Ref.TH.2133-CERN

A STUDY OF THE LANDAU HYDRODYNAMICAL MODEL PREDICTIONS
FOR HIGH ENERGY INCLUSIVE CHARGED PARTICLE PRODUCTION

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ABSTRACT

The predictions of the Landau hydrodynamical model are compared to the experimental results on the one - particle distributions from proton-proton collisions at the CERN ISR.

Data from all available collision energies, $23 < \sqrt{s} < 63$ GeV and longitudinal rapidity in the centre of mass less than 3 units are considered for transverse momenta below 0.8 GeV/c for pions, kaons and antiprotons. It is found that the model can account for the data in a satisfactory way, particularly for the pion distributions.

Both one - dimensional and three - dimensional versions of the model are considered. The parameters of the model are the velocity of sound in nuclear matter, c_0 , which describes the expansion and cooling of the fluid, and the break-up temperature, T_c , at which the particles emerge. The best agreement between the theoretical and experimental distributions is obtained for values of these parameters well below those from earlier investigations. The values are, however, compatible to those found in proton - nucleus investigations. The values of best agreement do, however, for the one - dimensional model, depend upon the range of phase space where the comparison is made. For the three - dimensional model, the parameter values are less sensitive to the range considered.

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1. INTRODUCTION

In this note we will compare the detailed predictions of the hydrodynamical model ¹⁾ for particle production to the experimental results which are available at present from the storage rings at CERN.

We will, in particular, consider the one-particle inclusive distributions ²⁾ for pions, kaons and antiprotons over the available energy range:

$$23 < \sqrt{s} < 63 \text{ GeV},$$

the rapidity range

$$|y_{cm}| < 3,$$

and for values of transverse momenta below 0.8 GeV/c.

We have excluded the proton spectrum because of the experimentally noticeable contributions from a diffractive component which is not accounted for in this model. Further, the large transverse momentum spectrum will not be discussed in this note.

The hydrodynamical model was introduced by Landau and his collaborators ^{1),3)}. It is a phenomenological device to describe the expansion and break-up into particles of an initial state, in which the whole available energy is assumed to be concentrated in a small Lorentz-contracted volume. The model contains two parameters: the velocity of sound c_0 which describes the expansion stage of the fluid, and the break-up temperature T_0 which determines the end of the expansion stage.

The model does not contain any restrictions due to the conservation of quantum numbers besides energy and momentum. In the language of thermodynamics, the influence of the chemical potentials on the energy is completely neglected. The results of our investigation are that the predictions of the model are systematically less in agreement with the data for kaons and antiprotons where such conservation laws may have large influence (e.g., "local compensation of strangeness and baryon number").

Some features of the hydrodynamical model are known to be qualitatively in agreement with the experimental results. In particular, the model predicts that

- 1) the inelasticity ⁴⁾, defined as the fraction of the total energy that is available for production of secondaries, is around 0.4 to 0.5 at present energies;

- 2) the distribution of particles in longitudinal rapidity space for a fixed value of transverse momentum is not flat, but exhibits curvature [i.e., like the "Landau Gaussian" distribution ¹⁾];
- 3) there is a rather fast rise, depending on c_0 , with energy of the one-particle distributions in the centre of the longitudinal rapidity space, compatible with a power dependence ^{1),5)} ("break of Feynman scaling");
- 4) the distribution in transverse momentum (for momenta below 1 GeV/c) is essentially given by the thermal motion, i.e., an exponential fall-off with a slope given by the inverse temperature T_c^{-1} at break-up of the fluid (with $T_c \simeq m_\pi$ in suitable units).

In the original papers of Landau ¹⁾ and his collaborators, the velocity of sound is determined from its value for a relativistic gas of massless particles in collision equilibrium ("black body radiation"): $c_0 = \sqrt{1/3}$.

Two possible models have been advanced to describe the residual dynamics of the particles after the fluid break-up. In the simplest model one assumes that the velocity distribution of the particles is equal to the corresponding fluid distribution at the time of the break-up into particles. (Actually, the isotherms are more general spacelike surfaces in space-time.) The analytical solution for the one-dimensional hydrodynamical equations then provides [cf. Eq. (27)] the well-known Gaussian rapidity distribution for the longitudinal motion of the bulk of the particles.

In order to account for the motion in the transverse direction, Landau ¹⁾ argued that the angular distribution of the emitted particles is essentially determined already at this one-dimensional stage. Because of the lack of compression in transverse directions in the original state and the corresponding smallness of the sidewise forces, the emission angle of the different parts of the fluid will not, according to Landau's ideas, change much in the final stages of the expansion. For that reason Landau exchanged the longitudinal particle rapidity to the pseudo-rapidity $\ln(\text{tg } \theta/2)$. This approximation should be valid for small masses. This picture, which was developed when only angular distributions from cosmic ray experiments were available, is however too crude to account for the presently available detailed experimental data and we will not consider it further.

In the second model the residual dynamics is determined by the thermal motion of a gas of free relativistic particles ^{3),5),6)}. This motion is superimposed upon the fluid motion along the critical isotherm (cf. Section 3). In such a picture, there will be a very definite coupling between the longitudinal and the transverse modes of motion.

The mean value of the transverse momentum will in both cases increase with the c.m.s. energy, $\sqrt{s}$, like a small power, $s^{1/12}$ or $s^{1/14}$ according to estimates ³⁾ [cf. Eq. (16)]. In later generalizations of the hydrodynamical model, different values of the velocity of sound c_0 have been considered.

