A PROGRAM TO COMPUTE BIRKHOFF NORMAL FORMS OF A
SYMPLECTIC MAP IN $\mathbb{R}^4$

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A program to compute the Birkhoff normal forms for symplectic maps in $\mathbb{R}^4$ is described. We consider the case of an elliptic fixed point, which is the most relevant for applications. We compute the normal forms both in the nonresonant and resonant case and we provide the interpolating hamiltonian and the normalizing transformation.

PROGRAM SUMMARY

Title of the program: ARES

Program obtainable from: Department of Physics, University of Bologna; e-mail: bazzani@bologna.infn.it

Computer: VAX 4000, HP 755.

Operating system: VMS, UNIX.

Programming language: Fortran 77.

High speed storage required: 1.2 Mb at perturbative order 10.

Peripheral used: Storage for input and output files.

No. of lines in combined program and test deck: 966.

Keywords: Symplectic maps, Birkhoff normal forms, resonances.

Nature of physical problem

The computation of normal forms of symplectic maps provides informations on the structure of the phase space in a neighborhood of a fixed point. This approach can be useful in beam dynamics to analyse the effects of nonlinearities of a magnetic lattice. The normal forms allow one to analytically optimize a magnetic lattice, constructing effective correction strategies for the multipolar errors.

Method of solution

The computation of normal forms is based on the solution of a conjugation equation for the initial map. It is convenient to represent both the normalizing transformation and the normal form by means of a Lie transformation. Then we can solve the equation by a perturbative method on the homogeneous orders of the Taylor expansion. The main algorithmic tool is the composition of two polynomial maps up to a given order in the Taylor expansion.

Typical running time: 160 sec. on VAX 4000 or 6.3 sec. on HP 755 at perturbative order 10 in the quasi-resonant case.

LONG WRITE-UP

1. Introduction

During the last decade, the theory of normal forms for symplectic maps has drawn the attention of the physicist community for its application to nonlinear beam dynamics [1]. The superconducting magnets, which will be used in the future particle accelerators, introduce unavoidable multipolar errors so that strong nonlinear effects appear in the betatronic motion [2] (i.e. the motion of a particle in the orthogonal plane to the reference orbit). The standard approach is based on the description of the betatronic motion in terms of the one-turn map, i.e. the map which transforms the coordinates and the momenta of a particle at a given orthogonal section of the reference orbit to the corresponding values after one turn in the magnetic lattice [3,4,5]: in this way, the problem of the stability of the betatronic motion is reduced to the analysis of the orbits of a symplectic map in the neighborhood of an elliptic fixed point. The Birkhoff normal forms for a symplectic map of $\mathbb{R}^4$ [6] has been successfully applied in the nonresonant case to calculate some relevant quantities in accelerator physics such as the dependence of the frequency on the distance from the reference orbit and the deformation of the phase space trajectories [7,8]. Moreover, effective strategies for the correction of the undesired nonlinear effects have been defined and carried out [5,9]. The approach based on resonant normal forms allows one to give a perturbative estimate of the relevant quantities of a resonance, such as the

position and the stability of the fixed points, their eigenvalues, and the width of the island in the phase space [10]. For this reason, the resonant normal forms are a useful tool to study an efficient strategy to sort the magnets to minimize the effects of random multipolar fields [11]. Another application is the analysis of beam diffusion due to the feeding currents of the quadrupoles [12].

In this article we describe a program to calculate the Birkhoff normal forms for symplectic maps of $\mathbb{R}^4$ both in the nonresonant and in the resonant case. The code is a generalization of previous programs [13] which compute the normal forms for area-preserving maps: it provides both the normalizing transformation and the interpolating hamiltonian in the case of a single or a double resonance [6]. The program can be interfaced with a software library [14] which computes the one-turn map for a lattice of a particle accelerator, starting from the input file of the MAD program [15]. The generalization of the code to the computation of normal forms in the neighbourhood of other types of fixed points (hyperbolic, complex instabilities), is straightforward, even if it is not given here for the sake of brevity. **Notations:** the boldface letters, latin or greek, indicate vectors of 4 components: for instance we have $\mathbf{z} = (z_1, z_1^*, z_2, z_2^*)$. The even components are canonically conjugated to the preceding odd components, i.e. (z_1, z_1^*) and (z_2, z_2^*) are pairs of conjugate variables. The scalar product of two vectors λ, ζ is denoted by $\lambda \cdot \zeta$, whereas $\lambda\zeta$ denotes the vector $(\lambda_1\zeta_1, \lambda_1^*\zeta_1^*, \lambda_2\zeta_2, \lambda_2^*\zeta_2^*)$.

