Ref.TH.2205-CERN

THE POLARIZABILITY SPIN-ORBIT POTENTIAL

R. Tarrach
CERN - Geneva

A B S T R A C T

The results known for spin and velocity independent forces due to the exchange of two photons are extended to spin-orbit forces. Special attention is devoted to the comparison with classical (external field) forces. A new effect has to be taken into account : the Thomas precession. A potential is obtained which allows a quick estimate of the polarizability corrections to the fine structure.

Ref.TH.2205-CERN
10 August 1976

1. INTRODUCTION

Recently, Bernabéu and the author ¹⁾ performed a complete study of the spin and velocity independent forces of electromagnetic origin which dominate the long distance interaction between any two systems (particles, nuclei, atoms, etc.) when one, at least, is neutral and one, at least, is spinless. These forces are of order e^4 , since they are due to the exchange of two photons. It was found that, expanding the potential in inverse powers of the distance, the two longest range terms were completely determined by the electromagnetic polarizabilities of the particle(s) (apart from charge, magnetic moment and the masses). A relation between forces due to virtual photons and parameters measured with real photons was thus established. For those cases in which a classical (external field) analogue exists, the first term turned out to be independent of the masses and therefore identical to its classical (static field) limit. In other words, the classical result, when it exists, is always the correct longest range force and not only in the static field limit (infinitely heavy field source). The range of validity of the obtained forces was also given.

A question arises immediately : whether one can extend these results to spin dependent forces - and if so, how far. The first part of the question will have an affirmative answer if : i) also for spin dependent forces the long-range (inverse power behaved) part is uniquely determined by real photon Compton scattering amplitudes, and ii) its longest range part is given by the value of these amplitudes at a physical point. This is indeed the case, since the dispersive approach followed in Ref. 1) [and used before for Van der Waal's forces by Feinberg and Sucher ²⁾] may be taken over to spin dependent forces without modification. We will outline the technique in Section 2.

Let us now turn to the second part of the question. We will not, as done in Ref. 1), perform a complete study of spin dependent forces, but instead choose a specific case of both theoretical and phenomenological interest. The following points will guide our choice. Spin dependent forces may be of several kinds (spin-spin, spin-orbit, tensorial, etc.). First, by forcing one particle to be spinless, only spin-orbit forces will be left. There is one main reason for this choice. Spin-orbit forces are necessarily velocity dependent, which is a qualitatively new feature. It therefore will show whether the identity of the largest range term and the classical potential is generalizable to both spin and velocity dependent forces. Second, by taking the particle with spin to be a lepton, no spin dependent polarizabilities (structure constants) will be introduced. This will enable us to carry out the comparison with classically

obtained potentials, since no classically undefined magnitudes will enter. Third, although the spinless particle may be charged, we will consider only forces not due to its charge. The justification for this last choice is that the forces due to the charge are well known [at least to order e^2 , see Eq. (20)], since they are responsible for the fine structure of hydrogenic atoms. Their effects, and not only to lowest order in the fine structure constant $e^2/4\pi$, are included in the theoretical expression for the fine structure energy difference [see Ref. 3]. To summarize, the consequences of our triple choice are the following :

- i) the forces will be both spin and velocity dependent. The comparison with classical (static field) limits is therefore extended to include both spin and velocity effects;
- ii) no classically undefined quantities (like spin dependent structure constants) are introduced. This makes the comparison with the classical calculation possible.
- iii) the obtained potential allows a quick estimate of the polarizability contribution to the fine structure of hydrogenic atoms for spinless nuclei.

Let us dwell more on this last point. The computation of dispersive or polarizability corrections to atomic levels is highly non-trivial. The example of muonic ^4He serves as an illustration of the complexity of these calculations ⁴⁾. But in many cases the potential

$$V(r) = - \frac{1}{2} \frac{e^2}{4\pi} \frac{\alpha}{r^4} \quad (1)$$

where α is the electric polarizability of the nucleus, allows a straightforward estimate of polarizability corrections to atomic levels [see Ref. 5]. This is the case for muonic ^4He , where an estimate with the potential (1) led to a more careful study and eventually turned out to be correct within a factor of two ⁶⁾. But the potential (1) cannot give polarizability corrections to fine structure, since being a spin independent interaction, its net effect on the fine structure energy difference will be zero. Our point iii) says that we will obtain, both classically and quantum-relativistically, the spin-orbit analogue to (1). Let us finally remind that the classical static field limit ($m \rightarrow \infty$, since the charge and magnetic moment are carried by the lepton) might be bad for atoms, which stresses the importance of the comparison of both classical and quantum relativistic approaches.

