Transverse Resistive Wall Wakefunction with Inductive Bypass

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Abstract

Charged particle beams in circular accelerators couple with their surroundings through induced electromagnetic fields. This interaction can be described by so-called coupling impedances. In the frequency domain the impedance in connection with the bunch spectrum allows for stability analysis and estimates. However, for simulation codes one usually needs the wakefunction, the equivalent of the impedance in time domain. Recently the transverse impedance of a cylindrical pipe with arbitrary surface impedance was given by L. Vos. An expression for the wakefunction of this transverse resistive wall impedance with inductive bypass is derived here.
1 Introduction

Charged particle beams interact electromagnetically with their vacuum chamber surroundings. The electromagnetic fields generated by an oscillating beam itself, through induced currents in the vacuum chamber walls, can drive the beam unstable by initiating transverse oscillations that grow in amplitude. The strength of the interaction of the beam with its environment is described in frequency domain by a coupling impedance. In time domain the interaction is conveyed by the same electromagnetic fields, and as the particles move at nearly the speed of light, those fields remain behind the exciting charge, therefore they are called 'wake' fields [1, 2].

In general one is not interested in the wake fields directly (as it is anyway too complicated to calculate them for a whole ring), but in the integrated or averaged effect those fields have on the particle trajectory during one revolution in the machine. The normalized integral of the EM force over one revolution period defines the so-called wake potential. In the case of an elementary charge exciting the wake fields, the wake potential is called wake function. Impedances and wake functions are related through Fourier transforms.

Recently L. Vos [3] has revised studies already performed by B. Zotter [4] and G. Nassibian and F. Sacherer [5] on the transverse impedance of a cylindrical pipe with (arbitrary) surface impedance. This has been found to be of major importance for LHC since the concept of so-called inductive bypass helps to cure instabilities due to the resistive wall impedance.

Here we restrict ourselves to the transverse case in circular-symmetric geometries.

2 Classical transverse resistive wall impedance and wakefunction

Various references [1, 2] show the derivation of the transverse resistive (thick) wall impedance in round pipes as well as the wake function. Therefore we give here only the expressions for the lowest dipole mode \( m = 1 \) (expressions for higher modes can be found in [6, p.204]) and recall the properties shortly:

\[
Z_{m=1}^\perp(\omega) = (1 + j \text{ sgn} \omega) \frac{c L}{\pi b^3} \sqrt{\frac{\mu_0 \mu_r}{2 \sigma_c}} \cdot \frac{\sqrt{|\omega|}}{\omega}
\]

\[
= (\text{sgn} \omega + j) \frac{\mu_r Z_0 L \delta_0}{2 \pi b^3} \cdot \frac{\sqrt{\omega_0}}{|\omega|}
\]

In the above formula \( c \) denotes the speed of light, \( L \) indicates the length over which the impedance is defined (e.g. \( L = 2\pi R \) = the circumference), \( b \) is the vacuum chamber radius, \( \mu_0 \) and \( \mu_r \) denote the free space and relative permeability, respectively, and finally \( \sigma_c \) designates the conductivity of the vacuum chamber wall. In the second expression the quantities \( \omega_0 = 2\pi/\tau_{rev} \), the angular revolution frequency, and \( Z_0 = c \mu_0 \), the free space impedance, were used. Consequently, \( \delta_0 \) denotes the skin depth at the angular revolution frequency. Any expression in this note is given in MKSA units.
Transverse impedances must fulfill certain properties [1, p.76], one of which requires the real part of the impedance to be anti-symmetric and the imaginary part to be symmetric.

\[
\begin{align*}
\text{Re} \, Z_m^\perp(\omega) &= -\text{Re} \, Z_m^\perp(-\omega) \\
\text{Im} \, Z_m^\perp(\omega) &= \text{Im} \, Z_m^\perp(-\omega)
\end{align*}
\]  

As a direct consequence of these properties, the \( \text{sgn} \)-function and the absolute value appear in the expressions. Identifying the cases \(+\omega\) and \(-\omega\) one can get rid of those functions in some cases, but generally not. (Hint: \textit{Mathematica} and probably other symbolic packages as well, have certain problems using discontinuous functions, especially in the routines for Fourier transforms. One way to overcome this, is to get rid of possible \( \text{sgn} \) and \( \text{abs} \) functions as mentioned before. Note however that one must be extremely careful about the definition of the sign of \( \sqrt{-1} \) and how this is treated in the internal routines of the symbolic package.)