2. Theory

The computation of the normal forms for a symplectic map $\mathbf{F}$ is based on the solution of the conjugation equation

$$\Phi \circ \mathbf{U} = \mathbf{F} \circ \Phi \quad (1)$$

where the unknown quantities Φ and $\mathbf{U}$ are respectively the normalizing transformation and the normal form. The normal form $\mathbf{U}$ is a map which is invariant under a symmetry group which is generated by the linear part of the map $\mathbf{F}$, or under subgroups of it. We assume that the initial map $\mathbf{F}$ has an elliptic fixed point at the origin so that, by using complex variables (z_1, z_1^*, z_2, z_2^*) [†], we can diagonalize the linear part

$$\mathbf{F}(\mathbf{z}) = \lambda_\omega \mathbf{z} + \sum_{n \geq 2} [\mathbf{F}]_n(\mathbf{z}) \quad (2)$$

where $\lambda_\omega = (e^{i\omega_1}, e^{-i\omega_1}, e^{i\omega_2}, e^{-i\omega_2})$ are the eigenvalues of the map at the origin and $[]_n$ denotes the homogeneous order n of the Taylor expansion. The map $\mathbf{F}$ has four components $\mathbf{F} = (F_1, F_1^*, F_2, F_2^*)$. The linear frequencies of the map are defined according to $\omega = (\omega_1, \omega_2)$. The particular form of the map (2) allows one to solve the conjugation

[†] In this case the asterisk denotes both the complex and the canonical conjugation.

equation (1) by using a standard order-by-order perturbative procedure. The solution (Φ, U) is given by a formal Taylor expansion up to a certain perturbative order. As it is well known, such expansions are usually divergent, but due to their asymptotic character, appropriate truncation of the series provides a very good approximation of the dynamics of the initial map F . Since we have diagonalized the linear part of F , the perturbative solution (Φ, U) has the form

$$\begin{aligned}\Phi(\zeta) &= \zeta + \sum_{n \geq 2} [\Phi]_n(\zeta) \\ U(\zeta) &= \lambda_\omega \zeta + \sum_{n \geq 2} [U]_n(\zeta).\end{aligned}\tag{3}$$

If we substitute the expansions (2) and (3) in the conjugation equation (1) and we define the operator $[\]_{\leq N}$ to retain only the terms up to order N , we obtain

$$\begin{aligned}\Delta_\omega[\Phi]_N(\zeta) + [U]_N(\zeta) &= \\ &= [[F]_{\leq N}([\Phi]_{<N}(\zeta))]_N - [[\Phi]_{<N}([U]_{<N}(\zeta))]_N\end{aligned}\tag{4}$$

where we have defined the operator Δ_ω according to

$$\Delta_\omega[\Phi]_N(\zeta) = [\Phi]_N(\lambda_\omega \zeta) - \lambda_\omega [\Phi]_N(\zeta);\tag{5}$$

we remark that the action of Δ_ω is different for the different components. Therefore the normal form U has to contain all the monomials of the r.h.s. which belong to the kernel of the operator Δ_ω , whereas the coefficients of the same monomials of the normalizing transformation Φ are left undetermined by eq. (4). We distinguish three different cases:

i) let k be an integer vector of $\mathbb{Z}^2$: the frequencies ω do not satisfy any resonance condition if one has

$$k \cdot \omega = 0 \bmod 2\pi \quad \Rightarrow \quad k = 0;\tag{6}$$

ii) the frequencies ω satisfy a single resonance condition if there exists an integer vector $r \in \mathbb{Z}^2$ such that

$$k \cdot \omega = 0 \bmod 2\pi \quad \Rightarrow \quad k = l r \quad l \in \mathbb{Z};\tag{7}$$

letting $r = (r_1, r_2)$, $|r_1| + |r_2|$ is called the order of the resonance;

iii) the frequencies ω satisfy a double resonance condition: there exist two integer vectors r and s such that

$$k \cdot \omega = 0 \bmod 2\pi \quad \Rightarrow \quad k = l_1 r + l_2 s \quad l_1, l_2 \in \mathbb{Z}.\tag{8}$$

