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ON THE EXISTENCE OF PHASE-SHIFT AMBIGUITIES IN ELASTIC
SPIN 0 - SPIN $\frac{1}{2}$ SCATTERING

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ABSTRACT

It is shown by construction that for any finite number of partial waves differential cross-sections and polarizations exist, which give rise to two distinct sets of phase-shifts.

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Suppose that, for an elastic spin 0 - spin $\frac{1}{2}$ scattering process two experimental quantities are known for all scattering angles θ and a fixed momentum k , namely the differential cross-section

$$k^2 \frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |g(\theta)|^2, \quad (1)$$

and the polarization ^{*)}

$$P = \frac{2 \operatorname{Re} [f(\theta)^* g(\theta)]}{|f(\theta)|^2 + |g(\theta)|^2}. \quad (2)$$

Once unitarity is imposed most cross-sections and polarizations will uniquely determine the amplitudes $f(\theta)$ and $g(\theta)$, except for the so-called generalized Minami ambiguity ¹⁾. It has, however, been shown for the case of S and P wave scattering that experimental quantities (1) and (2) exist, which give rise to two different sets of amplitudes, or, equivalently, two different sets of phase shifts ²⁾. Although it seems plausible from these examples that also for any finite number of partial waves exceptional cases may exist, so far such phase shift ambiguities have not been given. The purpose of this paper is to construct a simple example for an arbitrary, finite number of partial waves, thus proving that phase shift ambiguities persist for polynomial cross-sections and polarizations of any degree. For spinless elastic scattering the list of examples with a small number of partial waves ³⁾ has also recently been extended to an arbitrary finite number of waves ⁴⁾. In contrast to the spinless case, it is in spin 0 - spin $\frac{1}{2}$ scattering in principle possible to measure also $\operatorname{Im}(f^*g)$, which removes the ambiguities mentioned above.

For the construction of the example, two sets of amplitudes $\{f, g\}$ and $\{f', g'\}$ are considered, where

$$\begin{aligned} f'(\theta) &= \frac{A}{A^*} f^*(\theta), \\ g'(\theta) &= \frac{A}{A^*} g^*(\theta), \end{aligned} \quad (3)$$

^{*)} For convenience a factor i has been absorbed in $g(\theta)$.

with $A=f(0)$ the forward scattering amplitude. It can easily be seen that they lead to the same quantities (1) and (2). The question is whether the partial wave amplitudes in both cases obey elastic unitarity. That is, in the expansion

$$\begin{aligned} f(\Theta) &= \sum_{\ell=0}^L \left[(\ell+1) f_{\ell+} + \ell f_{\ell-} \right] P_{\ell}(\cos \Theta) = \\ &= \frac{1}{2i} \sum_{\ell=0}^L \left[(\ell+1) \zeta_{\ell+} + \ell \zeta_{\ell-} - (2\ell+1) \right] P_{\ell}(\cos \Theta), \end{aligned} \quad (4)$$

$$\begin{aligned} g(\Theta) &= i \sum_{\ell=1}^L (f_{\ell+} - f_{\ell-}) \sin \Theta \frac{d P_{\ell}}{d \cos \Theta} = \\ &= \frac{1}{2} \sum_{\ell=1}^L (\zeta_{\ell+} - \zeta_{\ell-}) \sin \Theta \frac{d P_{\ell}}{d \cos \Theta}, \end{aligned} \quad (5)$$

the quantities $\zeta_{\ell\pm}$ must have modulus 1. They represent then elastic scattering in terms of phase shifts $\delta_{\ell\pm}$:

$$\zeta_{\ell\pm} = \exp(2i \delta_{\ell\pm}). \quad (6)$$

Similarly, the quantities $\zeta'_{\ell\pm}$ corresponding to f' and g' should also take the form (6) with phase shifts $\delta'_{\ell\pm}$. The relation between $\zeta_{\ell\pm}$ and $\zeta'_{\ell\pm}$ can be found by means of the equations

$$\begin{aligned} \zeta_{\ell+} &= \frac{2A}{2\ell+1} \left[i a_{\ell} + \ell b_{\ell} \right] + 1, \quad \ell \geq 0, \\ \zeta_{\ell-} &= \frac{2A}{2\ell+1} \left[i a_{\ell} - (\ell+1) b_{\ell} \right] + 1, \quad \ell \geq 1, \end{aligned} \quad (7)$$

with

$$a_\ell = \frac{(\ell+1) f_{\ell+} + \ell f_{\ell-}}{A}, \quad \ell \geq 0,$$

$$b_\ell = \frac{i(f_{\ell+} - f_{\ell-})}{A}, \quad \ell \geq 1. \quad (8)$$

The coefficients a_ℓ are not independent, but satisfy

$$\sum_{\ell=0}^L a_\ell = 1, \quad (9)$$

as follows from expression (4) for $\theta=0$. Relation (3) implies that the substitution

$$a_\ell, b_\ell \rightarrow a_\ell^*, b_\ell^* \quad (10)$$

in Eq. (7) transforms $\zeta_{\ell\pm}$ into $\zeta'_{\ell\pm}$.

The next step in the construction of the ambiguity is the determination of coefficients a_ℓ and b_ℓ for which

$$|\zeta_{\ell\pm}(a_\ell, b_\ell)| = |\zeta_{\ell\pm}(a_\ell^*, b_\ell^*)|, \quad (11)$$

and

$$|\zeta_{\ell\pm}(a_\ell, b_\ell)| = 1. \quad (12)$$

Condition (11) requires a_ℓ to be real. If not, it can only be realized when A is purely imaginary. This in turn means that $\delta_{\ell+} = \delta_{\ell-}$ for all ℓ . The polarization will be identically zero, which is not an interesting case.

