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ABSENCE OF STATIC SOLUTIONS IN SOURCE-FREE
YANG - MILLS THEORY

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A B S T R A C T

The vanishing of self-stress for static systems excludes finite energy time-independent solutions of source-free Yang-Mills models (with compact gauge groups), except in five dimensions. Static solutions of Abelian self-interacting models of the Born-Infeld type are also forbidden.

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As is well known, there are no well-behaved source-free static solutions to Maxwell's or Einstein's equations in four dimensions. Furthermore, this property can be stated and established in a gauge independent way for both systems ^{*}). It is natural to conjecture that non-Abelian vector fields share this property since they also represent transverse massless excitations. The question is also of interest for general dimensionality in view of the recently obtained ¹⁾ well-behaved static solutions in five dimensions. We show here that $d=5$ is indeed the only space where they are allowed, and that for the case $d=4$ the argument is totally gauge-invariant, depending only on finiteness of energy and on the vanishing of the self-stress of static systems. For general d , existence of a gauge in which $\dot{\tilde{A}}_1=0$ is used, however.

We also show separately that there are no static solutions to Abelian self-interacting models of the Born-Infeld type.

The stress tensor for a Yang-Mills field $\tilde{F}_{\mu\nu}$ (in any dimension) has components

$$\begin{aligned} T_{\mu\nu} &= \tilde{F}_{\mu\alpha} \cdot \tilde{F}_{\nu}{}^{\alpha} - \frac{1}{4} \eta_{\mu\nu} \tilde{F}_{\alpha\beta}^2, \quad T_{\alpha}{}^{\alpha} = \frac{1}{4} (4-d) \tilde{F}_{\alpha\beta}^2 \\ T_{00} &= \frac{1}{2} (\tilde{F}_{0i}^2 + \frac{1}{2} \tilde{F}_{ij}^2). \end{aligned} \quad (1)$$

Compactness of the gauge group is required in order that $\tilde{F}_{0i}^2$, $\tilde{F}_{ij}^2$ be positive. Conservation implies that if the momentum density T^{0i} is static (or, more generally, if $\partial_0 \int d^{d-1} x x^i T^{0i} = 0$), then

$$\int d^{d-1} x T_i{}^i = \int d^{d-1} x \partial_j (x^i T^{ij}) = 0. \quad (2)$$

The right side of (2) is a divergence which vanishes since finite energy requires all $\tilde{F}_{\mu\nu}$ to fall off faster than $r^{-\frac{1}{2}(d-1)}$. Now from (1), we have ^{**)}

*) The vanishing of the divergence and curl of the field strength follow from the gauge-invariant requirement $\dot{\tilde{F}}_{\mu\nu}=0$ in the Maxwell case, while for relativity the time independence is equivalent to the existence of a timelike Killing vector in terms of which flatness can be demonstrated invariantly.

**) A similar expression is obtained in Ref. 2), from a quite different, variational, argument. However, it has been pointed out ³⁾ that the variations involved there may violate boundary conditions at infinity.

$$2 \int d^{d-1} x T_i^i = \int d^{d-1} x \left[(5-d) \frac{1}{2} \underline{F}_{ij}^2 + (d-3) \underline{F}_{oi}^2 \right] = 0. \quad (3)$$

Therefore, for $d=4$, $\underline{F}_{oi}$ and $\underline{F}_{ij}$ must all vanish. For $d=2$, $\underline{F}_{ij} \equiv 0$ of course, so $\underline{F}_{oi} = 0$ also follows. For $d=3$, only $\underline{F}_{ij} = 0$ follows, while for $d=5$, only $\underline{F}_{oi} = 0$ does, and for $d > 5$ we learn nothing further from (3) alone.

Thus, for our $d=4$ world (and for $d=2$), there was no need to involve any gauge dependent properties. Let us now derive $\underline{F}_{oi} = 0$ for general d , at the cost of the assumption that there exists a static gauge in which $\dot{\underline{A}}_i = 0$. In such a gauge, the electric field reduces to

$$\underline{F}_{oi} = -D_i \underline{A}_0 \quad (4)$$

where D_i is the usual gauge covariant derivative. Then since also

$$D^i \underline{F}_{oi} = 0 \quad (5)$$

it follows that, modulo surface terms $\sim \oint dS \cdot \nabla (\underline{A}_0^2)$,

$$0 = \int d^{d-1} x \underline{A}_0 \cdot D^i \underline{F}_{oi} = \int d^{d-1} x \underline{F}_{oi}^2 \quad (6)$$

and consequently that $\underline{F}_{oi} = 0$. With this result, (3) yields $\underline{F}_{ij} = 0$, except of course for $d=5$.

These arguments fail, as they should, in the presence of sources. For example, if there are Higgs fields, the stress-tensor (1) acquires contributions of the form

$$\underline{T}_{\mu\nu}^H = \underline{\pi}_\mu \cdot \underline{\pi}_\nu - \frac{1}{2} \gamma_{\mu\nu} \underline{\pi}_\alpha^2 + \gamma_{\mu\nu} (a \underline{\varphi}^2 - b \underline{\varphi}^4) \quad (7)$$

where $\underline{\pi}_\mu$ is the canonical momentum ($\underline{\pi}_\mu \equiv D_\mu \underline{\varphi}$). Thus, even if $\dot{\underline{\varphi}} = 0$, there are negative terms in the self-stress at $d=4$; sources can always always "support" static solutions.

We conclude with a sketch of the results for non-linear but Abelian gauge systems, whose Lagrangians depend on the two invariants $\alpha \equiv F^2$, $\beta \equiv \frac{1}{8} \tilde{F} F$ ($\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$). Clearly we are not interested in the general case $L(\alpha, \beta)$, since this includes models with negative energy. Consider rather the two "physical" Born-Infeld cases,

$$L_a = -(1 + F_{\mu\nu}^2)^{1/2} + 1 \quad (8.a)$$

$$L_b = -(1 + \alpha - \beta^2)^{1/2} + 1 \equiv -\gamma + 1 \quad (8.b)$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. The static field equations are then

$$\nabla \cdot \vec{B} = 0 = \nabla \times \vec{E} \quad (9)$$

together with the "constitutive" equations

$$\nabla \times (L'_a \vec{B}) = 0 = \nabla \cdot (L'_a \vec{E}), \quad L'_a \equiv \partial L_a / \partial F^2 \quad (10.a)$$

for (8.a), while (8.b) yields

$$\nabla \times (\gamma^{-1} \vec{B} + \gamma^{-1} \beta \vec{E}) = 0 = \nabla \cdot (\gamma^{-1} \vec{E} - \gamma^{-1} \beta \vec{B}). \quad (10.b)$$

From (10.a), we learn that $L'_a \vec{B}$ is a gradient and $L'_a \vec{E}$ a curl, and so they are respectively orthogonal to any transverse and longitudinal vectors. from (9), we deduce

$$\int d^3x L'_a \vec{B}^2 = 0 = \int d^3x L'_a \vec{E}^2 \quad (11)$$

and for monotonic non-singular L'_a , $\vec{B}=0=\vec{E}$ follows. A similar argument on (10.b), again using (9), implies that the fields vanish here as well, although the Lagrangians (8) are not scale invariant.

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