In order to extract the relevant informations on the dynamics contained in the normal form, it is convenient to introduce an interpolating hamiltonian, i.e. an hamiltonian whose phase-flow interpolates at integer times the iterations of the normal form U . This construction is necessary for two basic reasons: on the one hand, since we are using a perturbative

method, we obtain only the expansion of $\mathbf{U}$ up to a certain order N . As a consequence, the symplectic character can be preserved only up to the same order, and therefore $[\mathbf{U}]_{\leq N}$ cannot be directly iterated since deviations from symplecticity linearly increase with the number of iterations. The interpolating hamiltonian of $\mathbf{U}$ allows one to interpolate the orbits of the normal form with an exactly symplectic phase-flow. On the other hand, the interpolating hamiltonian explicitly provides the integrals of motion and the relevant indicators of the nonlinear motion to analyze the phase space structure.

In the nonresonant case i), the normal form is invariant under the group of continuous rotations and the interpolating hamiltonian has the form

$$H(\zeta) = i\omega_1 \zeta_1 \zeta_1^* + i\omega_2 \zeta_2 \zeta_2^* + \sum_{n \geq 3} [H(\zeta_1 \zeta_1^*, \zeta_2 \zeta_2^*)]_n, \quad (9)$$

i.e. it is a function of the amplitudes $\zeta_1 \zeta_1^*, \zeta_2 \zeta_2^*$ which represent a complete set of integrals of motion in involution; as a consequence, the normal form dynamics is completely integrable according to Arnold-Liouville's theorem.

In the case ii) it is not possible to find a time independent interpolating hamiltonian for the map $\mathbf{U}$, but if we factorize the linear part according to $\mathbf{U} = \lambda_\omega \hat{\mathbf{U}}$, we can calculate a time-independent interpolating hamiltonian for the map $\hat{\mathbf{U}}$ in the form

$$H(\zeta) = \sum_{n \geq 3} [H(\zeta)]_n \quad (10)$$

where the Taylor expansion contains only the monomials $\zeta_1^{n_1} \zeta_1^{*m_1} \zeta_2^{n_2} \zeta_2^{*m_2}$ whose exponents satisfy $\mathbf{n} - \mathbf{m} = l\mathbf{r}$, $l \in \mathbb{Z}$. Also in this case we refer to H as the interpolating hamiltonian of $\mathbf{U}$. By using the definition (7) it is easy to see that the hamiltonian (10) is invariant under the linear transformation $\zeta \rightarrow \lambda_\omega \zeta$ so that it is an integral of motion for the normal form $\mathbf{U}$, which can be interpolated by composing the phase-flow of the hamiltonian (10) at integer times n with the linear transformation $\lambda_\omega^n \zeta$. A second integral of motion is given by the scalar product $\mathbf{j} \cdot (\zeta_1 \zeta_1^*, \zeta_2 \zeta_2^*)$ where $\mathbf{j} \in \mathbb{Z}^2$ is orthogonal to the resonant vector $\mathbf{r}$. Therefore the normal form dynamics is completely integrable also in the single resonant case.

