Riemann Tensor of the Ambient Universe, the Dilaton and the Newton’s Constant

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We investigate a four-dimensional world, embedded into a five-dimensional spacetime, and find the five-dimensional Riemann tensor via generalisation of the Gauss (–Codacci) equations. We then derive the generalised equations of the four-dimensional world and also show that the square of the dilaton field is equal to the Newton’s constant. We find plausible constant and non-constant solutions for the dilaton.

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Since the pioneering works of Kaluza [1] and Klein [2], who unified gravitation with electromagnetism, the implications of possible extra dimensions to our world have been under intense investigation — see [3] for an extensive collection of papers on higher-dimensional unification. Jordan [4] and Thiry [5] used the equations of Kaluza-Klein’s theory to show that the gravitational constant solution for the Newton’s constant however is then allowed only if the square of the Maxwell electromagnetic tensor vanishes. Here we examine a plausible generalisation of gauge fixed Maxwell equations in the presence of a dilaton field. We also show how they affect the physics of the four-dimensional world. As a result we find a system of equations for the electromagnetic field, gravitational field and dilaton field. One of these equations is the Gauss (–Codacci) equations by utilising all degrees of freedom (the entire geometry) of the ambient spacetime and we also show how they affect the physics of the four-dimensional world. A generalisation of Gauss (–Codacci) equations by utilising all degrees of freedom (the entire geometry) of the ambient spacetime and we also show how they affect the physics of the four-dimensional world. As a result we find a system of equations for the electromagnetic field, gravitational field and dilaton field. One of these equations is a plausible generalisation of gauge fixed Maxwell equations in the presence of a dilaton field.

We assume that this transformation is invertible at each point. This means that the Jacobi matrices of the transformation and its inverse have non-vanishing determinants everywhere. Thus, to globally parametrise the foliated five-dimensional spacetime V, it is sufficient to use the coordinates of the four-dimensional world M and the foliation parameter s.

The vector normal to the surface is:

\[ N_\mu = \frac{\partial \psi}{\partial x^\mu}. \] (2)

Let us also define:

\[ e^\mu_j = \frac{\partial x^\mu}{\partial y^j}, \quad n^\mu = \frac{\partial x^\mu}{\partial s}, \quad E_\mu^i = \frac{\partial y^i}{\partial x^\mu}. \] (3)

The derivatives are related as follows:

\[ \partial_k = \frac{\partial}{\partial y^k} = e^\mu_k \partial_\mu, \] (4)

\[ \partial_{\nu} = \frac{\partial}{\partial x^\nu} = E_\mu^\nu \partial_\mu + N_\nu \partial_s. \] (5)

Obviously, if we denote \( e^\mu_k = n^\mu \) and \( E_\mu^5 = N_\mu \), then \( e^\mu_\nu \) and \( E_\mu^5 \) will be the Jacobi matrices of the transformation \( (x^\mu, s) \rightarrow (y^j, s) \) and its inverse. Therefore:

\[ e^\mu_\nu E_\mu^5 = \delta^\nu_\nu = E_\mu^\nu e^\mu_\nu. \] (6)

This orthogonality condition is equivalent to:

\[ e^\mu_i E_\mu^5 = \delta^i_1, \] (7)
$N_{\mu}n^{\mu} = 1,$ \hspace{1cm} (8)

$E_{\mu}^{i}e_{i}^{\mu} + N_{\mu}n^{\mu} = \delta_{\mu}^{i},$ \hspace{1cm} (9)

$N_{\mu}e_{j}^{\mu} = 0,$ \hspace{1cm} (10)

$n^{\mu} E_{\mu}^{i} = 0.$ \hspace{1cm} (11)

(Note that $\delta_{\mu}^{i}$ is $\dim V = 5$ and $\delta_{i}^{j}$ is $\dim M = 4$.)

Thus the bases $(e_{\mu}^{i}, n^{\mu})$ and $(E_{\mu}^{i}, N_{\mu})$ are dual. They do not depend on the metric of either spacetime, but only depend on the particular embedding chosen.

