I. INTRODUCTION

In 1951 David Bohm introduced an extension of quantum mechanics, now called Bohmian mechanics, in which particles have definite positions and velocities at all times, including between measurements. He also showed how the same idea can be applied to field configurations. Bohmian mechanics agrees with all of the experimental predictions of the Copenhagen interpretation of quantum mechanics but has the advantages of a smooth transition to classical mechanics, the absence of wave-function collapse, and not having to separate the universe into quantum systems and classical measuring devices. Furthermore, the dynamics is deterministic and time reversible. Randomness enters only in the initial conditions of the particle and field configurations. On the other hand, Bohm constructed his mechanics in first quantized form which is fine for the nonrelativistic Schrödinger equation but it can not handle fermion particle creation and annihilation in the second quantized representation of quantum field theory. In this paper we generalize Bohmian mechanics so that it can handle operators with discrete spectra, and thereby accommodate second quantized fermionic field theories while remaining deterministic.

In Bohmian mechanics, the quantum state and Hamiltonian do not constitute a complete description of physical reality as they do in the Copenhagen interpretation of quantum mechanics. Bohmian mechanics singles out a set of dynamical variables associated with a particular set of commuting operators (for example, the positions of all of the particles) that are needed in addition to the quantum state and Hamiltonian to constitute a complete description of physical reality at all times. J. S. Bell has coined the term 'beables', short for 'maybeables' for those operators promoted to reality status. The hesitancy is built into the word to indicate, somewhat analogously to gauge freedom, that the set of commuting operators is not uniquely determined by experiment. Bohm’s original formulation was expressed in first quantized notion, His formalism required that all of the beables have continuous spectra which is fine when electron position operators are the beables. However, the Dirac equation presents a problem in this formalism since it requires an infinite number of negative energy electrons in the vacuum, so the most naive extension of Bohmian mechanics to the relativistic realm requires the calculation of an infinite number of trajectories even when there is nothing really there. If instead one thinks of the vacuum as a vacuum, and moves to a second quantized representation, then one is faced with the possibility of electron-positron pair creation and annihilation. This can not be straightforwardly treated with continuous variables alone since the number of particles is not a continuous variable.

In 1984 J. S. Bell constructed a Bohmian-like mechanics for beables with discrete spectra, in particular he considered fermion configurations in relativistic field theories. He gave up on particle trajectories altogether and considered instead the fermion number density operators as the beables. Unlike Bohmian mechanics, Bell’s mechanics is stochastic. Bell was dissatisfied with this since "...the reversibility of the Schrodinger equation strongly suggests that quantum mechanics is not fundamentally stochastic in nature. However I suspect that the stochastic element introduced here goes away in some sense in the continuum limit." Since Bell there have been several other contributions to Bell-like dynamics for relativistic quantum field theory, some bringing back explicit particle trajectories, but like Bell regrettably sacrificing determinism. In this paper we introduce an alternative to Bell’s mechanics that is a deterministic time-reversible Bohmian mechanics for operators with both discrete and continuous spectra. The determinism and time reversible invariance is present at the course grained level. No continuous limit is necessary, and no modification of the Hamiltonian need be made. Since our extension of Bohmian mechanics allows for discrete beables it is able to handle particle creation and annihilation in second quantized field theories, and thereby dispense with one of the objections to Bohmian mechanics.

In section II of this paper we present a generalization of Bohmian mechanics for operators with continuous spectra...
that is also amenable to operators with discrete spectra. In section III we incorporate operators with discrete spectra into the formalism. In section IV we present the exactly solvable case of Bohmian mechanics for one beable, and an intriguing visualization of Bohmian mechanics for any number of beables. In section V contains a summary of our results.

II. BOHMian MECHANICS WITH PROJECTION OPERATORS

In this section we present Bohmian mechanics in a generalized way, making extensive use of projection operators, that will allow us to incorporate operators with discrete spectra. The generalization agrees with Bohm’s original formulation when the beables are particle position operators but also allows for any choice of commuting operators $\hat{\xi}_\ell$, $\ell = 1, 2, \ldots L$, $[\hat{\xi}_\ell, \hat{\xi}_\ell'] = 0$ for the beables. The operators can have continuous or discrete spectra. Our generalization of Bohmian Mechanics is not unique. In particular, we have some freedom to choose how many and which commuting operators we require to describe the status of all possible measurement devices. In this section we will deal only with operators with continuous spectra and extend the formalism to discrete operators in the next section.