The transverse wake function at a time \( t \) behind a point charge is the Inverse Fourier transform of the impedance

\[
W_m^\perp(t) = -j \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} Z_m^\perp(\omega) \exp(j\omega t).
\]  

Note the additional factor \(-j\) in the above formula, which describes the mere fact that the transverse force is 90° out of phase with the beam current [1, p.70]. Applying the Inverse Fourier transform to Eq. (1) leads to the classical transverse resistive thick wall wakefunction:

\[
W_{m=1}(t > 0) = \frac{cL}{\pi^{3/2} b^3} \sqrt{\frac{\mu_0 \mu_r}{\sigma_e}} \cdot \frac{1}{\sqrt{|t|}}
= \frac{\mu_r Z_0 L_0}{\pi b^3 \tau_{rev}} \cdot \sqrt{\tau_{rev}} \frac{1}{\sqrt{|t|}}
\]  

Remind yourself that the resistive part \( \text{Re} \, Z_m^\perp(\omega) \) causes instability for negative resistance and damping for positive resistance, because of:

\[
\Delta \Omega \propto j \, Z \quad \text{and} \quad x, y(t) \propto \exp(j(\Omega + \Delta \Omega)t)
\]

\( \Delta \Omega \) is caused by the impedance \( Z \), where \( \Delta \Omega \) is directly proportional to \( Z \) (this follows from the equations of motion, \( Z \) is part of the force on the RHS). From the equations of motion it follows that a negative resistance corresponds to a positive real-valued growth rate \( 1/\tau \).

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An instability in the above sense can be interpreted in frequency domain as follows: Wake fields due to the resistivity of the wall drive the beam unstable at a frequency where a line of the bunch spectrum happens to be in a region of large negative resistance.

Especially the \( 1/\sqrt{|t|} \) dependence of the wakefunction accounts for its importance concerning high intensity and/or multi-bunch machines. Due to the slow decay of the fields,
successive bunches 'see' each other or even a single bunch may see itself over consecutive
turns in the machine. Depending on the phase relation of the kick conveyed by the wake field
and the motion of the particle in concern this will lead to amplification or attenuation of the
particle betatron oscillation.

A direct consequence of the $1/\sqrt{t}$ dependence are furthermore long computation times
in simulations, as there is no straight-forward way of summing up wake contributions over
several turns form previous bunches in an efficient way. Fortunately a smart algorithm for
such summations has been found recently (Note in preparation), that reduces drastically the
time needed to execute the necessary summations.

Refer to Figure 2 and 3 (dashed lines) for plots of the transverse resistive wall impedance
and wakefunction in the classical thick wall case.

3 Transverse resistive wall Impedance and Wakefunction with
inductive bypass

First we will shortly recapitulate the concept of inductive bypass as described in [3]. If the
wall current finds a path with less resistivity it will certainly take that one, thus lowering
the impedance. Ref. [3] shows that such an alternative path acts like an inductance on the
beam.

Figure 3 shows the (longitudinal) equivalent circuit that has to be used for computing
the transverse impedance $Z_{\perp,\text{ibp}}$ of a pipe of unit length with longitudinal impedance $Z_{\parallel}$.

\begin{equation}
Z_{\perp,\text{ibp}}(\omega) = \frac{2c}{b^2\omega} \cdot \frac{Z_{\parallel} \cdot Z_{\text{ind}}}{Z_{\parallel} + Z_{\text{ind}}}
\end{equation}

3.1 Thick Wall

We are now ready to use the above mentioned equivalent circuit (6) to find the transverse resistive wall impedance with inductive bypass for thick walls (wall thickness $t \gg \delta_{\text{skin}}$). The longitudinal resistive wall impedance $Z_{\parallel}$ is

\begin{equation}
Z_{m=1}(\omega) = (1 + j \text{sgn} \omega) \frac{1}{\delta_{\text{skin}} \sigma_c} \frac{L}{2\pi b},
\end{equation}

where $\delta_{\text{skin}}$ is the skin depth, $L$ indicates the length over which the impedance is defined,
$b$ is the vacuum chamber radius, $\mu_0$ and $\mu_r$ denote the free space and relative permeability
respectively and $\sigma_c$ designates the conductivity of the vacuum chamber wall material.
We take directly the inductive bypass given in Ref. [3], which has a constant inductance of value $\mu_0/4\pi$, namely

$$Z_{\text{ind}}(\omega) = j \omega \cdot \frac{\mu_0}{4\pi} \cdot L,$$  \hspace{1em} (8)

where the same definitions as above apply.