Let us now introduce a scalar field $\phi(y^k, s)$, a vector field $A_t(y^k, s)$ and the metric tensor $g_{ij}(y^k, s)$ on $M$. We further define the metric $G_{\mu\nu}$ of the five-dimensional spacetime $V$ as an expansion over the basis vectors $E_{\mu}^{i}$ and $N_{\mu}$:

$$G_{\mu\nu} = E_{\mu}^{i}E_{\nu}^{j}g_{ij} + (N_{\mu}E_{\nu}^{j} + N_{\nu}E_{\mu}^{j})A_{i} + N_{\mu}N_{\nu}\phi.$$

(12)

Taking $x^i = y^i$, $x^5 = s = \text{const}$, i.e. $e_{\mu}^{i} = \delta_{\mu}^{i}$, $n^{\mu} = \delta_{\mu}^{5}$, $E_{\mu}^{i} = \delta_{\mu}^{i}$, $N_{\mu} = \delta_{\mu}^{5}$ in (12) corresponds to the original Kaluza’s model [1]. Klein’s modification [2] $g_{ij} \rightarrow g_{ij} + A_{i}A_{j}$, together with the identification of $\phi$ as a dilaton is the model put forward by Jordan [4] and Thiry [5].

We note that the metric (12) has the same form as the inverse of the metric of Klein’s model, thus the two theories are dual: Kaluza’s model corresponds to slicing, while Klein’s model corresponds to threading of the five-dimensional spacetime [15]. The case with $A_{t} = 0$ has also been considered (see, for example, [17], [18] and the references therein).

The lack of gauge invariance for the fields $A_{t}$, which we nevertheless will associate with the electromagnetic potentials, in view of the slicing–threading duality, is compensated by the freedom to fix $\phi$. This, as will become clearer later is the freedom to fix the dilaton field.

Returning to (12), we define $g^{ij}$ as the inverse of the metric $g_{ij}$. Thus $A^{i} = g^{ij}A_{j}$ and $A^{2} = g^{ij}A_{i}A_{j}$.

The inverse $G^{\mu\nu}$ of the metric $G_{\mu\nu}$ on $V$ is then given by:

$$G^{\mu\nu} = h^{ij}e_{i}^{\mu}e_{j}^{\nu} - \theta A_{i}(e_{i}^{\mu}n^{\nu} + e_{i}^{\nu}n^{\mu}) + \theta n^{\mu}n^{\nu},$$

(13)

where $\theta = (\phi - A^{2})^{-1}$ and $h^{ij} = g^{ij} + \theta A^{i}A_{j}$. One can easily check that $G^{\mu\nu}G_{\nu\sigma} = \delta_{\mu}^{\sigma}$. Using the inverse $G^{\mu\nu}$ (13) of the metric $G_{\mu\nu}$, we can raise and lower five-dimensional indexes to get:

$$N^{\mu} = G^{\mu\nu}N_{\nu},$$

(14)

$$N^{2} = N_{\mu}N^{\mu} = \theta,$$

(15)

$$n_{\mu} = G_{\mu\nu}n^{\nu} = A_{i}E_{\mu}^{i} + N_{\mu}\phi,$$

(16)

$$n^{2} = n_{\mu}n^{\mu} = \phi.$$ \hspace{1cm} (17)

Note that when $A_{i} = 0$ and $\phi = 1$, then $N^{\mu} = n^{\mu}$ as in the ADM approach [17].