In particular, the possibility that c_0 approaches the velocity of light has aroused some interest ⁷⁾. In that case the resulting distributions may exhibit certain similarities to the ones achieved in multiperipheral dynamics. Thus one may, in particular limits, achieve an energy-independent value of the mean transverse momentum and also obtain the flat "rapidity plateau". Such a limit must, however, be treated with the utmost care [cf. the discussion at Eq. (23)] because at the same time the inelasticity coefficient will vanish ⁴⁾ (i.e., there is no energy available for particle creation).

If, at the ISR energies, the relevant dynamics is governed by the resonance spectrum ^{5),8)} and the resonances exhibit narrow widths and a mass spectrum, $\rho(m)$, described by

$$\rho(m) = b (m/m_\pi)^a, \quad (1)$$

the velocity of sound becomes a constant:

$$c_0 = \sqrt{\frac{1}{4+a}}. \quad (2)$$

As for the residual dynamics, when $c_0 \neq \sqrt{1/3}$, the natural choice would be to consider the complete (three-dimensional) solution of the hydrodynamical model where the particle distribution is taken to be equal to the fluid motion. In the more general case, one may superimpose the thermal motion of an ideal gas upon the fluid motion (with appropriate continuity requirements). We will in this note use the latter possibility.

It is, in our opinion, not a valid test of the model to make use of only parts of its features and to neglect other properties. Thus the use of a Gaussian longitudinal rapidity distribution, together with an independent parametrization of the transverse motion ⁹⁾, is not a test of the properties of the hydrodynamical model.

We would finally like to comment upon one feature of the model that has raised some criticism in the past. In hadronic collisions the leading particles, in general, carry away half of the available energy, but in the hydrodynamical model the whole available energy is, in the initial state,

deposited in a small Lorentz contracted volume. Therefore, it has been argued that this initial state should be relevant only for the few very central collisions in which the incoming hadrons are stopped in the collision.

In the hydrodynamical model the time development is conveniently partitioned into three stages: the compression stage, the first emission stage and finally the expansion and cooling of the main part of the fluid.

The first two stages are terminated already after a time corresponding to a Lorentz contracted hadronic radius, while the third stage is characterized by a much longer time scale.

The solutions of the equations of motion which describe the first emission stage (the progressive wave solutions) exhibit a qualitatively different behaviour from those of the subsequent motion of the main part of the fluid. In particular, these waves contain some parts with large velocities. Estimates show that about half of the available energy is carried away in this way. The entropy content of the progressive waves, which is a measure of the mean number of particles carrying the energy, is however small and almost independent of the energy.

Therefore, unless a "peripheral" property of the collision can be ascertained on a time-scale of the order of a Lorentz contracted hadronic radius, the assumptions on the initial state in the hydrodynamical model are consistent with our knowledge of the energy distribution in hadronic interactions.

Explicit computations in the model exhibit that the predictions are, in general, independent of the precise properties assumed about the initial volume. There is, e.g., only a linear dependence upon the longitudinal parameter ("the hadron radius"). The only important geometrical parameter is, according to the numerical solution for the three-dimensional model, the ratio, R , between the transverse and the longitudinal widths of the volume where the energy is deposited.

It is a particular feature of the three-dimensional model that the contributions to the one-particle distributions decrease fast with decreasing ratio R . If the effect of different impact parameters is taken into account by an averaging procedure over R (assuming that the available energy for particle production is proportional to a power of the overlap region between the hadrons, i.e., to a power in R), then the contributions will come almost entirely from large values of R , i.e., from small impact parameters. In this way the model is "internally consistent" in the sense that of all

possible initial states to be averaged over, the ones giving the largest contributions are the ones close to central collisions (complete overlap). This picture is evidently rather different from the one emerging from, e.g., multiperipheral interactions.

It should be noted that the properties of the initial state may be tested on appropriate time scales in hadron-nucleus collisions ¹⁰⁾. In that case the impinging hadron, in general, performs several collisions, e.g., inside the Ag and Br nuclei, which are available in photographic emulsion. The properties of such collisions, according to the presently available experimental information, are well described by the hydrodynamical model ¹¹⁾.

2. THE HYDRODYNAMICAL MODEL

In this and the following sections we will present a few features of the hydrodynamical model that will be used in this investigation. We will only, in some cases where new results are presented, write detailed formulae. There are extensive reviews ⁵⁾, and most of the original work is published.

The model contains two sets of variables describing the internal state and the motion, respectively, of the fluid elements.

The thermodynamical variables which describe the internal state of the fluid elements are densities such as the energy densities and the entropy density, referred to the rest system of the fluid and further the temperature and the pressure. If the influence of viscosity and chemical potentials are neglected, then all of them may be expressed in terms of, e.g., the temperature which during the fluid expansion, in general, depends upon the space-time point. The initial temperature T_0 can be related to the c.m.s. energy $\sqrt{s}$ under the assumption that all the energy is initially deposited in a Lorentz contracted and compressed volume $V(s)$:

$$\begin{aligned} V(s) &= 2 \pi a^2 \Delta(s) , \\ \Delta(s) &= d \cdot \frac{c_0^2}{1 + c_0^2} \cdot \frac{2M}{\sqrt{s}} . \end{aligned} \tag{3}$$

The parameters d and a are measures of the longitudinal and the transverse dimensions of the hadrons at rest. In the "effective" longitudinal length $\Delta(s)$ the factor $c_0^2/(1+c_0^2)$ stems from the initial compression, while $2M/\sqrt{s}$ is the Lorentz contraction factor. The symbol M is used for the proton mass (m for the pion mass). We will, from now on, write the formulae in a way relevant to proton-proton collisions.