If $\hat{\xi}_\ell$ has continuous spectra we can express it as

$$\hat{\xi}_\ell = \int d\lambda \xi_\ell(\lambda) \hat{P}_\ell(\lambda)$$

(2.1)

where $\lambda_\ell$ parameterizes the eigenstates of $\hat{\xi}_\ell$, the integral is taken over the entire range of $\lambda_\ell$, and $\xi_\ell(\lambda_\ell)$ is the eigenvalue of $\hat{\xi}_\ell$ associated with the eigenstates labeled by $\lambda_\ell$.

$$\hat{\xi}_\ell |\lambda_\ell, q, \ell \rangle = \xi_\ell(\lambda_\ell) |\lambda_\ell, q, \ell \rangle$$

(2.2)

where $q$ distinguishes states with the same eigenvalue of $\hat{\xi}_\ell$. The projection operator for the eigenstates associated with $\xi_\ell(\lambda_\ell)$ is

$$\hat{P}_\ell(\lambda_\ell) = \sum_q \langle \lambda_\ell, q, \ell | \lambda_\ell, q, \ell \rangle.$$  

(2.3)

In this expression the sum over $q$ represents the sum or integral over states with the same eigenvalue of $\hat{\xi}_\ell$. The simplest $\xi_\ell(\lambda_\ell)$ function for an operator with continuous spectra is $\xi_\ell(\lambda_\ell) = \lambda_\ell$ in which case $\lambda_\ell$ has units of $\xi_\ell$. For this case the projection operator density, $\hat{P}_\ell(\lambda_\ell)$, takes on the particularly simple form

$$\hat{P}_\ell(\lambda_\ell) = \delta(\lambda - \hat{\xi}_\ell)$$

(2.4)

For operators with discrete spectra, which we cover in the next section, we will find it convenient to associate a range of lambda variables to a single eigenvalue.

Bohmian mechanics, describes the dynamics of a set of $\lambda_\ell(t)'s$ (that is, $\lambda_1, \lambda_2, \lambda_3, \ldots$, which we denote collectively by $\Lambda$) and thereby a set of $\xi_\ell(\lambda_\ell(t))'s$ (which we denote collectively by $\Xi$) which represents the physical values of these values at time $t$, regardless if a measurement is made or not. The quantum probability distribution of a particular $\lambda$ configuration is conveniently written in terms of the projection operators

$$P(\lambda, |t\rangle) = \langle t | \prod_{\ell=1}^L \hat{P}_\ell(\lambda_\ell) |t\rangle$$

(2.5)

This is a probability distribution since

$$\int d\lambda_1 \int d\lambda_2 \ldots \int d\lambda_L f(\xi_1(\lambda_1), \xi_2(\lambda_2), \ldots \xi_L(\lambda_L)) P(\lambda, |t\rangle) = \langle t | f(\hat{\xi}_1, \hat{\xi}_2, \ldots \hat{\xi}_L) |t\rangle$$

(2.6)

The result is unambiguous since we require that the $\hat{\xi}$ all commute with each other. The probability distribution has all the properties required of a classical probability distribution. The integral of the probability distribution taken over all of $\Lambda$ space is 1, and the probability distribution is real and non-negative provided that all of the projectors in the operator product commute with each other. The projectors will commute if the $\hat{\xi}_\ell$ all commute with each other, which is why we made this requirement for our set of beables.
The quantum state is in general not an eigenstate of the $\hat{\Xi}$. It is propagated forward in time as in conventional quantum mechanics
\[ i\hbar \frac{d|t>}{dt} = \hat{H}|t> \quad (2.7) \]
Additionally, the $\Lambda$ configuration is propagated forward in time with the first order equations
\[ \frac{d\lambda_{\ell}(t)}{dt} = v_{\ell}(\{\lambda(t)\}, t) \quad (2.8) \]
where the $v_{\ell}(\{\lambda(t)\}, t)$ are chosen so that the classical probability distribution of the $\lambda$ configuration of an ensemble of identical experiments
\[ P_c(\lambda, t) = \int d\lambda_1^0(0) \int d\lambda_2^0(0) \cdots \int d\lambda_L^0(0) P_c(\lambda', 0) \prod_{\ell=1}^L \delta(\lambda_\ell - \lambda'_\ell(t, \{\lambda'(0)\})) \quad (2.9) \]
agrees with the quantum probability distribution at all time provided they are in agreement at any one time. If the beables are chosen such that all measurements are measurements of the beables, this guarantees that the results of Bohmian mechanics are consistent with the results of conventional quantum mechanics.