The extrinsic curvature is:

$$K_{ij} = e_{j}^{\mu}e_{i}^{\nu}Q_{\mu\nu},$$

(18)

where $Q_{\mu\nu} = \nabla_{\mu}N_{\nu} - \nabla_{\nu}N_{\mu}$. Multiplying (18) across by $E_{\mu}^{i}E_{\nu}^{j}$ and applying the orthogonality conditions (7)–(11), one easily finds:

$$Q_{\alpha\beta} = E_{\alpha}^{i}E_{\beta}^{j}K_{ij} + (N_{\alpha}E_{\beta}^{j} + N_{\beta}E_{\alpha}^{j})f_{i} + N_{\alpha}N_{\beta}\chi,$$

(19)

with $\chi = n^{\mu}n^{\nu}Q_{\mu\nu}$ and

$$f_{j} = n^{\mu}e_{j}^{\beta}Q_{\mu\beta} = A^{i}K_{ij} + \frac{1}{N}\partial_{j}N,$$

(20)

where $N = \sqrt{N_{\mu}N^{\mu}}$.

The four-dimensional Christoffel symbols

$$\gamma_{jk}^{i} = \frac{1}{2}g^{ij}(\partial_{k}g_{ij} + \partial_{j}g_{ik} - \partial_{i}g_{jk}),$$

(21)

can then be expressed as:

$$\gamma_{jk}^{i} = (\partial_{k}e_{j}^{\alpha} + e_{j}^{\alpha}e_{k}^{\beta}\Gamma_{\alpha\beta}^{\mu})E_{\mu}^{i} - K_{ij}A_{i}^{k},$$

(22)

where $\Gamma_{\alpha\beta}^{\mu}$ are the five-dimensional Christoffel symbols.

Further, the four-dimensional Riemann curvature tensor

$$r_{ijkl} = \partial_{k}\gamma_{jl}^{i} - \partial_{l}\gamma_{jk}^{i} + \gamma_{jm}^{a}\Gamma_{al}^{k} - \gamma_{jm}^{k}\Gamma_{al}^{j},$$

(23)

using (22) and (7)–(11), becomes:

$$r_{ijkl} = E_{\alpha}^{i}e_{\gamma}^{\alpha}e_{k}^{\beta}e_{l}^{\gamma}R_{\lambda\mu}^{\alpha} + E_{\alpha}^{i}(\nabla_{\mu}n^{\alpha})(e_{k}^{\beta}K_{ij} - e_{k}^{\gamma}K_{ij}) + \nabla_{i}(A^{k}K_{jk} - K_{jk}A^{k}) - \nabla_{k}A^{i}(K_{jkl} - K_{kl}A_{j}) + A^{i}A^{m}(K_{jkl}K_{jk} - K_{mk}K_{jk}),$$

(24)

where $R_{\lambda\mu}^{\alpha}$ is the five-dimensional Riemann curvature tensor. The above is a generalisation of the Gauss equations. Taking $A^{i} = 0$ and $\phi = 1$, one simply recovers the well known Gauss equations (see, for example, [19]):

$$r_{ijkl} = E_{\alpha}^{i}e_{\gamma}^{\alpha}e_{k}^{\beta}e_{l}^{\gamma}R_{\lambda\mu}^{\alpha} + K_{i}^{k}K_{ij} - K_{i}^{j}K_{k}.$$

(25)

Let us now expand the five-dimensional Riemann curvature tensor over our basis. Using its symmetries we can write:

$$R_{\lambda\mu\sigma\tau} = E_{\alpha}^{i}E_{\mu}^{j}E_{\nu}^{k}E_{\tau}^{l}U_{ijkl} +$$

$$+ [(N_{\alpha}E_{\mu}^{j} - N_{\mu}E_{\alpha}^{j})E_{\nu}^{k}E_{\tau}^{l} +$$

$$+ (N_{\alpha}E_{\mu}^{j} - N_{\mu}E_{\alpha}^{j})E_{\nu}^{k}E_{\tau}^{l}V_{ijkl}],$$

(26)

where the coefficients in this expansion satisfy:

$$U_{ijkl} = U_{klij} = -U_{ijlk} = -U_{jikl},$$

(27)

$$V_{ijkl} = -V_{ijk}, \quad W_{ijkl} = W_{ijlk}.$$ \hspace{1cm} (28)

