective co-ordinates ¹⁾. The purpose of this letter is to show that, in fact, there is no pole in $(H+i\epsilon)^{-1}$ and that G can be defined in such a way that the ordinary Feynman rules are valid.

We shall illustrate this point in the example of one and two loop calculations with u_{cl} taken to be a one-soliton solution, rather than give a detailed formal proof in the general case. Our version of such a proof will be published elsewhere. We hope that after the comment is made it will become evident, so that each specialist will be able to supply his own proof, using his favourite methods.

The main observation is that u_α are not square integrable functions over space-time, in spite of the fact that they vanish rapidly when $|x| \rightarrow \infty$. This shows that they belong to the continuous spectrum of H , so that G is non-unique rather than non-defined.

More specifically, let

$$u_d(x, t) = \psi \left(\frac{x - vt - a}{\sqrt{1-v^2}} \right)$$

be a one-soliton solution; $\psi(r)$ assumes its constant asymptotic values rapidly when $|r| \rightarrow \infty$. The function

$$\psi_a = \psi'(r) / N, \quad N^2 = \int_{-\infty}^{\infty} |\psi'(r)|^2 dr$$

is an eigenfunction of the operator $\hat{h}$

$$\hat{h} = - \frac{d^2}{dr^2} + Q(r), \quad Q(r) = v''(u_d)$$

with zero eigenvalue; we suppose for simplicity that besides this eigenvalue $\hat{h}$ has only pure continuous spectrum with eigenfunctions $\psi_k(r)$

$$\hat{h} \psi_k = (k^2 + m^2) \psi_k, \quad -\infty < k < \infty,$$

$$\int_{-\infty}^{\infty} \psi_k^*(r'') \psi_k(r') dk = \delta(r'' - r') - \psi_a(r'') \psi_a(r'),$$

where m is a mass of the field $u(x)$. The resolvent $R(\epsilon) = (H + i\epsilon)^{-1}$ has a kernel

$$R(x'', t''; x', t'; \epsilon) = \frac{i}{2\sqrt{i\epsilon}} e^{-i\sqrt{i\epsilon} |\tau'' - \tau'|} \psi_a(\tau'') \psi_a(\tau') + \\ + \frac{i}{2} \int_{-\infty}^{\infty} \frac{dk}{k_\epsilon} e^{-ik_\epsilon |\tau'' - \tau'|} \psi_k^*(\tau'') \psi_k(\tau') dk,$$

$k_\epsilon = (k^2 + m^2 + i\epsilon)^{1/2}$, $\tau = x \sinh \gamma + t \cosh \gamma$, $\tau = (x-a) \cosh \gamma + t \sinh \gamma$, $\cosh \gamma = (1 - v^2)^{-1/2}$
and when $\epsilon \rightarrow 0$

$$R(x'', t''; x', t'; \epsilon) = \frac{\psi_a(\tau'') \psi_a(\tau')}{2i\sqrt{i\epsilon}} + G_0(x'', t''; x', t'),$$

where

$$G_0 = G_d + G_c = \frac{1}{2} |\tau'' - \tau'| \psi_a(\tau'') \psi_a(\tau') + \\ + \frac{i}{2} \int_{-\infty}^{\infty} \frac{dk}{k_0} e^{-ik_0 |\tau'' - \tau'|} \psi_k^*(\tau'') \psi_k(\tau')$$

It is easy to see that G_0 is an honest Green's function of H .

$$H G_0 = \delta(x'' - x') \delta(t'' - t')$$

This is so because of the fact that T and G_0 differ by a $1/\sqrt{\epsilon}$ term rather than by a pole. We propose to use

$$G = c \psi_a(\tau'') \psi_a(\tau') + G_0$$

as a propagator and claim that physical answers do not depend on the constant c . Alternatively, it can be said that $1/\sqrt{\epsilon}$ term cancels in the calculations.

The one-loop contribution is given in terms of the determinant $\Delta = \det H$ and the only way to calculate Δ is to differentiate it first. For example, we have in the notation $\psi_r = \partial_\gamma u_d = \tau \psi'(\tau)$

$$\frac{d}{d\gamma} \ln \Delta = \text{tr} (H^{-1} \psi'''(u_d) \psi_r) = \\ = \text{tr} (G_0 \psi'''(u_d) \psi_r) + c \int \psi'''(u_d) \psi_a^2 \psi_r dx dt. \quad (2)$$

Now ψ_a and ψ_r satisfy equations of the form (1). Combining them with the following equation

$$\square \psi_{aa} + v'''(u_d) \psi_a^2 + v''(u_d) \psi_{aa} = 0 \quad (3)$$

we get that

$$v'''(u_d) \psi_a^2 \psi_r = \square \psi_r \psi_{aa} - \square \psi_{aa} \psi_r = \tau \frac{d}{d\tau} (\psi_a' \psi_{aa} - \psi_{aa}' \psi_a)$$

so that the integral in (2) vanishes.