The total entropy is conserved for a non-viscous gas during the expansion, and in this case is proportional to the mean multiplicity of secondaries $\bar{N}$ ¹⁾. Computed in the initial state one attains from the volume dependence in Eq. (3) and from the thermodynamical relationship between energy density, entropy density and the temperature

$$\bar{N}(s) = k \left(\frac{s}{M_m} \right)^{\frac{1}{2} \frac{1-c_0^2}{1+c_0^2}}, \quad (4)$$

with a suitable constant k ⁵⁾.

Except for the set of internal variables, the fluid state is described by a velocity field u^μ (conveniently parametrized by the three-velocity $\vec{v}$):

$$\begin{aligned} u_0 &= (1 - \vec{v}^2)^{-1/2}, \\ \vec{u} &= \vec{v} (1 - \vec{v}^2)^{-1/2}. \end{aligned} \quad (5)$$

The initial state develops in space and time during expansion and cools in such a way that the energy momentum tensor of the fluid is a conserved quantity:

$$\partial_\mu T^{\mu\nu} = 0. \quad (6)$$

Equation (6) constitutes a system of four hyperbolic first-order differential equations for the unknown T and $\vec{v}$.

There is an analytic solution in terms of elementary functions only in the case when the transverse motion is neglected⁷⁾. This solution is most easily described in terms of a function of two variables $\psi = \psi(\tau, \eta)$. The fluid state at the point in space-time (x_ℓ, t) is then characterized by the temperature T and the longitudinal velocity v_ℓ :

$$\begin{aligned} T &= T_0 \exp \tau; \quad \tau \leq 0, \\ v_\ell &= \tanh \eta, \\ x_\ell &= \exp(-\tau) \cdot \left(\frac{\partial \psi}{\partial \tau} \sinh \eta - \frac{\partial \psi}{\partial \eta} \cosh \eta \right), \\ t &= \exp(-\tau) \cdot \left(\frac{\partial \psi}{\partial \tau} \cosh \eta - \frac{\partial \psi}{\partial \eta} \sinh \eta \right). \end{aligned} \quad (7)$$

An explicit expression for Ψ can be derived in terms of an integral over the Bessel function I_0 . For our purpose only, the following combination is of interest [this result can be derived for arbitrary values of c_0 in a way similar to the one employed in Ref. 7) for $c_0 = \sqrt{1/3}$]:

$$\chi = \partial_{\tau} \Psi - \Psi = \frac{\Delta(s)}{c_0} \exp\left(-\tau \frac{1-c_0^2}{2c_0^2}\right) I_0\left(\frac{1-c_0^2}{2c_0^2} \sqrt{\tau^2 - c_0^2} \eta\right). \quad (8)$$

The critical surface where the particles emerge is determined by the requirement that the internal energy of the fluid corresponds to the value for a free particle in its rest system, i.e., for pions $\sim m/\pi a^2 d$. In terms of hydrodynamical concepts, this corresponds to a fixed value of the temperature, i.e., of the variable τ . One obtains as required that

$$\tau \equiv \tau_c = -\frac{c_0^2}{1+c_0^2} \ln\left(\frac{s}{mM\Lambda}\right) \equiv -\frac{c_0^2}{1+c_0^2} \ell. \quad (9)$$

Here $\Lambda=1$ if the break-up temperature is the pion mass [cf. Eqs. (34)-(35)]. In Eq. (9) it is implicitly assumed that the original volume is Lorentz contracted. If this is not the case, then τ_c should be exchanged to (τ_c') .

$$\tau_c' = -\frac{c_0^2}{1+c_0^2} \ln\left(\frac{2\sqrt{s}}{m\Lambda}\right). \quad (9')$$

A very accurate approximation to this analytic solution is derived in Landau's original paper ¹⁾ (it is called the quasiscaling approximation by later authors).

$$\alpha = \frac{1}{2} \ln\left(\frac{t+x_\ell}{t-x_\ell}\right), \quad t' = \sqrt{t^2 - x_\ell^2}, \quad (10)$$

$$\beta = \ln\left(\frac{t'}{\Delta(s)}\right),$$

$$\tau = \tau_c(t', \alpha) = \frac{1}{2}(1-c_0^2)\sqrt{\beta^2 - \alpha^2} - \frac{1}{2}(1+c_0^2)\beta,$$

$$\eta = \alpha - \frac{1}{4} \ln\left(\frac{\beta + \alpha}{\beta - \alpha}\right) \equiv \alpha - f. \quad (11)$$

Equation (10) can be solved for t' as a function of α and τ_c along the critical isotherm:

$$t' = \Delta(s) \exp \beta = \Delta(s) \left(\frac{s}{mM\Lambda} \right)^{1/2} \exp \left(\frac{1-c_0^2}{2c_0^2} \sqrt{\tau_c^2 - c_0^2 \alpha^2} \right). \quad (12)$$

Landau's approximate solution is an accurate description of the space-time development of that part of the fluid which contains most of the entropy (and therefore the part from which most of the final state particles emerge). It is evident from Eq. (10) that the motion of the fluid elements in the longitudinal direction occurs approximately with a constant velocity, i.e., the quantity $f \sim 0$:

$$\tanh \eta \approx \tanh \alpha = \frac{x_L}{t}. \quad (13)$$

Milekhin ³⁾ has taken this fact as a starting point to investigate the influence of the transverse expansion of the fluid. He assumes that the transverse expansion and cooling is independent of this longitudinal velocity of the fluid elements, i.e., of the variable α . With the further assumption of azimuthal symmetry the transverse development of the fluid then only depends upon the corresponding "proper time" t' in Eqs. (11) and (12) and the transverse radial co-ordinate r .

This is written

$$\begin{aligned} \tau(x_L, r, t) &= \tau_L(x_L, t) + \tau_\perp(r, t'), \\ \eta(x_L, r, t) &= \alpha + f(x_L, t) f'(r, t), \\ \xi &= \xi(r, t'). \end{aligned} \quad (14)$$

Here τ_L and f are determined in Eq. (10), while the quantity $\tau_\perp$ describes "the cooling" of the fluid stemming from its transverse expansion.