Using Bohmian mechanics Eq.(2.8), the time dynamics of the classical probability distribution is
\[ \frac{dP_c(\{\lambda\}, t)}{dt} = -\sum_{\ell=1}^L \frac{\partial P_c(\{\lambda\}, t)}{\partial \lambda_\ell} v_{\ell}(\{\lambda\}, t) \quad (2.10) \]
Whereas the time derivative of the quantum probability distribution on the other hand is
\[ \frac{\partial P(\lambda, t)}{\partial t} = \sum_{\ell=1}^L <t | \left( \prod_{j=1}^{\ell-1} \hat{P}_j(\lambda_j) \right) \frac{1}{i\hbar} [\hat{P}_\ell(\lambda_\ell), \hat{H}] \left( \prod_{k=\ell+1}^L \hat{P}_k(\lambda_k) \right) |t>. \quad (2.11) \]
Following David Bohm’s insight, we note that if the classical and quantum probability distributions agree at any particular time then they agree for all time provided that the $v_{\ell}(\{\lambda\})$ are chosen so that the two time derivatives Eqs. (2.10) and (2.11) are equal. This is what we do now.

Our goal is to rewrite Eq.(2.11) in the form
\[ \frac{\partial P(\lambda, t)}{\partial t} = -\sum_{\ell=1}^L \frac{\partial J_\ell(\lambda, t)}{\partial \lambda_\ell}, \quad (2.12) \]
with $J_\ell(\lambda, t)$ real. If this can be accomplished then we can set
\[ v_{\ell}(\{\lambda\}) = \frac{J_\ell(\lambda, t)}{P(\lambda, t)} \quad (2.13) \]
and we will have determined a consistent Bohmian dynamics. Associating the $\ell$ terms in both expressions we have
\[ \frac{\partial J_\ell(\lambda, t)}{\partial \lambda_\ell} = -<t | \left( \prod_{j=1}^{\ell-1} \hat{P}_j(\lambda_j) \right) \frac{1}{i\hbar} [\hat{P}_\ell(\lambda_\ell), \hat{H}] \left( \prod_{k=\ell+1}^L \hat{P}_k(\lambda_k) \right) |t>. \quad (2.14) \]
Ignoring for the moment the possibility that the right hand side of Eq.(2.14) is not real we write
\[ J_\ell(\lambda, t) = <t | \left( \prod_{\ell'=1}^{\ell-1} \hat{P}_{\ell'}(\lambda_{\ell'}(t)) \right) \hat{J}_\ell(\lambda_\ell(t)) \left( \prod_{\ell''=\ell+1}^L \hat{P}_{\ell''} \right) |t>, \quad (2.15) \]
in which
\[ \frac{d\hat{J}_\ell(\lambda)}{d\lambda} = -\frac{1}{i\hbar} [\hat{P}_\ell(\lambda), \hat{H}] \quad (2.16) \]
which is easily integrated to
\[ \dot{J}_\ell(\lambda_\ell(t)) = \frac{1}{i\hbar} [\dot{G}_\ell(\lambda_\ell(t)), \hat{H}] = -\frac{1}{i\hbar} [\dot{L}_\ell(\lambda_\ell(t)), \hat{H}] \tag{2.17} \]
where
\[ \dot{G}_\ell(\lambda_\ell(t)) = \int_{\lambda_\ell(t)} d\lambda' \hat{P}_{t}(\lambda') \tag{2.18} \]
and
\[ \dot{L}_\ell(\lambda_\ell(t)) = \int_{\lambda_\ell(t)} d\lambda' \hat{P}_{t}(\lambda'). \tag{2.19} \]
are projection operators for all states greater than or less than \( \lambda_\ell(t) \) respectively. The current operator, \( \dot{J}_\ell \), can be generalized to periodic beables such as the position of a bead on a ring with
\[ \dot{J}_\ell(\lambda_\ell(t)) = \frac{1}{i\hbar} \int d\lambda' \int d\lambda'' \hat{P}_{t}(\lambda'_\ell) \hat{H} \hat{P}_{t}(\lambda''_\ell) f(\lambda'_\ell, \lambda_\ell(t), \lambda''_\ell) \tag{2.20} \]
where \( f(\lambda'_\ell, \lambda_\ell(t), \lambda''_\ell) = 1(-1) \) if there is a non-crossing path that goes from \( \lambda'_\ell \) to \( \lambda''_\ell \) through \( \lambda_\ell(t) \) in the positive (negative) direction and \( f(\lambda'_\ell, \lambda_\ell(t), \lambda''_\ell) = 0 \) if there is no such path. We will not use this generalization in this paper.