In the case iii) we proceed as before, i.e. we factorize the linear part of $\mathbf{U}$ in order to build a time-independent interpolating hamiltonian of the form (10) for the map $\hat{\mathbf{U}}$; the expansion in the r.h.s. contains the monomials $\zeta_1^{n_1} \zeta_1^{*m_1} \zeta_2^{n_2} \zeta_2^{*m_2}$ such that $\mathbf{n} - \mathbf{m} = l_1 \mathbf{r} + l_2 \mathbf{s}$, where $l_1, l_2 \in \mathbb{Z}$. Still the hamiltonian (10) is an integral of motion for the normal form $\mathbf{U}$ since it is invariant under the transformation $\zeta \rightarrow \lambda_\omega \zeta$ and gives informations on the structure of the phase space (for instance, it gives the position and the stability of the fixed points), but no other integral of motion can be constructed and the normal form dynamics is then non-integrable.

In order to explicitly calculate the interpolating hamiltonian, we use the definition of Lie-transformation associated to an hamiltonian field $H(\zeta)$. Let $A(\zeta)$ be any function, we define the operator D_H according to

$$D_H A(\zeta) = \sum_{j=1}^2 \frac{\partial A}{\partial \zeta_j} \frac{\partial H}{\partial \zeta_j^*} - \frac{\partial A}{\partial \zeta_j^*} \frac{\partial H}{\partial \zeta_j} = \{A, H\} \quad (11)$$

i.e. the Poisson bracket of H and A . The Lie-transformation generated by the Hamiltonian H is then defined by

$$e^{D_H} A(\zeta) = \sum_{l \geq 0} \frac{D_H^l}{l!} A(\zeta), \quad (12)$$

where D_H^l denotes the power l of the operator D_H , i.e. the operator itself applied successively l times. These are standard definitions in the hamiltonian theory: as it is well known, the phase-flow generated by H is given by $e^{tD_H} \zeta$, where $t \in \mathbb{R}$. Therefore, in order to construct the interpolating hamiltonian of $\hat{U}$ we have to solve the functional equation

$$e^{D_H} \zeta = \hat{U}(\zeta) \quad (13)$$

(where we identify $\hat{U} = U$ in the non-resonant case). If we substitute the perturbative solution (3) obtained from the functional equation, we can solve Eq. (13) in all the cases we considered before. Indeed, if the map $\hat{U}$ starts with the identity, the hamiltonian H defined by Eq. (13) cannot contain any quadratic term and the operator D_H increases the order of any monomial; then, if we consider the homogeneous order N , we obtain

$$D_{[H]_{N+1}} \zeta = [U]_N - \sum_{l \geq 2}^{N-1} \left[\frac{D_H^l}{l!} \zeta \right]_N, \quad (14)$$

where the r.h.s. depends only on $[H]_{\leq N}$; this shows that Eq. (13) can be solved order by order. In all the cases of interest, the computation of the interpolating hamiltonian H reduces to the solution of Eq. (13) for a map $\hat{U}$ which starts with the identity (in the non-resonant case the linear part can be easily eliminated from both sides of Eq. (13)).

Finally, we have to fix the undetermined coefficients of the transformation Φ by requiring the symplectic character for the transformation itself. This is achieved by representing Φ by means of a Lie transformation $e^{D_G} \zeta = \Phi$ so that the coefficients of Φ are determined order by order from the equation

$$[\Phi]_N = \sum_{l \geq 1}^N \left[\frac{D_G^l}{l!} \zeta \right]_N \quad (15)$$

(we stress the fact that Φ starts with the identity). Nevertheless, some coefficients of Φ are yet not fixed by Eq. (15); such coefficients can be chosen arbitrarily without affecting the symplectic character of Φ and represent an intrinsic gauge of the normalizing transformation. These remaining terms correspond to the coefficients of G which enter the interpolating hamiltonian of the normal form and are set to zero by the program.