The variable ξ in Eq. (14) is the transverse rapidity. In terms of η and ξ the fluid velocity field is given by:

$$\begin{aligned} u_0 &= \cosh \eta \cdot \cosh \xi, \\ u_L &= \sinh \eta \cdot \cosh \xi, \\ u_\perp &= \sinh \xi. \end{aligned} \quad (15)$$

The derivatives of f and f' are neglected in this approximation. Even for this simplified problem, it is not possible to find an analytical solution, and one must take recourse to numerical calculations. Milekhin argued that there is no strong dependence on the variable r . Therefore, he converted the partial differential equations into a set of ordinary differential equations in t for the r averaged quantities $\bar{\tau}_1$, $\bar{\xi}$ and $\bar{f}$. These equations are then solved and the solution for $\bar{\tau}_1$, $\bar{\xi}$ and $\bar{f}$ is presented in parametric form in his original paper ³⁾. He finds for $c_0 = \sqrt{1/3}$, along the critical isotherm, that the transverse motion exhibits a slow energy dependence and a noticeable dependence upon the value of the longitudinal rapidity:

$$\sinh \xi = 0.53 \left(\frac{s}{2M^2} \right)^{1/14} \exp \left(-\frac{\eta^2}{6L} \right). \quad (16)$$

Here the parameter L stands for

$$L = 0.56 \ell, \quad (17)$$

with ℓ defined in Eq. (9).

The original system of partial differential equations can also be solved without the r averaging procedure in the same approximation as Milekhin uses, by means of finite difference methods along the characteristics. A detailed report of such an investigation and the dependence on the initial conditions of the results for arbitrary values of the velocity of sound c_0 will appear elsewhere ¹¹⁾.

This investigation shows that though the r dependence of the quantities used, in general, is slow (cf. below and Fig. 1) the one-particle distributions are nevertheless influenced. Therefore, the averaging procedure in r introduced by Milekhin ³⁾ is, in our opinion, not justified.

3. THE ONE-PARTICLE INCLUSIVE DISTRIBUTION

In this Section we will present the explicit expressions that are used in the comparison between the experimental and theoretical distributions. According to well-known formulae ⁸⁾, the one-particle distributions are described by integrals along the critical isotherms of the distributions that are used for the break-up dynamics.

The formulae used by Milekhin ³⁾ for the superposition of fluid motion and thermal motion have been criticized because they do not fulfil the energy sum rules. The difference between his formulae and the ones employed here ⁸⁾ is, in general, however more a matter of principle than of numbers.

We will always assume that the fluid elements in their rest systems can be treated as a relativistic gas in collision equilibrium and with a thermal motion appropriate to the break-up temperature T_c . Then the inclusive particle distribution of each element is given by

$$F(X) = \left(\exp \frac{X}{T_c} \pm 1 \right)^{-1}, \quad (18)$$

$$X = \bar{E} = u_0 E - \vec{u} \cdot \vec{p}.$$

Here $\bar{E}$ is the particle energy in the rest system of the gas fluid element, $(\vec{p}, E)$ is the particle energy momentum vector when the fluid moves with the four-velocity U_μ , while the signs $\pm$ in the formula for F correspond to fermions and bosons, respectively.

The one-particle distribution is given by

$$E \frac{d^3 N}{d p^3} = \int_{S(T_c)} d s_\mu (\vec{x}, t) \cdot p^\mu F(X(\vec{x}, t)). \quad (19)$$

Here the integration volume $S(T_c)$ is the isotherm corresponding to the break-up temperature T_c . Neglecting small terms in the integrand and introducing the longitudinal fluid rapidity variable η as a parameter along $S(T_c)$, the integral is transformed into

$$E \frac{d^3 N}{d p^3} = k' \int d^2 r \int d\eta \left(E \frac{dx_L}{d\eta} - p_L \frac{dt}{d\eta} \right) F(\mu_T \cosh(y - \eta) \cdot \cosh \xi - p_T \cos(\theta - \varphi) \sinh \xi). \quad (20)$$

Here k' is a constant and y and μ_T are the longitudinal rapidity and transverse mass of the particle:

$$y = \frac{1}{2} \ln \left(\frac{E + p_L}{E - p_L} \right), \quad (21)$$

$$\mu_T = \sqrt{p_T^2 + m^2}.$$

The derivatives indicated in the integrand can easily be expressed in terms of derivatives of the function χ in Eqs. (7) and (8). Then the integration element in Eq. (20) becomes

$$G \equiv E \frac{dx_L}{d\eta} - p_L \frac{dt}{d\eta} = -\mu_T \exp(-\tau_c) \cdot (\cosh(y-\eta) c_0^2 \cdot \partial_{\tau} \chi + \sinh(y-\eta) \partial_{\eta} \chi). \quad (22)$$

Explicit expressions for the derivatives of χ are

$$\begin{aligned} \exp(-\tau_c) \partial_{\tau} \chi &= \frac{1-c_0^2}{2c_0^2} \cdot \frac{\Delta(s)}{c_0} \left(\frac{s}{mM\Lambda}\right)^{1/2} \cdot \left(I_0(z) - \frac{\tau_c}{\sqrt{\tau_c^2 - c_0^2 \eta^2}} I_0'(z)\right), \\ \exp(-\tau_c) \partial_{\eta} \chi &= \frac{1-c_0^2}{2c_0^2} \cdot \frac{\Delta(s)}{c_0} \left(\frac{s}{mM\Lambda}\right)^{1/2} \cdot \frac{\eta c_0^2}{\sqrt{\tau_c^2 - c_0^2 \eta^2}} I_0'(z), \\ z &= \frac{1-c_0^2}{2c_0^2} \cdot \sqrt{\tau_c^2 - c_0^2 \eta^2} \equiv \frac{1}{2} \cdot \frac{1-c_0^2}{1+c_0^2} \cdot \sqrt{\ell^2 - \eta^2 \frac{(1+c_0^2)^2}{c_0^2}}. \end{aligned} \quad (23)$$

We note, in particular, that the terms in Eq. (23) will all vanish in the limit $c_0 \rightarrow 1$, which indicates the necessity to treat this limit carefully⁹⁾.