At least one of the b_ℓ 's must be complex in order to get different sets of phase-shifts from the transformation (10). Consider for this particular ℓ value ($\ell > 0$), Eqs. (11) and (12). In terms of

$$A = X + i Y \quad (13)$$

the conditions can be satisfied, when

$$a_\ell = \frac{1}{2} (2\ell+1) \frac{Y}{|A|^2} , \quad (14)$$

$$b_\ell = |b_\ell| \exp(i \beta_\ell) , \quad (15)$$

where

$$|b_\ell| = \frac{2\ell+1}{2[\ell(\ell+1)]^{1/2}} \frac{Y}{|A|^2} , \quad (16)$$

$$\cos \beta_\ell = \frac{1}{2[\ell(\ell+1)]^{1/2}} \frac{Y}{X} . \quad (17)$$

The X and Y values must be such that β_ℓ is a real angle, i.e.,

$$0 \leq Y \leq 2[\ell(\ell+1)]^{1/2} |X| . \quad (18)$$

The positivity of Y stems from the fact that it gives the total cross-section.

For $\ell=0$ one obtains two possibilities to satisfy the conditions (11) and (12). Either

$$a_0 = 0 , \quad (19)$$

or

$$a_0 = \frac{Y}{|A|^2} . \quad (20)$$

When one wants to take b_n real for a specific n value, several solutions for a_n and b_n can be found from (11) and (12), e.g., $a_n = b_n = 0$. For non-trivial solutions, conditions like Eq. (18) will have to be obeyed. Since the expression for a_n will be different from Eq. (14), the requirement (9) will turn out to be more involved than the case where all b_ℓ are complex. Since our principal aim is to demonstrate the existence of ambiguities, we do not present results for the case where some b_n are real. Also for a_0 a choice will be made, namely Eq. (20). The interdependence of the a_ℓ 's through Eq. (9) gives a quadratic relation between X and Y . Choosing X to be positive (which means a choice of sign for the polarization P) one obtains

$$X = \left\{ \frac{1}{2} [L(L+2) + 2] Y - Y^2 \right\}^{1/2}, \quad (21)$$

The Y values will be in an interval which follows from the most restrictive condition in Eq. (18) :

$$0 \leq Y \leq \frac{4}{9} [L(L+2) + 2]. \quad (22)$$

Taking a Y value in this interval, X is determined by Eq. (21) and the phase shifts follow from

$$\begin{aligned} \zeta_0 &= 1 + 2i \frac{A Y}{|A|^2}, \\ \zeta_{\ell+} &= \frac{A}{|A|} \left\{ \frac{X}{|A|} + \left(\frac{\ell}{\ell+1} \right)^{1/2} \frac{Y}{|A|} \exp \left[i \arccos \left(\frac{1}{2[\ell(\ell+1)]^{1/2}} \frac{Y}{X} \right) \right] \right\}, \\ \zeta_{\ell-} &= \frac{A}{|A|} \left\{ \frac{X}{|A|} - \left(\frac{\ell+1}{\ell} \right)^{1/2} \frac{Y}{|A|} \exp \left[i \arccos \left(\frac{1}{2[\ell(\ell+1)]^{1/2}} \frac{Y}{X} \right) \right] \right\}, \end{aligned} \quad (23)$$

for $\ell = 1, \dots, L$. The $\zeta'_{\ell\pm}$ are obtained by taking the complex conjugate of the expression inside the brackets $\{ \}$. Since the $\zeta_{\ell\pm}$ in Eq. (23) only depend on the ratio Y/X the L dependence apparent in Eqs. (21) and (22) can be scaled away by choosing

$$\bar{Y} = \frac{2 Y}{L(L+2) + 2} \in \left[0, \frac{8}{9} \right], \quad (24)$$

and

$$\bar{X} = \frac{2 X}{L(L+2) + 2} = (\bar{Y} - \bar{Y}^2)^{1/2}, \quad (25)$$

instead of X and Y in Eq. (23). Of course, the total cross-section Y does depend on L , but the phase shifts of the example only depend on ℓ . For large ℓ values one obtains $\delta_{\ell\pm} \simeq \delta'_{\ell\pm}$.

The behaviour at the end points of the interval (22) or (24) deserves some attention. For $\bar{Y} \rightarrow 0$ (and consequently $\bar{Y}/\bar{X} \rightarrow 0$) all phase shifts $\delta_{\ell\pm}$ and $\delta'_{\ell\pm}$ become zero, as it should be for a vanishing total cross-section. In the other end point of the interval one could expect also an equality (mod π) of the phase shifts since this is the case for the various S and P wave examples ²⁾. Here it turns out not to be true anymore for D and higher waves. For completeness, the S , P and D wave case is shown in Figs. 1a and 1b.

Summarizing, polynomial cross-sections and polarizations of any degree exist for which the phase shift solution shows a non-trivial twofold ambiguity.

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FIGURE CAPTION

The phase shift ambiguity for S, P and D waves, as a function of the parameter $\tilde{Y}$. The solid lines represent $\delta_{\ell\pm}$, the dashed lines $\delta'_{\ell\pm}$.

FIG.1a

FIG.1.b