A well-known result of number theory states that the set of frequencies ω which do not satisfy any resonant condition has full measure; moreover, any ω can be approximated with an arbitrary precision by resonant frequencies, i.e. for any $\delta \geq 0$ it exists an integer vector k such that

$$|(k \cdot \omega) \bmod 2\pi| \leq \delta; \quad (16)$$

indeed, there is an infinite set of $\mathbf{k}$ which satisfy the inequality (16). The quantities defined by Eq. (16) are the well-known small denominators of the classical perturbation theory and are given by the operator Δ_ω in the functional equation (1). In many physical situations the linear frequencies ω are so close to a low order resonance that the asymptotic character of the series (3) appears after few perturbative orders: in these cases we are close to a resonance (quasi-resonant case) and it is convenient to compute the resonant normal forms (even if the linear frequency are not exactly resonant) in order to get relevant information on the dynamics. Let α be the resonant frequencies, $\lambda_\alpha = (e^{i\alpha_1}, e^{-i\alpha_1}, e^{i\alpha_2}, e^{-i\alpha_2})$ and let $\omega - \alpha = \epsilon$ be the distance from the resonance, with the hypothesis $\|\epsilon\| \ll 1$; we construct the normal form with respect to the frequencies α in order to avoid the presence of the small divisors due to this resonance. Contrary to the previous case, we cannot use Eq. (14) to construct the interpolating hamiltonian: in fact the map $\hat{U} = \lambda_\alpha^{-1}U$ does not start with the identity (now the linear frequencies are ϵ). Therefore, we proceed by using the following strategy: under the hypothesis that ϵ does not satisfy any resonance condition in common with α , we determine the non-resonant normal form V and the normalizing transformation $\hat{\Phi}$ of the map $\hat{U}$, and we compute the interpolating hamiltonian K of V ; then we define the interpolating hamiltonian H of $\hat{U}$ according to

$$H(\zeta) = K(\hat{\Phi}^{-1}(\zeta)). \quad (17)$$

We can prove that H is correctly defined since it is continuous in the limit $\|\epsilon\| \rightarrow 0$ (passage through the resonance); indeed both the hamiltonian K and the transformation $\hat{\Phi}^{-1}$ diverge when we pass through the resonance, but the divergent terms cancel in the composition. The same procedure can be applied when α satisfies a single resonant condition or a double resonant condition.

3. Algorithms

The program ARES has an internal representation of the normal form U and of the normalizing transformation Φ by means of Lie transformations according to

$$\begin{aligned} \Phi(\zeta) &= e^{D_G} \zeta \\ U(\zeta) &= \lambda_\omega e^{D_A} \zeta; \end{aligned} \quad (18)$$

therefore we can use the following property to calculate the compositions in the conjugation equation (1)

$$\begin{aligned} F(\Phi(\zeta)) &= e^{D_G} F(\zeta) \\ \Phi(U(\zeta)) &= e^{D_A} \Phi(\lambda_\omega \zeta). \end{aligned} \quad (19)$$

The computation of a Lie transformation essentially requires an efficient algorithm for the product of two polynomials up to a given homogenous order: this is obtained by using

two integer vectors containing the addresses of the coefficients of the factor polynomials which contribute to a fixed monomial of the result, so that the product of two polynomials is equivalent to as many scalar products as the number of the coefficients of the results; this algorithm is vectorizable. By using the product, we define the operator $D_H A$ for any function A according to the definition (11), and the Lie transformation $\exp(D_H)A$ is calculated by a recursive algorithm, which compute the powers of D_H .

The fundamental equation (4) can be written in the form

$$\Delta_\omega D_{[G]_{N+1}} \zeta + \lambda_\omega D_{[A]_{N+1}} \zeta = [\exp(D_{[G]_{\leq N}})[\mathbf{F}]_{\leq N}(\zeta)]_N - [\exp(D_{[A]_{\leq N}})[\Phi]_{< N}(\lambda_\omega \zeta)]_N \quad (20)$$

and can be explicitly solved since the operator Δ_ω is inverted according to (see def. (5))

$$\Delta_\omega^{-1} \zeta_1^{n_1} \zeta_1^{*m_1} \zeta_2^{n_2} \zeta_2^{*m_2} = \frac{\zeta_1^{n_1} \zeta_1^{*m_1} \zeta_2^{n_2} \zeta_2^{*m_2}}{e^{i(n_1-m_1)\omega_1 + i(n_2-m_2)\omega_2} - \lambda_\omega} \quad (21)$$

for any monomial not belonging to the kernel of Δ_ω . Conversely, the monomials in the kernel of Δ_ω , which define the hamiltonian A_{N+1} , remain undetermined in G_{N+1} and can be chosen arbitrarily without affecting the symplectic character of the normalizing transformation: the program sets these coefficients to zero.