The classical (static field) limit corresponds to an infinitely heavy lepton which is the source of a static electric field $\vec{E}$ and a static magnetic field $\vec{B}$, due to its charge and magnetic moment, respectively. The spinless particle, with electric and magnetic polarizabilities α and β , respectively, moves with velocity $\vec{v}$ in these static fields. The longest range interaction is then

$$V(\vec{r}) = -\frac{1}{2} \alpha \vec{E}'^2 - \frac{1}{2} \beta \vec{B}'^2 \quad (2)$$

where $\vec{E}'$ and $\vec{B}'$ are the electric and magnetic fields in the rest system of the moving particle and which may be written in terms of the static electric and magnetic fields $\vec{E}$ and $\vec{B}$ as

$$\begin{aligned} \vec{B}' &= \vec{B} - \vec{v} \times \vec{E} \\ \vec{E}' &= \vec{E} + \vec{v} \times \vec{B} \end{aligned} \quad (3)$$

where only terms linear in $\vec{v}$ have been kept. Substituting Eq. (3) into Eq. (2) and $\vec{E}$ and $\vec{B}$ by their expressions, the potential linear in $\vec{v}$ which is obtained is

$$V = -\frac{e^2}{4\pi} \frac{\alpha + \beta}{M} \frac{\vec{\mu} \cdot \vec{L}}{r^6} \quad (4)$$

where $\vec{\mu}$ is the magnetic moment of the lepton (its charge has been taken negative) and $\vec{L}$ is the orbital angular momentum, $\vec{L} = \vec{r} \times M\vec{v}$. Introducing now the spin through the relationship $\vec{\mu} = -\frac{1}{2m} \vec{\sigma}$, we obtain the classical spin-orbit potential

$$V_{so} = \frac{e^2}{4\pi} \frac{\alpha + \beta}{2mM} \frac{\vec{\sigma} \cdot \vec{L}}{r^6} \quad (5)$$

which acts between a neutral particle and an infinitely heavy lepton (but $\vec{\mu}$ fixed).

We would like to make some comments on the result, Eq. (5). Equation (5) should be reproduced by the quantum relativistic calculation in the $m \rightarrow \infty$ limit, as then the lepton might be substituted by a static external field which is the classical limit. On the other hand, the classical derivation for static fields leading to Eq. (5) may be equally done for two particles of finite masses;

it is enough to substitute M in Eq. (5) by the reduced mass $Mm/(M+m)$ and to take into account that the angular momentum will be defined in terms of relative distance and velocity and the reduced mass. Finally, let us remark that the combination of polarizabilities which appears in Eq. (5), $\alpha+\beta$, is precisely the one given by the total photoproduction sum rule and is therefore determined by the total cross-section of real photons on the spinless particle (nucleus).

In the next section, the quantum relativistic derivation of the analogue to the classical potential of Eq. (5) will be presented. Section 3 is devoted to an effect which is present only for spin-orbit forces and which was not included in the previous sections. In the last section, the results are commented from both theoretical and phenomenological points of view.

2. THE QUANTUM RELATIVISTIC CALCULATION

The calculation of the quantum relativistic analogue of the potential (5) is much more involved. A detailed description of the dispersive approach which we shall follow is given in Ref. 1). Here, only the main features of the technique will be outlined, stressing somewhat more the new aspects due to the spin dependence.

i) The incoming and outgoing momenta of the lepton (of mass m) will be denoted by p and p' , r and r' being the corresponding quantities of the spinless particle of mass M . Calling $\vec{s}_\alpha$ the three four-vectors of the spin quantization basis, the spin dependent amplitude may be written as

$$S = i \epsilon_{\alpha\beta\gamma\delta} \vec{s}_\alpha \vec{s}_\beta p_\gamma r_\delta T(s, t) \quad (6)$$

where q is the momentum transfer. This is the most general invariant spin dependent amplitude which does not violate either time reversal or parity. $T(s, t)$ depends only on the c.m. energy squared s and the momentum transfer squared $t = q^2$.