The next step is to insert these expressions (7–8) into (6) in order to get a formula for the impedance with inductive bypass, which leads us to

$$Z_{\perp m=1, \text{ibp}}(\omega) = (1 + j \text{ sgn } \omega) \frac{c \mu_0 L}{2 \pi b^2} \left( \frac{1}{-j + \text{ sgn } \omega \left( 1 + b \sqrt{\frac{\sigma_c \mu_0}{2 \mu_r} \sqrt{\omega} \right)} \right),$$  \hspace{1em} (9)

which is an already simplified expression where the properties (2) of the impedance are retained. Finally, to keep better track of the general form we introduce the constants $c_1$ and $c_2$,

$$Z_{\perp m=1, \text{ibp}}(\omega) = c_1 \cdot \frac{(1 + j \text{ sgn } \omega)}{\text{ sgn } \omega - j + c_2 \text{ sgn } \omega \sqrt{\omega}},$$

$$c_1 = \frac{c \mu_0 L}{2 \pi b^2} = \frac{Z_0 L}{2 \pi b^2},$$

$$c_2 = b \cdot \sqrt{\frac{\sigma_c \mu_0}{2 \mu_r}} = \frac{b}{\mu_r \delta_0 \sqrt{\omega_0}}. \hspace{1em} (10)$$

Figure 2 shows the impedance with inductive bypass. The important difference is immediately visible:

$$\lim_{\omega \to 0} \text{Re } Z_{\perp, \text{ibp}}(\omega) = 0$$

$$\lim_{\omega \to 0} \text{Im } Z_{\perp, \text{ibp}}(\omega) = \text{const.} = c_1$$

$$\lim_{\omega \to \pm \infty} Z_{\perp, \text{ibp}}(\omega) = 0.$$  \hspace{1em} (11)
For $\omega \to 0$ the imaginary part of the impedance approaches a finite value and what is even more important, the real part goes to zero for $\omega \to 0$. The effect is that the (multi-)bunch mode $m$ which is closest to the origin (usually modes $0, -1, \ldots$) becomes stabilized now, and a higher mode, which is much harder to drive, will become the most unstable one. Hence, overall the beam will be more stable.

In order to find the wake function we have to apply the inverse Fourier transformation on Eq. (10). Although expression (10) seems to be harmless, the transformation is not straightforward, rather we have to split the impedance (10) into two terms, which can be done by extending the denominator with a pseudo-complex conjugate form. We get

$$Z_{m=1,\text{ibp}}(\omega) = -c_1 \frac{2}{c_2^2 \omega + 2 j} + c_1 c_2 \frac{(\omega + j |\omega|)}{c_2^2 \omega^2 + 2 j \omega}.$$

Using this technique we can perform the inverse Fourier transformation on the two terms independently. The first term gives zero, only the second term contributes to the result (for $t > 0$):

$$W_{m=1,\text{ibp}}(t > 0) = \sqrt{\frac{2}{\pi}} \frac{c_1}{c_2 \sqrt{|t|}} - \frac{2 c_1}{c_2^2} \exp \left[ \frac{2}{c_2^2} |t| \right] \cdot \left( 1 - \text{Erf} \frac{\sqrt{2}}{c_2} \right).$$

After inserting the constants $c_1$ and $c_2$ we arrive at the final result, the transverse resistive thick wall impedance with inductive bypass,

$$W_{m=1,\text{ibp}}(t > 0) = \frac{c L}{\pi^{3/2} b^4} \sqrt{\frac{\mu_0 \mu_r}{\sigma_c}} \cdot \frac{1}{\sqrt{|t|}} - \exp \left[ \frac{4 \mu_r}{b^2 \sigma_c} |t| \right] \frac{2c L \mu_r}{b^4 \pi \sigma_c} \cdot \left( 1 - \text{Erf} \frac{4 \mu_r}{b^2 \sigma_c \mu_0} |t| \right).$$

Thus we have found an expression where the first term reveals the classical result plus an additional term which defines the correction to it. This correction is active over the whole
Figure 4: Transverse resistive thick wall wakefunction in the classical case (dashed) and with inductive bypass (solid) in logarithmic scale. The relative difference is also drawn (long dashed).