The two-loop answer is given by

$$-\int V_4(x, t) G^2(x, t; x, t) dx dt + \int V_3(x_1, t_1) V_3(x_2, t_2) \times \\ \times \left[\frac{2}{3} \left(G(x_1, t_1; x_2, t_2) \right)^3 + G(x_1, t_1; x_2, t_2) G(x_2, t_2; x_1, t_1) G(x_2, t_2; x_2, t_2) \right] dx_1 \dots dt_2$$

and contains contributions proportional to c^3 , c^2 and c . Now the c^3 term is proportional to the integral

$$C_3 = \int v'''(u_d) \psi_a^3 dx dt,$$

which was shown to vanish. The c^2 term is given by

$$C_2 = - \int v^{[IV]}(u_d) \psi_a^4 dx dt + \\ + 3 \int V_3(x_1, t_1) \psi_a^2(x_1, t_1) G_0(x_2, t_2; x_2, t_2) \psi_a^2(x_2, t_2) V_3(x_2, t_2) dx_1 \dots dt_2$$

The last integral can be rewritten by means of Eq. (3) for ψ_{aa}

$$\int G_0(x_1, t_1; x_2, t_2) \psi_a^2(x_2, t_2) V_3(x_2, t_2) dx_2 dt_2 = -\psi_{aa}(x_1, t_1),$$

so that

$$C_2 = - \int (v^{[IV]}(u_d) \psi_a^4 + 3 v'''(u_d) \psi_a^2 \psi_{aa}) dx dt$$

which is equal to a derivative of c_3 and also vanishes.

The term linear in c is proportional to $(d^2/da^2) \ln \Delta$ and also vanishes because Δ does not depend on a . This concludes our illustration.

Several comments may be made.

1. The expression for G we have used is not the most general one compatible with the radiation boundary conditions. We can use

$$G = c \psi_a(r'') \psi_a(r') + d \left(\psi_a(r'') \psi_v(x', t') + \psi_v(x'', t'') \psi_a(r') \right) + G_0$$

and the physical answers will not depend on c and d .

2. One can observe that the first term in G_0 is an increasing function of $|\tau'' - \tau'|$. Nevertheless, it does not lead to divergences in time integration. Indeed, this term vanishes when $(x'', t'') = (x', t')$, so it does not give any contribution in one loop. Contributions in two loops, not including the continuous spectrum part of the propagator, i.e., G_c , are proportional to $(d/da) \ln \Delta$ or $\int v''' (u_a) t_a^3 dx dt$ and vanish. There remain terms of the form

$$\int (|\tau_1 - \tau_2| \psi_a(r_1) \psi_a(r_2))^2 G_c(x_1, t_1; x_2, t_2) dx_1 dt_2,$$

which do converge in $(\tau_1 - \tau_2)$ because of the oscillations in G_c .

3. In the collective co-ordinates' approach one uses G_c as a propagator and introduces some additional vertices besides V_n . Integrals similar to the last one substitute for the contributions of these vertices in our approach.

4. An analogous superficial zero mode problem exists in the non-relativistic quantum mechanical scattering problem. Let us illustrate it with the example of a particle with one degree of freedom. The action functional has the form

$$A_{t'}^{t''} [q(t)] = \int_{t'}^{t''} \left(\frac{1}{2} \dot{q}^2 - V(q) \right) dt$$

and the scattering amplitude is proportional to

$$\lim_{\substack{t'' \rightarrow \infty \\ t' \rightarrow -\infty}} \int \exp \left\{ \frac{i}{\hbar} A_{t'}^{t''} [q] \right\} \prod_{t' \leq t \leq t''} dq(t) \quad \begin{aligned} q(t) &\rightarrow v'' t'' + q_0'', \quad t \rightarrow \infty; \\ q(t) &\rightarrow v' t' + q_0', \quad t \rightarrow -\infty, \end{aligned}$$

where the limit does not depend on q_0'' and q_0' [see, e.g., Ref. 3]. Quantum corrections to the transmission are to be calculated with the classical solution $q_{cl}(t)$ for which $v'' = v'$. The definition of the propagator H^{-1} , where

$$H = \frac{d^2}{dt^2} + V''(q) \Big|_{q=q_d}$$

apparently faces the zero mode problem because $\dot{q} = (d/dt)q_{cl}(t)$ is an eigenfunction of H with zero eigenvalue. But the function $\dot{q}$ is not a bound state of H but a virtual state (we use the usual quantum scattering terminology to describe eigenfunctions of H). It is easy to show that

$$(H + i\varepsilon)^{-1}(t'', t') = \text{const} \frac{\dot{q}(t'')\dot{q}(t')}{\sqrt{\varepsilon}} + G_0(t'', t'),$$

where G_0 is well-defined for $\varepsilon=0$. The Feynman rules with

$$G = c \dot{q}(t'')\dot{q}(t') + G_0(t'', t')$$

as a propagator and $V_n = v^{[n]}(q_{cl}(t))$ as vertices can be shown to lead to an ordinary quasiclassical expansion for the transmission coefficient. In particular, this expansion is independent of c .

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