The interpolating hamiltonian H can be immediately computed from A in the non-resonant or in the exactly resonant case. In the quasis resonant case, we proceed by calculating the non-resonant normal form for the map $\hat{\mathbf{U}} = \lambda_\alpha^{-1} \mathbf{U}$ by using the same algorithm described before up to the same order. Let K be the interpolating hamiltonian of the non-resonant normal form and e^{D_B} the Lie transformation which represents the new normalizing transformation $\hat{\Phi}$; then we obtain the interpolating hamiltonian for the normal form $\mathbf{U}$ according to Eq. (17), which can be written in the form

$$[H]_{\leq N+1}(\zeta) = [\exp(-D_{[B]_{\leq N+1}})[K]_{\leq N+1}(\zeta)]_{\leq N+1}. \quad (22)$$

4. Program description

The program ARES reads two input files, MAP.DAT and RES.DAT and writes three output files: HAMIL.DAT, PHI.DAT and PSI.DAT.

The file MAP.DAT contains the coefficients of the initial map $\mathbf{F}$ with the linear part in a diagonal form according to Eq. (2). It is written as

1. A single record containing the order of the map $\mathbf{F}$ (free format).
2. A series of records, containing the polynomial expansion of the map. Each record contains the complex coefficients of a monomial, and the relative index of the powers

of the coordinates (format 1h ,2d24.16,4x,'(' ,4i2,')'); records are ordered according to the components of the map and, inside each component, to increasing powers of the coordinates. For instance in the test run input file (see section 5), the eighth record corresponds to the coefficient of the monomial $z_2 z_2^*$ of the first component of **F**. Double complex precision is assumed.

The file RES.DAT contains the informations to calculate the normal forms: it is written as

1. a single record containing two integers (free format): the first integer denotes the dimension of the space of the resonant vectors: it is set to 0 in the non-resonant case, 1 in the case of a single resonance and 2 in the case of a double resonance. The second integer has to be set to 1 in the nonresonant or in the exactly resonant case, and to 2 in the quasisresonant case.
2. four records, each containing a complex number: these are the four resonant eigenvalues $e^{i\alpha_1}$, $e^{-i\alpha_1}$, $e^{i\alpha_2}$, $e^{-i\alpha_2}$ [see Section 2].
3. A couple of integers (free format), which are the two components of the first resonant vector **r** [see Eq. (7)].
3. A couple of integers (free format), which are the two components of the second resonant vector **s** [see Eq. (8)].

In the non-resonant case the resonant frequencies and the resonant vectors are read but not used; similarly, in the single resonance case the second vector is read but not used. The order for the perturbative solution of the conjugation equation is asked interactively during the execution of the program.

The file HAMIL.DAT contains the complex coefficients of the interpolating hamiltonian; it is written as

1. A single record (free format) containing an integer that is the order of the interpolating hamiltonian of the normal form **U**.
2. A series of records that contain the polynomial expansion of the hamiltonian. Each record contains the complex coefficient of a monomial, and the relative index of the powers of the normal coordinates $(\zeta_1, \zeta_1^*, \zeta_2, \zeta_2^*)$ in the same format of MAP.DAT. Only coefficients different from zero are printed out. Records are ordered according to the same rules of the file MAP.DAT.

The file PHI.DAT contains the complex coefficients of the normalizing transformation; it is written as

1. A single record (free format) containing an integer that is the order of the normal form series.
2. A series of records, containing the polynomial expansion of Φ Each record contains the complex coefficient of a monomial of a single component, and the relative index of the powers of the normal coordinates $(\zeta_1, \zeta_1^*, \zeta_2, \zeta_2^*)$, in the same format of MAP.DAT. All the coefficients are printed out. Records are ordered according to the same rules of the file MAP.DAT.

The file PSI.DAT contains the complex coefficients of the inverse of Φ ; it is written in the same format as PHI.DAT.