Table 1: Parameters used for plotting Figures 2–5

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>$1 \cdot 10^{-2}$ m</td>
<td>tube radius</td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>$18 \cdot 10^8$ (Ωm)$^{-1}$</td>
<td>conductivity</td>
</tr>
<tr>
<td>$c$</td>
<td>$3 \cdot 10^8$ m/s</td>
<td>speed of light</td>
</tr>
<tr>
<td>$L$</td>
<td>$1$ m</td>
<td>(= unit length) length for impedance definition</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>$4\pi 10^{-7}$ Vs(Am)$^{-1}$</td>
<td>permeability of free space</td>
</tr>
<tr>
<td>$\mu_r$</td>
<td>$1$</td>
<td>relative permeability $\mu_r = \mu/\mu_0$</td>
</tr>
<tr>
<td>$\tau_{rev, LHC}$</td>
<td>$88.9245 \cdot 10^{-6}$ s</td>
<td>revolution time</td>
</tr>
</tbody>
</table>

range $t \in [0, \infty]$, but at the limit $t \to 0$ it approaches a finite value, therefore it does not change the overall value of the wakefunction, which approaches infinity there. On the other hand, in the limit $t \to \infty$, the correction term vanishes. Thus the intermediate range is most interesting, where the correction term gives its dominant contribution.

Figure 3 shows the behavior of the wakefunction in comparison to the classical result. For small time distances the difference is marginal, but as soon as the time distance grows the effect of the inductive bypass becomes visible: The wake falls off much faster than $1/\sqrt{t}$ causing less effect on succeeding bunches. The course of the two wake functions (classical and with inductive bypass) is better seen in a logarithmic plot, see figure 4.
3.2 Thin Wall

In the case of a thin wall, wall thickness \( t_w < \delta_{\text{skin}} \), the same exercise can be conducted: We start with a simple thin wall formula, that can be found e.g. in [7]. This formula describes the physical situation, that all the wall current flows in the wall of cross-section \( 2 \pi b t_w \):

\[
Z_{m=1}^\perp(\omega) = \frac{c L}{\pi b^3 \sigma_c t_w \cdot \omega} \tag{15}
\]

Again \( c \) denotes the speed of light, \( L \) indicates the length over which the impedance is defined (e.g. \( L = 2\pi R = \) the circumference), \( b \) is the vacuum chamber radius and \( \sigma_c \) designates the conductivity of the vacuum chamber wall. The vacuum chamber has the wall thickness \( t_w \).

It is already stated in the above reference, that this expression is only valid for sufficiently low frequencies, \(|\omega| < 2/(\sigma_c \mu_0 \mu_r t_w^2)\). The introduction of the inductive bypass, by applying (6), rectifies (15) in the sense, that it becomes meaningful over the whole range of frequencies. Inclusion of the inductance from Ref. [3] gives

\[
Z_{m=1}^\perp(\omega) = \frac{c \mu_0 L}{2 \pi b^2} \cdot \frac{1}{\frac{1}{2} b t_w \sigma \mu_0 \cdot \omega - j},
\]

\[
= \frac{Z_0 L}{2 \pi b^2} \cdot \frac{1}{\frac{b t_w}{\delta_0^2 \mu_r \omega_0} - j},
\]

which is a known thin metal wall formula, derived in various forms already early in the literature [4, 5, 8], but is not often used, for reasons unknown to the author.

![Figure 5: Real and imaginary part of the transverse resistive thin wall impedance in the classical case (dashed) and with inductive bypass (solid)](image)

Applying the inverse Fourier transformation on this expression is straightforward. We get the resistive thin wall wakefunction

\[
W_{m=1,ibp}^\perp(t > 0) = \frac{c L}{\pi b^3 \sigma_c t_w} \cdot \exp \left[ -\frac{2}{b t_w \sigma_c \mu_0} |t| \right],
\]

\[
= \frac{\mu_r Z_0 L \delta_0^2 \omega_0}{2 \pi b^3 t_w} \cdot \exp \left[ -\frac{\mu_r \delta_0^2}{b t_w \omega_0} |t| \right].
\]

(17)
4 Conclusion

A wakefunction for the transverse resistive wall impedance with inductive bypass has been derived and discussed. The main use is for programs working in time domain, simulating instabilities and collective effects in accelerators with multi-bunch operation. It should be noted again that the inclusion of the inductive bypass in the stability analysis of LHC (mainly concerning the collimators) leads to more optimistic predictions.

References