Using a well-known approximation for the Bessel functions, we can rewrite the integration element to order η^2/ℓ^2 :

$$G \equiv G_1 = \mu_T \exp(z) \cdot (\cosh(\eta-y) - \frac{\eta(1+c_0^2)}{2c_0^2 \ell} \sinh(\eta-y)). \quad (24)$$

The quantity z is defined in Eq. (23).

In case the contributions from the term proportional to $\sinh(\eta-y)$ are neglected (which turn out to be appropriate according to explicit computations for the range of experimental values investigated by us), we can, for small values of η , expand the square root in the exponent to get

$$G \equiv G_2 = \mu_T \cosh(y-\eta) \left(\frac{s}{mM\Lambda}\right)^{1/2} \frac{1-c_0^2}{1+c_0^2} \exp\left(-\frac{\eta^2(1-c_0^4)}{4c_0^2 \ell}\right). \quad (25)$$

Except for the factor corresponding to the energy of the particle in the fluid rest system $[\mu_T \cosh(y - \eta)]$ this expression is seen to contain the total multiplicity of secondaries [cf. Eq. (4)] as well as a Gaussian rapidity distribution. The width of the Gaussian is seen to grow with the logarithm of the c.m.s. energy in a similar way as the available particle phase space.

It is instructive to note that the results in Eq. (25) can be achieved from the Landau solution if the small quantity f is neglected in Eqs. (10). Then

$$G = \mu_T t' (\cosh(\alpha - y) \frac{\partial \alpha}{\partial \eta} + \sinh(\alpha - y) \frac{\partial t'}{\partial \eta} \cdot \frac{1}{t'}) \approx G_1. \quad (26)$$

In addition to the two quantities G_1 and G_2 defined above, we have also made use of the more accurate expressions.

$$G_3 = \mu_T (\cosh(y - \eta) \cdot (I_0(z) + \frac{L}{\sqrt{L^2 - \frac{(1 + c_0^2)^2}{c_0^2} \eta^2}} I_1(z)) - \sinh(y - \eta) \cdot \frac{\eta (1 + c_0^2) \frac{1}{c_0^2}}{\sqrt{L^2 - \frac{(1 + c_0^2)^2}{c_0^2} \eta^2}} I_1(z)), \quad (27)$$

in the comparison of the predictions from the one-dimensional hydrodynamic model to the experimental data. In all these cases the r integral in Eq. (20) can be performed directly and we have

$$\left(\frac{d^3 N}{dy dp_T^2} \right)_j = A_j \int d\eta G_j F(\mu_T \cosh(y - \eta)), \quad j = 1, 2, 3, \quad (28)$$

where A_j is a constant and F and G_j are determined by Eqs. (18), (24), (25) and (27).

For the three-dimensional hydrodynamical expansion we have considered both the results of Milekhin described above and the explicit solution without r averaging.

The following formula for the integration element G is relevant to Milekhin's work:

$$G \equiv G_M = \mu_T \cosh(y - \eta) \exp(-0.113\ell + 0.363\sqrt{\ell^2 - 4.92\eta^2}). \quad (29)$$

One is then left with an integral over the fluid longitudinal rapidity and the angle φ between the transverse motion of the fluid and the particle momenta:

$$\left(\frac{d^3N}{dy dp_T^2}\right)_M = \frac{A_M}{2\pi} \int_0^{2\pi} d\varphi \int d\eta G_M F(\mu_T \cosh(y - \eta) \overline{\cosh \xi} - p_T \overline{\sinh \xi} \cos \varphi). \quad (30)$$

The expressions for F , G_M and for $\overline{\sinh \xi}$ are given by Eqs. (18), (29), and (16), respectively, while A_M is a constant.

In the case where the r dependence is taken into account, it is necessary to perform the integrals in a different way. In particular, it is necessary to know the rapidity distribution of the fluid along the iso-term for the integral in Eq. (20). From the numerical solution mentioned above ¹¹⁾, the following parametrization of the surface as well as the transverse rapidity distribution can be derived

$$\begin{aligned} r &= x \exp(\beta \sqrt{\ell^2 - \gamma \eta^2}), \\ t' &= h(r) \exp(\alpha \sqrt{\ell^2 - \gamma \eta^2}), \\ \sinh \xi &= g(r) \exp(\alpha' \sqrt{\ell^2 - \gamma \eta^2}). \end{aligned} \quad (31)$$

The parameters α , α' , β and γ depend upon the value of c_0 and the ratio R between the transverse and the longitudinal dimensions of the volume in which the initial energy is deposited. For the case when $R=1$ one has, inside the ISR range of energies, the values given in Table 1.

The functions h and g are depicted in Fig. 1 for given values of the sound velocity c_0 and for $R=1$.