5. Test run

In the test run, we compute the resonant normal form of a quadratic symplectic mapping whose frequencies are close to a single resonance:

$$\begin{aligned} F_1 &= e^{i\omega_1} \left(z_1 - \frac{i}{4} [(z_1 + z_1^*)^2 - (z_2 + z_2^*)^2] \right) \\ F_2 &= e^{i\omega_2} \left(z_2 + \frac{i}{2} [z_1 z_2 + z_1^* z_2 + z_1 z_2^* + z_1^* z_2^*] \right) \end{aligned} \quad (23)$$

whereas the components F_1^* and F_2^* are given by the complex conjugated equations. The input file MAP.DAT contains the coefficients of the map; the linear frequencies are $\omega_1 = 0.57$ and $\omega_2 = 0.28$, and the resonant frequencies are $\alpha_1 = 0.568$ and $\alpha_2 = 0.284$. The resonant vector $\mathbf{k}$ which satisfies $\mathbf{k}\alpha = 0$ is $\mathbf{k} = (1, -2)$. These data are given in the input file RES.DAT. The order up to which the normal form series are computed is given interactively, and in the test run is 3. Two output files are given: HAMIL.DAT, which contains the coefficients of the interpolating hamiltonian, PHI.DAT, which contains the coefficients of the direct conjugating function. Only the first part of the file PHI.DAT, containing the first component of the conjugating function, has been printed out. The file PSI.DAT, which contains the coefficients of the inverse conjugating function, is not given for the sake of brevity.

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File input

MAP.DAT

```

2
0.0000000000000000D+00 0.0000000000000000D+00 ( 0 0 0 0 )
0.0000000000000000D+00 0.0000000000000000D+00 ( 0 0 0 1 )
0.0000000000000000D+00 0.0000000000000000D+00 ( 0 0 1 0 )
0.0000000000000000D+00 0.0000000000000000D+00 ( 0 1 0 0 )
-0.9048270524660195D+00 -0.4257792915650727D+00 ( 1 0 0 0 )
0.1064448228912682D+00 -0.2262067631165049D+00 ( 0 0 0 2 )
0.2128896457825363D+00 -0.4524135262330098D+00 ( 0 0 1 1 )
0.1064448228912682D+00 -0.2262067631165049D+00 ( 0 0 2 0 )
0.0000000000000000D+00 0.0000000000000000D+00 ( 0 1 0 1 )
0.0000000000000000D+00 0.0000000000000000D+00 ( 0 1 1 0 )
-0.1064448228912682D+00 0.2262067631165049D+00 ( 0 2 0 0 )

```

0.0000000000000000D+00 0.0000000000000000D+00 (1 0 0 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (1 0 1 0)
 -0.2128896457825363D+00 0.4524135262330098D+00 (1 1 0 0)
 -0.1064448228912682D+00 0.2262067631165049D+00 (2 0 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 0 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 1 0)
 -0.9048270524660195D+00 0.4257792915650727D+00 (0 1 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (1 0 0 0)
 0.1064448228912682D+00 0.2262067631165049D+00 (0 0 0 2)
 0.2128896457825363D+00 0.4524135262330098D+00 (0 0 1 1)
 0.1064448228912682D+00 0.2262067631165049D+00 (0 0 2 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 1 0 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 1 1 0)
 -0.1064448228912682D+00 -0.2262067631165049D+00 (0 2 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (1 0 0 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (1 0 1 0)
 -0.2128896457825363D+00 -0.4524135262330098D+00 (1 1 0 0)
 -0.1064448228912682D+00 -0.2262067631165049D+00 (2 0 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 0 1)
 -0.1873813145857246D+00 0.9822872507286887D+00 (0 0 1 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 1 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (1 0 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 0 2)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 1 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 2 0)
 -0.4911436253643443D+00 -0.9369065729286230D-01 (0 1 0 1)
 -0.4911436253643443D+00 -0.9369065729286230D-01 (0 1 1 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 2 0 0)
 -0.4911436253643443D+00 -0.9369065729286230D-01 (1 0 0 1)
 -0.4911436253643443D+00 -0.9369065729286230D-01 (1 0 1 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (1 1 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (2 0 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 0 0)
 -0.1873813145857246D+00 -0.9822872507286887D+00 (0 0 0 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 1 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 1 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (1 0 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 0 2)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 1 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 2 0)
 -0.4911436253643443D+00 0.9369065729286230D-01 (0 1 0 1)
 -0.4911436253643443D+00 0.9369065729286230D-01 (0 1 1 0)

```

0.0000000000000000D+00 0.0000000000000000D+00 ( 0 2 0 0)
-0.4911436253643443D+00 0.9369065729286230D-01 ( 1 0 0 1)
-0.4911436253643443D+00 0.9369065729286230D-01 ( 1 0 1 0)
0.0000000000000000D+00 0.0000000000000000D+00 ( 1 1 0 0)
0.0000000000000000D+00 0.0000000000000000D+00 ( 2 0 0 0)