The right hand side of Eq. (2.14) is not guaranteed to be real unless \([\hat{P}_\ell, \hat{H}], \hat{P}_{\ell'} = 0 \) for \( \ell \neq \ell' \) which is true if \([\hat{\xi}_\ell, \hat{H}], \hat{\xi}_{\ell'} = 0 \) for \( \ell \neq \ell' \). If this is not the case we can make it real simply by taking the real part
\[ J_\ell(\lambda, t) = \text{Re} \left( <t| \left( \prod_{\ell'-1}^{\ell-1} \hat{P}_{\ell'}(\lambda_{\ell'}(t)) \right) \hat{J}_\ell(\lambda_\ell(t)) \left( \prod_{\ell''=\ell+1}^{L} \hat{P}_{\ell''} \right) |t> \right). \tag{2.21} \]
but this picks out a particular order of the operators for special treatment. A more democratic way to guarantee that the current is real is the symmetric average
\[ J_\ell(\lambda, |t>) = <t|S \left\{ \left( \prod_{\ell'-1}^{\ell-1} \hat{P}_{\ell'}(\lambda_{\ell'}(t)) \right) \hat{J}_\ell(\lambda_\ell(t)) \left( \prod_{\ell''=\ell+1}^{L} \hat{P}_{\ell''} \right) \right\} |t> \tag{2.22} \]
where \( S\{\ldots\} \) implies a symmetric average of all of the operators inside the braces. For example if \( L=3 \),
\[ J_1(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{6} <t|2\hat{J}_1 \hat{P}_2 \hat{P}_3 + \hat{P}_3 \hat{J}_1 \hat{P}_2 + \hat{P}_2 \hat{J}_1 \hat{P}_3 + \hat{P}_3 \hat{P}_2 \hat{J}_1 |t> \tag{2.23} \]
where we have used the fact that the \( P \)'s commute to combine some terms. This freedom in the choice of \( J_\ell \) is a particular example of a more general freedom. We can add any function \( Q_\ell \) to the probability current density, \( J_\ell \), such that \( \sum_\ell d/d\lambda_\ell Q_\ell = 0 \). This is the second way that the dynamics are not unique, (the first being the choice of which operators to anoint to beable status).

Our final expression for the \( \lambda \) dynamics is
\[ \frac{d\lambda_\ell(t)}{dt} = v_\ell(\{\lambda(t)\}, t) = \frac{<t|S \left\{ \left( \prod_{\ell'=1}^{\ell-1} \hat{P}_{\ell'}(\lambda_{\ell'}(t)) \right) \hat{J}_\ell(\lambda_\ell(t)) \left( \prod_{\ell''=\ell+1}^{L} \hat{P}_{\ell''} \right) \right\} |t>}{<t| \prod_{\ell=1}^{L} \hat{P}_{\ell}(\lambda_{\ell})|t>} \tag{2.24} \]

An equivalent way to write the equations of Bohmian mechanics, is to use the Heisenberg representation in which the quantum state does not change with time but any operator \( \hat{A}(t) \) depends on time via
\[ \hat{A}(t) = e^{-\hat{H}t/\hbar} \hat{A} e^{\hat{H}t/\hbar}. \tag{2.25} \]
The equations of Bohmian Mechanics in the Heisenberg representation are
\[ < 0|S \left\{ \left( \prod_{\ell'=1}^{\ell-1} \hat{P}_{\ell'}(\lambda_{\ell'}(t), t) \right) d\lambda_\ell(\lambda_\ell(t), t) \left( \prod_{\ell''=\ell+1}^{L} \hat{P}_{\ell''} \right) \right\} |0> = 0 \tag{2.26} \]
where

\[ d\hat{L}_\ell(\lambda_\ell(t), t) = \frac{\partial \hat{L}_\ell(\lambda_\ell(t), t)}{\partial t} + \frac{d\lambda_\ell(t)}{dt} \frac{\partial \hat{L}_\ell(\lambda_\ell(t), t)}{\partial \lambda_\ell(t)}. \]  \tag{2.27}

in which

\[ \frac{\partial \hat{L}_\ell(\lambda_\ell(t), t)}{\partial t} = \frac{1}{i\hbar} [\hat{L}_\ell(\lambda_\ell(t), t), \hat{H}] = -\hat{J}_\ell(\lambda_\ell(t), t), \]  \tag{2.28}

and

\[ \frac{\partial \hat{L}_\ell(\lambda_\ell(t), t)}{\partial \lambda_\ell(t)} = \hat{P}_\ell(\lambda_\ell(t), t). \]  \tag{2.29}