For small values of t' (i.e., less than proper times of the order of a few Lorentz contracted hadronic radii), the parametrizations above should be modified. This is relevant for the fluid motion in the first moments after the collision, and involves a small amount of matter fluid being ejected with large transverse rapidity (the energy involved is, however, much smaller than the one carried by the progressive wave in the longitudinal direction according to the discussion in the Introduction). In contrast to

the part described by Eq. (32) the results depend upon the particular "geometrical form" that is assumed for the hadrons. Fortunately, these corrections do not contribute to the one-particle distributions for small values of the transverse momentum and we will neglect them in the following. The formula corresponding to Eqs. (28) and (30) is, in this case, after appropriate scaling according to the results in Eq. (31):

$$\left(\frac{d^3N}{dy dp_T^2}\right)_c = \frac{A_c}{2\pi} \int d\eta d\varphi \exp((2\beta + \alpha)\sqrt{\ell^2 - \gamma\eta^2}) \cdot \int x dx h(x) F(p_T \cosh(y - \eta)V(x) - p_T W(x) \cos \varphi), \quad (32)$$

with

$$W(x) = \exp(\alpha' \sqrt{\ell^2 - \gamma\eta^2}) \cdot g(x), \quad (33)$$

$$V(x) = \sqrt{1 + W(x)^2}.$$

In all the formulae above we have made use of the expression for ℓ defined by Eq. (9). Whenever temperatures T_c , essentially different from the pion mass m , have been considered we have used a temperature dependent value for the quantity Λ which occurs in Eq. (9). For values of Λ below the pion mass, the following expression is used:

$$\Lambda = (0.15 + 0.82 \frac{T_c}{m}) \cdot (\frac{T_c}{m})^4. \quad (34)$$

This value of Λ is computed from the energy density of an ideal pion gas at temperature T_c and is valid for $1/3(m) < T_c < m$. For values of $T_c > m$ one should use:

$$\Lambda = (\frac{T_c}{m})^4. \quad (35)$$

We have also investigated the effect of neglecting Lorentz contraction for the initial volume. This corresponds to exchanging the quantity $\Delta(s)$ in Eq. (3) to $\hat{\Delta} = d(c_0^2/1 + c_0^2)$. Effectively it will mean that the one-particle distribution should be multiplied by the factor $(s/2M)^{1/4}$, while in the implicit energy dependence the quantity τ_c in Eq. (9) is exchanged for τ'_c in Eq. (9').

4. COMPARISON WITH THE EXPERIMENTAL DATA

The model predictions have been compared to data on the inclusive charged hadron production in proton-proton collisions at the CERN ISR. The model is expected to give good predictions for central production and not too large transverse momenta. Fits have, therefore, been restricted to centre-of-mass rapidities less than 3 units, and transverse momenta less than 0.8 GeV/c. The experimental data ^{2a-d} come from four different experimental set-ups covering different angular regions. Normalization errors are taken into account by applying a 5% over-all uncertainty to the data from three of the experiments ^{2a-c}, and 3% to one more accurate experiment ^{2d} at 90° in the centre of mass. From experiments 2a-c) come data on production of pions, kaons and antiprotons, while from experiment 2d) only pion data were available. We have compared the model predictions of the distributions $E(d^3N/dp^3)$ to the experimental results for the invariant cross-section $E(d^3\sigma/dp^3)$ and also to the cross-section renormalized by σ_{inel} . There is little difference between the results obtained in the two cases. This is due to the relatively large uncertainties in the normalization of the experiments. In general, the uncorrected case will provide for a somewhat better χ^2 in the comparison. For the case of the one-dimensional formula, Eq. (24), the best agreement between theory and experiment gives a $\chi^2 = 1591$ for the renormalized and $\chi^2 = 1530$ for the uncorrected cross-section for 418 data points on π^+ . The difference in parameter values for best agreement is less than 1%. All model fits mentioned below have been made to the uncorrected cross-sections.

In order to compare the predictions of the one-dimensional model to the experimental data, we have made use of the formulae in Eqs. (24), (25) and (27), combined with the main expression in Eq. (28) (the different cases will henceforth be referred to as $j=1,2,3$, respectively). We have allowed the parameters A_j , T and c_0 to vary freely and minimized the χ^2 to find the best agreement.

In Table 2 the results are gathered from a comparison between the experimental results for π^+ using the different formulae for the one-dimensional model. The first line is the results for $j=1$ for all data points, the second for the approximation $j=2$. The third and fourth lines are the results for $j=3$ with and without the term containing $\sinh(y-\eta)$ in Eq. (27). The formula for $j=2$ is expected to be accurate only for small values of the fluid rapidity η , while the formula for $j=3$ should be accurate also for large values of η . There is evidently no essential difference in the use of the different formulae, although the full expression is slightly better.

We have tested the significance of the Lorentz contraction of the initial volume as indicated at the end of the last Section. We find that without this feature the fits are impossible ($\chi^2 > 10^6$).

In Table 3 the results of fits (for $j=3A$) to the distributions of different kinds of particles are given.

The value of the break-up temperature T_c , corresponding to the best agreement between one-dimensional hydrodynamical theory and experiment, is close to the pion mass independent both of the exact formula used and of the particle type. The value of the velocity of sound c_0 is smaller than the ones proposed so far for the over-all data.

The quality of the fits are clearly better for pions than for the other particle types. This may be due to the fact that the model does not contain any features that take into account the conservation laws which could play a role for the production of these particles with special quantum numbers. It should, however, also be noted that the data are less complete and of poorer quality for kaons and antiprotons than for pions.

The model accounts fairly well for the pion data. However, a number of systematic differences should be noted. These are clearly seen if the fits are made to restricted regions of phase space.

Thus, if the comparison is restricted to the π^+ data from the most accurate experiment (2d) measured at $y=0$, an excellent fit is obtained with $\chi^2/\text{NDF} = 96/93$ for $c_0 = 0.67 \pm 0.05$ ($T_c = 127$ MeV). (The fit is shown as solid lines on Fig. 2.) In this case the value of c_0 is determined only by the observed rise in cross-section with energy.