```

RES.DAT

```

1 2
(-0.9101059706849957,-0.4143755809932842)
(-0.9101059706849957, 0.4143755809932842)
(-0.2120071099220547, 0.9772681235681935)
(-0.2120071099220547,-0.9772681235681935)

```

```

1 -2

```

```

0 0

```

File output

HAMIL.DAT

```

4
0.4163336342344337D-16 -0.2513274122871842D-01 ( 0 0 1 1)
0.5551115123125783D-16 0.1256637061435919D-01 ( 1 1 0 0)
-0.7853981633974517D-02 0.2499177478845327D+00 ( 0 1 2 0)
0.7853981633974517D-02 0.2499177478845327D+00 ( 1 0 0 2)
0.2857146535290464D-15 -0.4361909421149224D-01 ( 0 0 2 2)
0.4140256339886388D-16 -0.3473579159003464D+00 ( 1 1 1 1)
-0.2602085213965211D-17 0.4519559249190815D-01 ( 2 2 0 0)

```

PHI.DAT

```

3
0.0000000000000000D+00 0.0000000000000000D+00 ( 0 0 0 0)
0.0000000000000000D+00 0.0000000000000000D+00 ( 0 0 0 1)
0.0000000000000000D+00 0.0000000000000000D+00 ( 0 0 1 0)
0.0000000000000000D+00 0.0000000000000000D+00 ( 0 1 0 0)

```

0.1000000000000000D+01 0.0000000000000000D+00 (1 0 0 0)
 -0.2888579567353014D+00 -0.1250000000000000D+00 (0 0 0 2)
 0.5588162072428725D-01 -0.2500000000000000D+00 (0 0 1 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 2 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 1 0 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 1 1 0)
 -0.9695993888120168D-01 0.1250000000000000D+00 (0 2 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (1 0 0 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (1 0 1 0)
 -0.5588162072428725D-01 0.2500000000000000D+00 (1 1 0 0)
 0.2794081036214364D-01 0.1250000000000000D+00 (2 0 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 0 3)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 1 2)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 2 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 0 3 0)
 -0.8022837080524574D-01 0.1104248790851976D+00 (0 1 0 2)
 0.6732316492049053D+00 0.3167987670974450D+00 (0 1 1 1)
 0.2750900537653966D-01 -0.1071405021365049D+00 (0 1 2 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 2 0 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (0 2 1 0)
 -0.1427439718982138D-01 -0.1725478212976450D-01 (0 3 0 0)
 0.1244315190110875D-01 -0.6522928659328943D-01 (1 0 0 2)
 0.1981278383385864D+00 -0.8239936510889834D-17 (1 0 1 1)
 0.1777314653594189D-01 0.9317009695543304D-01 (1 0 2 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (1 1 0 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (1 1 1 0)
 -0.9604552299523627D-01 -0.4519559249190815D-01 (1 2 0 0)
 0.0000000000000000D+00 0.0000000000000000D+00 (2 0 0 1)
 0.0000000000000000D+00 0.0000000000000000D+00 (2 0 1 0)
 0.8620540864153090D-02 0.4282598581317743D-17 (2 1 0 0)
 0.3666827853000940D-01 -0.1725478212976453D-01 (3 0 0 0)