In the Schrodinger representation, since the quantum state changes with time, one is tempted to think of it as a
dynamic variable just like the beables and wonder why the beables depend on the quantum state but not the other way
around. In the Heisenberg representation the quantum state is not a dynamic variable so this apparent asymmetry
does not arise.

If an ensemble of identical experiments are performed, in which the Λ configurations at a particular time for each
experiment are taken at random from the quantum distribution Eq.(2.5) then if the Λ configurations are propagated
forwards and backwards in time via Eq.(2.24) or Eq.(2.26) then for all other times the probability distribution of the
Λ configurations over the ensemble Eq.(2.9) will equal the quantum distribution. Provided that the set of anointed
operators is sufficient to describe the status of all measurement devices, Bohmian mechanics will agree with all
results of conventional quantum mechanics without resorting to wavefunction collapse or some other alternative to
the Schrodinger dynamics to describe the measurement process as is done in the orthodox Copenhagen interpretation.
Bohmian mechanics replaces this with the mystery of how to explain why the classical and quantum probability
distributions should agree at any time at all.\footnote{2.5}

If all of the λ_\ell’s correspond to position coordinates in nonrelativistic quantum mechanics then the conventional
form of Bohmian mechanics is recovered from Eq.(2.24) or Eq.(2.26). For then the current operators are

\[ \hat{J}_\ell(\lambda_\ell) = \frac{1}{2i} \left[ \frac{\hat{p}_\ell}{m_\ell}, \delta(\lambda_\ell - \hat{x}_\ell) \right]_+. \]  \tag{2.30}

This is the current operator in conventional Bohmian mechanics so the equivalence with the traditional formalism
is proved. The present formalism is more flexible than the traditional formalism and can be easily generalized to
account for beable operators with discrete spectra, which is what we consider in the next section.

III. FITTING THE DISCRETE SQUARE PEG INTO THE CONTINUOUS ROUND HOLE

In this section we incorporate beables derived from operators with discrete spectra into Bohmian mechanics. Doing
so, we are immediately faced with the question of how to retain determinism which Bell sacrificed with regret. The
problem is most apparent when the initial quantum state is an eigenstate of each ξ_\ell for then the initial Ξ configuration
is uniquely determined so there appears to be no room for any randomness initially. The quantum time dynamics
will immediately make the quantum state a superposition of Ξ states so the classical probability distribution must
develop some spreading to agree with its quantum counterpart. Since there is no randomness in the initial conditions,
it appears that randomness must enter through the dynamics and therefore determinism must be sacrificed. Note
that this is only a problem if all of the ξ_\ell operators have discrete spectra. If even just one of the ξ_\ell operators have
continuous spectra then that is enough to make the initial state not unique so there is the possibility of a deterministic
dynamics producing the correct future probability distributions. Strictly speaking the same problem can arise for
operators with continuous spectra but for continuous spectra one can wiggle one’s way out by asserting that in any
actual experiment the quantum state is never exactly in an eigenstate of all of the operators. There is always some
spreading for whatever reason.

A way to retain deterministic dynamics with discrete operators is to assign finite ranges of λ_\ell to the same eigenvalue
of ξ_\ell. ξ_\ell sits still when λ_\ell is moving smoothly through a region that corresponds to the same eigenvalue and ξ_\ell makes
a sudden hop to a new eigenvalue when λ_\ell smoothly moves from one eigenvalue region to another. The problem of
determinism is solved by this devise since there are now many Λ configurations corresponding to the same physical
state. It might appear that for operators with discrete spectra, λ_\ell can legitimately be called a hidden variable since
the particular value of $\lambda_\ell$ inside an eigenvalue range is unobservable and therefore "hidden". But $\lambda_\ell$ is observable at those times when the $\xi_\ell$ hops to a new value since there is a unique value of $\lambda_\ell$ for each transition. Therefore $\lambda_\ell$ encodes observable information about previous transitions and the times that they occurred. We now construct the explicit Bohmian mechanics for the discrete case.