ii) A fixed s dispersion relation is written for the two-photon exchange amplitude $T(s, t)$,

$$T(s, t) = \frac{1}{\pi} \left[\int_0^\infty - \int_{-\infty}^{(M-m)^2 - s} \right] dt' \frac{\text{Abs } T(s, t')}{t' - t} \quad (7)$$

Possible subtractions are irrelevant to what follows.

iii) The long range (inverse power of r) part of the c.m. effective potential is given by the Fourier transform of (6) but substituting $T(s,t)$ by Eq. (7) and keeping only its two photon t channel cut contribution. Since for this contribution the Fourier and the dispersive integrations commute, we obtain

$$V_{SO} = -\left(\frac{e^2}{4\pi}\right)^2 \frac{M+m}{Mm} \frac{\vec{\sigma} \cdot \vec{L}}{r} \frac{\partial}{\partial r} \left[\frac{1}{r} \int_0^\infty dt \text{Abs } T(s_0, t) e^{-\sqrt{t} r} \right] \quad (8)$$

where $\vec{L}$ is the c.m. orbital angular momentum, and $\text{Abs } T(s_0, t)$ is taken at the threshold $s = s_0 = (M+m)^2$ since we will keep only terms whose complete velocity dependence comes from the orbital angular momentum $\vec{L}$.

iv) By unitarity, the two-photon absorptive part of $T(s,t)$ is given in terms of the real photon Compton scattering tensors $T^{\mu\nu}$ of both particles in an unphysical region starting at $t=0$. That is

$$\text{Abs } S = -\frac{1}{128\pi^2} \int d\hat{K} T_{\mu\nu}^{(m)S}(p, q, K) T^{(M)\mu\nu}(r, q, K) \quad (9)$$

where K is the sum of the momenta of the two photons and $d\hat{K}$ represents the integration over the angles of the three-vector $\vec{K}$. Equation (9) has to be understood in the following sense: the spin dependent part of $T_{\mu\nu}^{(m)}(p, q, K)$ is taken, which is done here in a covariant fashion à la Bouchiat and Michel⁷⁾. Then the spin dependence is factorized as shown in Eq. (6), leaving the absorptive part of $T(s,t)$.

v) Taking into account gauge invariance, crossing and analyticity, which is done using Bardeen and Tung's Compton scattering amplitudes⁸⁾, an expansion of these amplitudes around $t=0$ allows one to obtain the longest range terms of an expansion of the potential in $1/r$ with parameters measured in real Compton scattering as coefficients.

A calculation along these lines gives for a negatively charged lepton and a spinless neutral system the following spin-orbit potential

$$V_{SO} = \frac{e^2}{4\pi} \frac{M+m}{Mm} (\alpha + \beta) \frac{\vec{\sigma} \cdot \vec{L}}{2m} \left(\frac{1}{r^6} - \frac{5}{\pi} \frac{1}{m r^7} \right) + O\left(\frac{1}{r^8}\right) \quad (10)$$

which, for $m \rightarrow \infty$, reproduces the classical potential (5). But more; taking into account that the angular momentum in (10) is $\vec{L} = (Mm/M+m)\vec{r} \times \vec{v}$, $\vec{r}$ and $\vec{v}$ being the relative distance and velocity, the largest range part of (10) turns out to be identical to the classical result, exactly as happens for spin independent forces. The second term of (10) is also determined by the polarizabilities (in fact, the same combination $\alpha+\beta$) and has no classical counterpart. It carries, for dimensional reasons, and abandoning natural units to show this, a factor $\hbar/c$ with respect to the first term, which shows both its quantum and relativistic origin. Shorter range terms are no longer determined by the polarizabilities alone. All the results obtained in Ref. 1) for spin and velocity independent forces are thus of more general validity; they may be extended to spin-orbit forces.