Using (24), one can further find:

$$U_{ijkl} = e_{i}^{\mu}e_{j}^{\nu}e_{k}^{\sigma}R_{\lambda\mu\sigma\tau}$$

$$= \tau_{ijkl} - \{\pi_{ik}\pi_{lj} - \pi_{il}\pi_{kj}\},$$

(29)

$$V_{ijkl} = n^{\mu}e_{j}^{\nu}e_{k}^{\sigma}R_{\lambda\mu\sigma\tau}$$

$$= A^{i}U_{ijkl} - \frac{1}{N}(\nabla_{k}\pi_{lj} - \nabla_{l}\pi_{kj}),$$

(30)
where \( \pi_{jl} = \frac{1}{2} K_{jl} \).

Finding the remaining tensor \( W_{jl} \) is more complicated. One can easily see that

\[
W_{jl} = n^\mu n^\sigma e^\lambda_j e^\ell_l R_{\lambda\mu\sigma\ell} = S_{jl} + A^k V_{ikj} ,
\]

where

\[
S_{jl} = \frac{1}{N^2} n^\sigma e^\lambda_j e^\ell_l (\nabla_{\lambda} Q_{\sigma\ell} - \nabla_{\ell} Q_{\sigma\lambda}) .
\]

In the above we identify derivatives in the direction of \( n^\sigma \).

To handle this type of terms, we will have to explicitly invoke the dependence on \( s \).

Firstly, in virtue of (5), we get the following expression for the extrinsic curvature (18):

\[
K_{jl} = - \frac{N^2}{2} \left[ (\nabla_j A_l + \nabla_l A_j) + \frac{N^2}{2} \partial_s g_{jl} \right] + \frac{1}{N} \left[ \nabla_j \left( A^k \pi_{kl} \right) + \nabla_l \left( A^k \pi_{kj} \right) \right] - \frac{1}{N} A^k \nabla_k \pi_{lj} + \frac{1}{N} \pi_{jl}^k + \frac{1}{N} \partial_s \pi_{jl} + \Omega_{jl} ,
\]

where \( \Omega_{jl} \) contains only terms which are proportional to derivatives of the basis vectors and their duals with respect to \( (y^k, s) \).

The five-dimensional Ricci tensor can easily be calculated from (26):

\[
R_{\mu\nu} = \sum_j E_{\mu j} E_{\nu j} \left[ h^{ik} U_{ijkl} - N^2 A^k (V_{jkl} + V_{klj}) + N^2 W_{jl} \right] + \left( N_\mu E_{\nu} + N_\nu E_{\mu} \right) (-h^{ik} V_{jkl} + N^2 A^j W_{jkl}) + \sum_j N_j N_l h^{ik} W_{jkl} .
\]

Then the five-dimensional Einstein’s equations in vacuum

\[
R_{\mu\nu} = 0
\]

reduce to

\[
h^{ik} U_{ijkl} - N^2 A^k (V_{jkl} + V_{klj}) + N^2 W_{jl} = 0 ,
\]

\[
h^{ik} V_{jkl} - N^2 A^j W_{jkl} = 0 ,
\]

\[
h^{jl} W_{jkl} = 0 .
\]

Multiplying (37) by \( A^j \) and adding it to (38) allows us to exclude \( W_{jl} \) from equation (38). Then, using the expressions (29) and (30) for \( U_{ijkl} \) and \( V_{jkl} \), (38) becomes:

\[
\nabla_k \pi^k_l - \nabla_l \pi^k_k = 0 ,
\]

Equations (40) (as we will see below) are a generalisation of Maxwell’s equations in a fixed gauge.

One has to make a very important point here. Klein’s theory corresponds to a threading decomposition of the five-dimensional spacetime [15]. Rigorous analysis [16] shows that the curvature tensor of the hypersurface formed is given by Zelmanov’s curvature tensor, which differs from the ordinary Riemann curvature tensor by additional terms containing \( s \)-derivatives of the four-dimensional metric. The cylinder condition forces the two curvature tensors equal and thus represents a surface forming condition. In Kaluza’s theory, the foliation of the five-dimensional spacetime corresponds to slicing [15]. Then the four-dimensional metric \( g_{ij} \) naturally appears as the slicing metric and imposing a cylinder condition is not at all necessary.