On the other hand, for the over-all fit the value of c_0 (0.48) is determined mainly by the widths of the rapidity distribution (because of the fairly large normalization errors). Therefore, in the one-dimensional model the observed width of the rapidity distribution corresponds to a larger s dependence at $y=0$ than is actually found (see Figs. 2 and 3). Also the best fitted value of the break-up temperature T_c depends on the p_T range considered. As can be seen from Table 4 there is a systematic increase in T_c with increasing p_T . A feature of the model which seems to be in agreement with experiment is the prediction of a turnover of the invariant cross-section for small values of p_T . The effect is noticeable in the experimental results, stemming from Böggild et al. ^{2d)} where the smallest values of p_T are explored. The model predicts that the effect should be more pronounced the larger the mass of the particle (because the effective variable is the longitudinal mass μ_T instead of p_T in the

formulae for small p_T). In the experiment of Alper et al., a turnover of the antiproton cross-section is seen at p_T values where there is still no turnover in the pion cross-section. At large p_T the hydrodynamical model predictions fall increasingly below the data.

The three-dimensional formula which contain integrations over both the radial variable r , the angle φ , and the fluid rapidity η , is much more complicated to compare to the experimental data. We have only considered the results for π^+ and we have further only investigated a set of discrete values of c_0 . Thus, only the temperature T_c and the normalization constant A are allowed to vary in Eq. (32).

From a comparison with the more accurate data of Böggild et al. at $y=0$, we obtain the following χ^2 for 93 degrees of freedom: $\chi^2(c_0 = \sqrt{1/3}) = 6354$, $\chi^2(c_0 = 0.5) = 805$, $\chi^2(c_0 = 0.4) = 166$, and $\chi^2(c_0 = 0.32) = 282$.

The break-up temperature T_c for $c_0 = 0.4$ is 93.0 ± 0.7 MeV, i.e., essentially below the results for the one-dimensional model. We have not tried to vary c_0 in order to find the exact position of the minimum in χ^2 . Figure 4 shows the fit to the data of Böggild et al., at $y=0$ for $c_0 = 0.4$.

If the formula in Eq. (32) is used for a fit to all the data from all four experiments for $p_T < 0.8$ GeV, we find $T_c = 96.4 \pm 0.3$ MeV and $\chi^2/\text{NDF} = 1289/416$. In this case the model gives too small a width for the observed rapidity distribution (Fig. 5), but too strong an s dependence (Fig. 4). The fit would probably be better for a value of c_0 slightly larger than 0.4. The goodness of the fit is sensitive to the parameters of Eq. (31) given in Table 1. In particular, if the parameter γ , which is only calculated to an accuracy of ± 1 , is left as a free parameter, the fit improves to give an χ^2/NDF of 1033 for $\gamma = 7.2$ (see Table 2).

From the appearance of the functions in the integrand of Eq. (32), it can be seen that there is a rapid dependence upon the radial variable r in the neighbourhood of $r = r_{\text{max}}$. In order to test the sensitivity of the parameters for best agreement to different sets of data (similarly as for the one-dimensional formula in connection with Table 4), we have considered the results of performing the integral in r only up to $r = 0.85 r_{\text{max}}$ for different ranges in p_T . In Table 5 the results of this investigation are exhibited. It is evident that the three-dimensional results are much less sensitive than the one-dimensional one to the range in p_T , and that the rapid variation with r is not of importance.

Milekhin's results discussed above are only derived for $c_0 = \sqrt{1/3}$. It is evident from Fig. 1 that the $\sinh \xi$ is sensitive to the value of c_0 and that the experimental data favour essentially smaller values of $\sinh \xi$ than the ones found for $c_0 = \sqrt{1/3}$.

Milekhin's formulae lead to an unacceptable fit (we find a χ^2/NDF of 2877/416 for $T_c = 65.7 \pm 0.2$ MeV). As an indication of the sensitivity of the results to the value of $\sinh \xi$, i.e., to the size of the fluid transverse velocity, we have considered Milekhin's formulae but with Eq. (16) exchanged for

$$\overline{\sinh \xi} = b \left(\frac{s}{2M\pi} \right)^{1/4} \cdot \exp \left(-\frac{\eta^2}{6L} \right).$$

If the parameter b is allowed to vary together with the break-up temperature T_c and the normalization constant A_M in Eq. (30), we find in a comparison with the experimental data that for $b = 0.288 \pm 0.008$ and $T_c = 101.9 \pm 1.4$ MeV there is a $\chi^2/\text{NDF} = 953/415$. Thus, the change of b from $b = 0.53$ in Eq. (16) to $b = 0.29$ means a dramatic change in agreement between the formulae and the experimental data (see Table 4).

5. CONCLUSION

The Landau hydrodynamic model allows for a qualitatively correct description of the one-particle distributions over the range in phase space which is considered here, i.e., centre-of-mass rapidities less than 3 units, transverse momenta less than 0.8 GeV/c and $23 < \sqrt{s} < 63$ GeV. The description is considerably better for pions than for other particle types. Even for pions there are, however, in the one-dimensional approximation of the hydrodynamical model, systematic differences between the model predictions and the data.

The dependence upon the c.m.s. energy, determined here mainly by the width of the rapidity distributions, tends to provide a larger rise with energy at $y=0$ than the one observed. For increasing p_T the data show a less rapid fall-off than the model prediction. For small values of p_T the model predicts a turnover in agreement with the experiment, probing the smallest values of p_T .

The values of the parameters for best agreement are, in the one-dimensional model, more sensitive to the range in phase space which is used for comparison with the experiment than in the three-dimensional description. This has a significantly improved rapidity dependence.