3. THE THOMAS PRECESSION

The potential (10) will not yet give the total spin-orbit force between the two particles we are considering. This is due to a well-known kinematical effect, the Thomas precession, and which is specific for spin-orbit forces [see Ref. 9]. Its dynamical reason is the existence of a longer range spin independent force which acts between the two particles [in our case the spin independent polarizability potential whose classical expression is (1)]. This makes that the incoming and outgoing lepton states used to obtain the potential (10) are not free states; their rest system is not Lorentzian. This leads to an additional interaction energy of spin-orbit type and which, if the ratio of the spin independent to the directly obtained spin-orbit force goes like r^2 , will have the same behaviour in r as this last one. The neglect of this effect is well known at order e^2 ; the classical spin-orbit potential is twice the one obtained from the Dirac equation with a Coulomb potential by performing a Foldy-Wouthuysen transformation⁹⁾. Both our classical and quantum relativistic calculations have to be corrected for this effect, since their incoming and outgoing states were taken to be free. We will do it only for the second one, Eq. (10), since nothing new is learnt from the comparison of both approaches. We will therefore have to add to the potential (10) a term¹⁰⁾

$$V_{Th} = - \frac{1}{4} \frac{M+m}{Mm} \frac{\vec{\sigma} \cdot \vec{L}}{mr} \frac{d}{dr} V(r) \quad (11)$$

$V(r)$ being the spin independent polarizability potential and whose classical expression is (1). Its quantum relativistic counterpart was obtained in Ref. 1) and reads

$$V(r) = \frac{e^2}{4\pi} \left[-\frac{1}{2} \frac{\alpha}{r^4} + \frac{1}{4\pi} \frac{11\alpha + 5\beta}{m r^5} \right] + O\left(\frac{1}{r^6}\right) \quad (12)$$

Putting together Eqs. (10), (11) and (12), the total spin-orbit potential which is obtained is given by

$$V_{SO} = \frac{e^2}{4\pi} \frac{M+m}{Mm} \frac{\vec{\sigma} \cdot \vec{L}}{2m} \left[\frac{\beta}{r^6} + \frac{15}{8\pi} \frac{\alpha - \beta}{m r^7} \right] + O\left(\frac{1}{r^8}\right) \quad (13)$$

which is our main result. The comparison of Eq. (13) with Eq. (10) shows the drastic change in the dependence on the polarizabilities caused by the inclusion of the Thomas precession.

As the Thomas precession correction is due to an incomplete treatment of the incoming and outgoing states in the derivation of Eq. (10), the result (13) should be obtained at once if all forces which are present are taken into account and the states are treated in a relativistic way. In other words, the approach equivalent to the one which gives the correct spin-orbit potential at order e^2 , i.e., Dirac equation plus Coulomb potential plus Foldy-Wouthuysen transformation, should also work at order e^4 . We will show this by obtaining in that way the first term of Eq. (13), which is (taking $M \rightarrow \infty$ in order to avoid unessential complications due to the reduced mass)

$$V_{SO} = \frac{e^2}{4\pi} \frac{\beta}{2m^2} \frac{\vec{\sigma} \cdot \vec{L}}{r^6} \quad (14)$$

The main difference between the calculation at order e^2 and at order e^4 is that in this last one there is both a spin independent and a spin dependent (Dirac matrices dependent) potential to be fed into the Dirac equation, whereas in the first one only the Coulomb potential (spin independent) enters. The spin independent potential one has to put into the Dirac equation is clearly Eq. (1). To obtain the spin dependent (Dirac matrices dependent) one is a much harder task. We will not do it, but instead obtain directly from Eq. (10) the spin-orbit potential to which it leads. It is enough to realize that the potential (10) has two distinct parts. Let us remind that when the spin dependent part of a matrix element in the Dirac spinor space is obtained, part

of it comes from big components of the Dirac spinors times spin dependent matrix and the rest from small components of the Dirac spinors times spin independent matrix. As the spin independent matrix is known, since it leads, with the big components, to the potential (1), its contribution to Eq. (10) may easily be obtained. This is done as follows. From Ref. 7) one has the following decomposition of the Dirac spinors in spin independent and dependent parts

$$u(p, \lambda) u^\dagger(p', \lambda') \approx 2m^2 I_{\lambda'\lambda} - i \epsilon_{\alpha\beta\gamma\delta} p^\alpha p'^\beta \vec{s} \cdot \vec{\sigma}_{\lambda'\lambda} \quad (15)$$