To simplify the analysis of the physics described by the fields \( A_i, \phi, \) and \( N \), we will, however, put aside the \( s \)-dependent terms. Also for simplicity, we will assume that the basis elements and their duals are constant (thus recovering the original Kaluza’s theory). The tensor \( \Omega_{jl} \) will then vanish from (34).

Equation (40) becomes:

\[
\nabla_k F^{kl} = -2 A^k \pi^l + \frac{2}{N^2} (\pi^k - \pi^k g^{kl}) \partial_k N .
\]

Here \( F_{kl} = \nabla_k A_l - \nabla_l A_k \) is the Maxwell electromagnetic tensor with \( A_k \) being the electromagnetic potential.

The first term on the right-hand-side of (41) describes an interaction between electromagnetic and gravitational fields. We assume that it is much smaller than the remaining terms, so that we can neglect it. Note that \( A_k \) cannot be “gauged up” to increase the scale of \( A_k \).

Furthermore, if \( N \) is a constant, then (41) becomes the usual Maxwell’s equations:

\[
\nabla_k F^{kl} = 0 .
\]

The remaining two equations are:

\[
\nabla_k \left( \frac{f^k}{N} \right) = 0 ,
\]

\[
r_{jl} - \frac{1}{2} g_{jl} r = \frac{N^2}{2} T_{jl} ,
\]

where \( r = g^{ik} r_{ik} \) and \( r_{jl} \) are the four-dimensional scalar curvature and four-dimensional Ricci tensor, respectively.

The energy-momentum tensor \( T_{jl} \) is therefore given by:

\[
T_{jl} = T_{jl}^{\text{Maxwell}} + g^{ik} \nabla_i B_{jkl} + C_{jl} + D_{jl} ,
\]

where:

\[
T_{jl}^{\text{Maxwell}} = g^{ik} F_{ij} F_{kl} - \frac{1}{4} g_{jl} F_{ik} F^{ik} ,
\]

\[
B_{jkl} = A_k \nabla_l A_j - A_l \nabla_k A_j - A_j F_{kl} + \nabla_j (A_k A_l) + g_{jl} (A^i \nabla_i A_k - A_k \nabla_i A^l) ,
\]

(46)
\[
C_{jl} = g_{jl} A^i A_k r_{ik} - 2 A^i A_l r_{ij} - 2 A^i A_j r_{il},
\]
\[
D_{jl} = \frac{4}{N^4} (\partial_j N)(\partial_l N) - \frac{2}{N^2} \nabla_j \nabla_l N - \frac{2}{N^2} \pi^k_k (A_l \partial_j N + A_j \partial_l N) + \frac{2}{N^2} \left[ -A^k \pi_{jl} + A_{rj}^k + A_{lj}^k \right] - g_{jl} \left( A^k \pi^t_k - A^t \pi^k_k \right) \partial_k N.
\]

We will analyse each of these terms separately. The first one, \(T_{jl}\), is the Maxwell energy-momentum tensor. The tensor \(C_{jl}\) describes interaction between electromagnetic and gravitational fields. From (44) we see that if \(N^2\) is very small (as we will confirm later), then \(r_{jl}\) will be of the order of \(N^2\), which justifies the neglect of the interaction terms in (41) and the tensor \(C_{jl}\).