The velocity of sound in nuclear matter for best agreement between theory and experiments is found to be close to 0.4 c, rather than $\sqrt{1/3}$ c or $\sqrt{1/2}$ c¹²⁾ assumed previously.

The temperature at which the particle formation occurs is close to the pion mass in the one-dimensional model but assumes a much lower value, around 95 MeV, in the three-dimensional description.

We thank Maurice Jacob for reading the manuscript and for useful suggestions.

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Table 1

Parameters used in the three-dimensional model Eq. (31), for different values of velocity of sound, c_0 ($c = 1$)

c_0	α	β	α'	γ
$\sqrt{1/3}$	0.055	0.083	0.072	8.2
0.5	0.076	0.092	0.080	9.0
0.4	0.082	0.11	0.092	9.0
0.32	0.097	0.14	0.10	9.5

Table 2

Comparison between the fits to the π^+ data from experiments 2a-d within the range $23 < \sqrt{s} < 63$ GeV, $p_T \leq 0.8$ GeV/c and $|y_{cm}| < 3$ using the different one- and three-dimensional hydrodynamical formulae. $j = 3A$ corresponds to the full Eq. (27), whereas in $j = 3B$ the term proportional to $\sinh(y - \eta)$ is neglected

Formula	Eq.	Velocity of sound	Temperature (MeV)	χ^2/NDF
$j = 1$	(24)	0.547 ± 0.006	133.8 ± 0.3	1530/415
$j = 2$	(25)	0.480 ± 0.002	131.5 ± 0.3	1640/415
$j = 3A$	(27)	0.480 ± 0.002	133.9 ± 0.3	1551/415
$j = 3B$	(27)	0.438 ± 0.002	131.7 ± 0.3	1618/415
	(32)	0.4 ($\gamma = 9.0$)	96.4 ± 0.3	1289/416
	(32)	0.4 ($\gamma = 7.2 \pm 0.2$)	95.4 ± 3.5	1033/415
	(30)	$1/\sqrt{3}$ ($b = 0.53$)	65.7 ± 0.2	2877/416
	(30)	$1/\sqrt{3}$ ($b = 0.288 \pm 0.008$)	101.9 ± 1.4	953/415

Table 3

Fits to the one-dimensional formula, Eqs. (27), (28), for the different particle types using data from experiments 2a - c within the range $23 < \sqrt{s} < 63$ GeV, $p_T < 0.8$ GeV/c and $|y_{cm}| < 3$.

Particle type	Velocity of sound	Temperature (MeV)	χ^2/NDF
π^+	0.457 ± 0.005	135.7 ± 0.4	1304/320
π^-	0.407 ± 0.003	131.8 ± 0.3	1618/311
K^+	0.550 ± 0.022	139.4 ± 3.5	766/118
K^-	0.410 ± 0.006	133.3 ± 0.8	952/113
$\bar{p}$	0.400 ± 0.004	128.4 ± 2.2	876/186

Table 4

Fits to the one-dimensional formula, Eqs. (24), (28), for π^+ using data from experiments 2a - c within the range $23 < \sqrt{s} < 63$ GeV, $|y| < 3$ and p_T as given below.

p_T range (GeV/c)	Velocity of sound	Temperature (MeV)	χ^2/NDF
< 0.25	0.540 ± 0.015	100.8 ± 3.1	142/89
$0.25 - 0.45$	0.531 ± 0.009	131.3 ± 1.4	206/110
$0.45 - 0.85$	0.478 ± 0.003	151.6 ± 1.2	293/127

Table 5

Fits to the three-dimensional formula, Eq. (32), with $c_0=0.4$ using the same data as in Table 3.

Upper limit for r ($r_{\max} = 1$)	p_T range (GeV/c)	Temperature (MeV)	χ^2/NDF
1	< 0.25	91.4 ± 0.8	505/182
0.85	< 0.25	97.8 ± 0.8	519/182
1	$0.65 - 0.85$	99.0 ± 0.7	684/165
0.85	$0.65 - 0.85$	105.1 ± 0.6	673/165

FIGURE CAPTIONS

- Fig. 1 : The dependence of the functions h and g on x as defined in Eq. (31).
- Fig. 2 : The full formulation of the one-dimensional model as given in Eqs. (27), (28) is used for a comparison with the data of the experiment of Böggild et al. 2d), using that data only in the fit (solid line, $\chi^2/\text{NDF} = 96/92$), and all available data 2a-d) with $|y| < 3$, $p_T < 0.8 \text{ GeV}/c$ (hatched line).
- Fig. 3 : The full formulation of the one-dimensional model as given in Eqs. (27), (28), is used for a comparison with the data of experiments 2a-c) at $\sqrt{s} = 45 \text{ GeV}$. The solid line is the extrapolated fit to the data of (2d) only at $y=0$ and $p_T < 0.42 \text{ GeV}/c$ ($\chi^2/\text{NDF} = 2546/415$). The hatched line is the over-all fit to the data 2a-d) ($\chi^2/\text{NDF} = 1551/415$). The wiggles of the fitted lines at large $|y|$ are due to differences in p_T values of these points.
- Fig. 4 : The three-dimensional model as given in Eq. (32) with $c_0 = 0.4$ is used for a similar comparison as in Fig. 2. The solid line represents an $\chi^2/\text{NDF} = 166/93$.
- Fig. 5 : The three-dimensional model as given in Eq. (32) with $c_0 = 0.4$ is used for a similar comparison as in Fig. 3. The solid line represents an $\chi^2/\text{NDF} = 1408/416$ and the hatched line an $\chi^2/\text{NDF} = 1289/416$.

FIG.1

Fig. 2

Fig. 3

Fig. 4

Fig. 5