Its first part, together with the spin independent matrix element, leads to the potential (1). The corresponding absorptive part of the two-photon exchange amplitude may be found in Ref. 1). The same absorptive part together with the second part of the second term of expression (15) leads to the part of the potential (10) we are trying to find. It is obtained from Ref. 1) and Eq. (15) and reads

$$Abs \ T(s_0, t) = \frac{4\pi}{e^2} \propto \frac{\sqrt{t}}{16m} \quad (16)$$

This absorptive part put into Eq. (8) gives the part of the potential (10) due to the reduction from Dirac spinors to bispinors. It is

$$V = \frac{e^2}{4\pi} \frac{\alpha}{m^2} \frac{\vec{\sigma} \cdot \vec{L}}{r^6} \quad (17)$$

The difference of the first part of the potential (10) (for $M \rightarrow \infty$) and Eq. (17) is therefore the spin-orbit potential whose origin is the Dirac matrices dependent potential we should have put into the Dirac equation. It reads

$$V = \frac{e^2}{4\pi} \frac{\beta - \alpha}{2m^2} \frac{\vec{\sigma} \cdot \vec{L}}{r^6} \quad (18)$$

The spin-orbit potential coming from the spin independent potential (1) has to be added to this. That part being equivalent to the order e^2 situation, the expressions of Ref. 9) may be used, leading immediately to

$$V = \frac{e^2}{4\pi} \frac{\alpha}{2m^2} \frac{\vec{\sigma} \cdot \vec{L}}{r^6} \quad (19)$$

The total spin-orbit potential is thus given by the sum of Eqs. (18) and (19) and coincides, as we wished to show, with the potential (14).

4. COMMENTS AND CONCLUSIONS

As happens for spin independent forces ¹⁾ and due to the expansions which have been done in order to get the result (13), its region of validity is both limited by the conditions $rm \gtrsim 1$ and $rE^* \gtrsim 1$, where E^* is a typical excitation energy of the spinless particle (nucleus). Shorter range terms in the expansion in $1/r$ are no longer determined by the electric and magnetic polarizabilities.

As emphasized in the Introduction, Eq. (13) gives the correction due to the polarizabilities to the spin-orbit potential. At dominant order e^2 this potential is known to be (for $M \gg m$) ⁹⁾

$$V_{so} = \frac{e^2}{4\pi} \frac{Z}{4m^2} \frac{\vec{\sigma} \cdot \vec{L}}{r^3} \quad (20)$$

Let us compare Eqs. (13) and (20). This will be done by choosing a specific application. We therefore will estimate the ratio of the contributions of Eq. (13) to Eq. (20) for the fine structure p wave energy difference in hydrogenic atoms. The result is, using the $n=2$, $\ell=1$ hydrogenic wave functions and a cut-off R , of the order of

$$Z \left(\frac{e^2}{4\pi} \right)^2 m^2 (\alpha, \beta) R^{-1} \quad (21)$$

Let us explain expression (21). The first three factors are Z^{-1} times [coming from Eq. (20)] the inverse of the Bohr radius squared, which essentially is the p wave contribution of the potential (20). The last factor, the inverse of the cut-off, comes from the potential (13), since for p waves it gives a linear divergence at the origin. Taking, as customary, R to be the nuclear radius (and therefore not much different from E^{*-1}), the presence of the cut-off in expression (21) limits its applicability to muonic atoms. In effect, for electronic atoms $Rm \ll 1$ and this is far from the region where Eq. (13) is valid. For higher waves, the relevant distance being the Bohr radius instead of the cut-off, Eq. (13) is also valid for electronic atoms. But then the correction is further suppressed, since R^{-1} in Eq. (21) will be substituted by the inverse of the Bohr radius. On the other hand, for muonic atoms, where $R \sim m^{-1}$, since the nuclear radius is of the order of the Compton wavelength of the muon, Eq. (13) may be used. But, since shorter range terms in Eq. (13) start being important too, the fourth factor in Eq. (21) has to be understood as a coefficient of the order of the electromagnetic polarizabilities of the nucleus, and not just as β .