Using (33) in (20) and then (20) in equation (43), we see that a constant solution for \(N\) is allowed by equation (43) if \(\phi\) satisfies:
\[
\nabla^k \partial_k \phi = \frac{1}{2} F^{ik} F_{ik},
\]
where \(F^{ik}\) is a solution of (42). For the constant solution for \(N\), the tensor \(D_{jl}\) vanishes. Moreover:
\[
0 \equiv g^{ij} \nabla_i T_{jl} = g^{ij} \nabla_i T_{jl}^{\text{Maxwell}},
\]
\[
g^{ml} g^{nk} \nabla_m \nabla_n B_{jk} = -\frac{2}{N} \nabla_j \nabla_k \left( \frac{\alpha}{N} \right) = 0 \text{ in view of (43).}
\]
In other words, the conservation law (50) is given by the usual Maxwell energy-momentum tensor \(T_{jl}^{\text{Maxwell}}\) and \(N^2\) plays the role of the Newton’s constant \(G_N\):
\[
N^2 = \frac{8\pi G_N}{c^2}.
\]
One has to point out here that in the set-up of Thiry [5], and in [14], \(G_N \approx \phi^2\), where \(\phi\) satisfies (49). This implies that a constant solution for \(\phi\) and, respectively, \(G_N\) is only possible when the unphysical constraint \(F^{ik} F_{ik} = 0\) is satisfied.

In contrast, in the dual set-up, a constant solution is possible. However, \(N\) (together with \(A_t\) and \(g_{ij}\)) is a solution to the system of equations (41), (43), and (44) and, in general, does not need to be a constant. Then it plays the role of a dilaton field.

To illustrate this, consider the standard cosmological metric [11] with \(E_{\mu} = \delta_{\mu}^t\):
\[
ds^2_{(5)} = -s^2 dt^2 + t^{2/(1-\alpha)} (dr^2 + r^2 d\Omega^2) + \alpha^2 (1-\alpha)^{-2} \tilde{t}^2 ds^2.
\]
Changing variables by: \(r \rightarrow s^4 e^{\tilde{t}r}\) with \(\gamma = -(1/2)(1+\alpha)/(1-\alpha)\) and \(t^{1/\alpha} \rightarrow a(t)\), we get:
\[
ds^2_{(5)} = -s^2 a(t) A_t^2 (2-\alpha^2) \tilde{t}^2 dt^2 + 2 A_t dtds + sa^2(t) e^{\tilde{t}r} (\beta^2 dr^2 + d\Omega^2) + \phi ds^2,
\]
where \(A_t = \gamma / (2^{3/2} \pi^2 / (1-\alpha)^2)\), \(A_t = A_r = A_\theta = 0\), and \(\phi = (\gamma / s) a(t) (2-\alpha^2) (\beta^2 dr^2 + d\Omega^2) + \phi ds^2\). We take \(\beta\) to be a negative constant, so that the field \(A_t\) will fall off towards infinity. Since \(a(t)\) describes the inflation, we note that the field \(A_t\) expands as \(a^2(t)\). The dilaton (which models the Newton’s constant) varies as:
\[
N^2 = (1-\alpha)^2 a^{-2} (a(t))^{-2\alpha}.
\]
Thus:
\[
\frac{G}{\dot{G}} = -2\alpha \frac{\dot{a}}{a} = -2\alpha H,
\]
where \(H\) is the Hubble’s constant. Observational limits [7] put \(\alpha < 10^{-3}\).

One should note that the four-dimensional metric is now s-dependent, but this does not pose a problem in the slicing formulation. Only the term \(\partial_k g_{ij}\) from the extrinsic curvature (33) should be recovered.

Finally, we note that the general solution for the dilaton field can be written as:
\[
N^2 = \frac{\left( \det g_{ij} \right) \left( \det E_{\mu}^{\nu} \right)^2}{\det G_{\mu\nu}},
\]
for a solution \(G_{\mu\nu}\) of (36) and embedding specified with \(E_{\mu}^{\nu}\).

To recapitulate, we have found plausible generalisations of Einstein–Maxwell equations and explained the origin of the constant solution for the dilaton (representing the Newton’s constant \(G_N\)) as well as the possibilities for modelling non-constant solutions for different cosmologies (representing time-varying \(G_N\)) in relation to the gauge freedom of our model.

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E.M.P. dedicates this work to his son Emanuel.

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