As an example, and in order to give a number, we will take muonic ${}^4\text{He}$ where a cut-off of $R \sim 2 \text{ fm}$ (which, incidentally, is m^{-1}) gave within a factor two the correct polarizability correction to the $2s-2p$ energy difference ⁶⁾. Taking a typical value of $5 \times 10^{-2} \text{ fm}^3$ for the polarizability factor Eq. (21) turns out to be of the order of 10^{-6} ! In view of our result given in Eq. (13), the smallness of its contribution is easily understood : it is a short range correction to a long range potential giving thus in Eq. (21) a $(e^2/4\pi)^2$ depressing factor. For muonic atom physics, we may therefore conclude by stating that the polarizability corrections to fine structure are far below present attainable accuracies [see, e.g., Ref. 11].

It should also be clear from the previous estimate that the condition of validity of the classical limit of Eq. (13), its first term, is given (for p waves) by $mR > 1$ which, even for muonic atoms, is hardly true. Therefore, for muonic atoms, where Eq. (13) may be used to estimate polarizability effects for p waves, the precise dependence on the polarizabilities is somewhat irrelevant and thus no numerical estimate should be too strongly based on it (e.g., on the smallness of β which appears in the classical limit.)

Before ending, we would like to make a remark on antiprotonic atoms. All the results obtained for muonic atoms may be taken over to antiprotonic atoms. Of course, now Eq. (13) gives only part of the polarizability correction, the part which does not depend on spin dependent polarizabilities. The part of the polarizability effect which may be estimated from Eq. (13) and which is given by Eq. (21) is two orders of magnitude bigger than for muonic atoms, as a result of the bigger mass of antiprotons. For high Z atoms (which will also increase the numerical value of the polarizabilities) the polarizability contribution may thus easily reach the one per cent level !

Let us conclude by answering some questions posed in the Introduction. All results obtained for spin independent forces may be extended to spin-orbit ones. Again the classical field result coincides exactly with the longest range quantum relativistic one, even without taking the static field limit. A next order term without classical counterpart and determined again by the electromagnetic polarizabilities is also obtained. A new effect, which did not appear for spin independent forces, had to be taken into account : the Thomas precession. It is shown that it is accounted for by a proper treatment of the incoming and outgoing states based on the Dirac equation. The potential (13) is a tool which allows a quick estimate of the polarizability correction to fine structure. As shown for the case of muonic ${}^4\text{He}$ this correction is negligible, but increases both with Z and the mass of the bound particle (antiprotonic atoms).

ACKNOWLEDGEMENTS

The author wishes to thank sincerely Professors J. Bernabéu and T.E.O. Ericson for invaluable help and fruitful discussions. He is indebted to the CERN Theoretical Physics Division for kind hospitality and to the GIFT for financial support.

REFERENCES

- 1) J. Bernabéu and R. Tarrach - to appear in Ann.Phys.
- 2) G. Feinberg and J. Sucher - Phys.Rev. 2A, 2395 (1970).
- 3) B.E. Lautrup, A. Peterman and E. de Rafael - Physics Reports 3C, 195 (1972).
- 4) J. Bernabéu and C. Jarlskog - Phys.Letters 60B, 197 (1976).
- 5) T. Ericson and J. Hüfner - Nuclear Phys. B47, 205 (1972).
- 6) J. Bernabéu and C. Jarlskog - Nuclear Phys. B75, 59 (1974).
- 7) C. Bouchiat and L. Michel - Nuclear Phys. 5, 416 (1958).
- 8) W.A. Bardeen and W.K. Tung - Phys.Rev. 173, 1423 (1968).
- 9) J.D. Bjorken and S.D. Drell - "Relativistic Quantum Mechanics", McGraw Hill (1964).
- 10) J.D. Jackson - "Classical Electrodynamics", John Wiley and Sons, New York (1962).
- 11) A. Bertin et al. - Phys.Letters 55B, 411 (1975).