If $\xi_\ell$ has discrete spectra we can express it as

$$\xi_\ell = \sum_n \xi_{\ell n} \hat{P}_\ell(n)$$  \hspace{1cm} (3.1)$$

where $\xi_{\ell n}$ is the nth eigenvalue of $\hat{\xi}_\ell$ and the projection operator is

$$\hat{P}_\ell(n) = \sum_q |n, q, \ell < n, q, \ell|$$  \hspace{1cm} (3.2)$$

We now seek to express Eq.(3.1) in the continuous form Eq.(2.1) and define an appropriate $\xi_\ell(\lambda_\ell)$ function and projection operator $\hat{P}_\ell(\lambda_\ell)$ so that we may carry over Eq.(2.24) or Eq.(2.26) unchanged for the dynamics. There are many ways to do this. Technically, $\theta_\ell(\lambda_\ell) = \lambda_\ell$ and $\hat{P}_\ell(\lambda) = \delta(\lambda - \hat{\xi}_\ell)$ as in the continuous case does the job. But this leads to zero probability for $\lambda_\ell$ not equal to an eigenvalue. We can correct for this by smearing out the delta function over a range of $\lambda_\ell$ so that $\hat{P}_\ell(\lambda_\ell)$ is not zero between eigenvalues and a range of $\lambda_\ell$ corresponds to the same state. There are innumerable ways to parameterize $\xi_\ell$ to achieve this. Here is one way. Define the $\xi_\ell$ function

$$\xi_\ell(\lambda_\ell) = \xi_{\ell n}, \hspace{0.5cm} n = n(\lambda_\ell)$$  \hspace{1cm} (3.3)$$

where $\lambda_\ell$ has no units and $n(\lambda_\ell)$ is the closest integer to $\lambda_\ell$. With this $\xi_\ell(\lambda_\ell)$ function we achieve agreement between Eq.(3.1) and Eq.2.1) using

$$\hat{P}_\ell(\lambda_\ell) = \hat{P}_\ell(n(\lambda_\ell)).$$  \hspace{1cm} (3.4)$$

where $\hat{P}_\ell(n(\lambda_\ell)) = 0$ if there is no eigenvalue associated with the integer $n(\lambda_\ell)$. For discrete spectra with integer eigenvalues we can use the explicit forms

$$\hat{G}_\ell(\lambda_\ell(t)) = (n_\ell(t) + 1/2 - \lambda_\ell(t)) \hat{P}_\ell(n_\ell(t)) + \sum_{j=n_\ell(t)+1}^{n_\ell(t)+1} \hat{P}_\ell(j)$$

$$\hat{L}_\ell(\lambda_\ell(t)) = (\lambda_\ell(t) - n_\ell(t) + 1/2) \hat{P}_\ell(n_\ell(t)) + \sum_{j=-\infty}^{n_\ell(t)-1} \hat{P}_\ell(j),$$  \hspace{1cm} (3.5)$$

in the expressions for the current operator $\hat{J}_\ell$.

Using these definitions in the velocity expression Eq.(2.24) or Eq.(2.26) and keeping in mind that the physical values of $\xi_\ell$ with discrete spectra are determined by Eq.(3.3) we have defined a deterministic Bohmian mechanics for operators with discrete and continuous spectra. The initial $\Lambda$ configuration is taken from the quantum probability distribution Eq.(2.5) with Eq.(3.4) used for projectors for discrete operators. This means that the particular $\lambda_\ell$ for a given $\xi_{\ell, n}$ is chosen at random from a distribution spread uniformly from $n - 1/2$ to $n + 1/2$. We are free to order the eigenvalues along the $\lambda_\ell$ line any way we like, a freedom that also exists in the continuous case. This is the third way that the dynamics are not unique. The choice of ordering of the eigenvalues in $\Lambda$ space for each beable operator profoundly effects the dynamics since, for example a system in eigenvalue state 2 can only get to eigenvalue state 4 by first passing through eigenvalue state 3. For some systems the Hamiltonian determines a particular order, but perhaps there are Hamiltonians with transition elements between eigenvalues that are far apart in $\Lambda$ space, for any ordering that one chooses. This is a possible disadvantage of this method which is not shared by the stochastic Bell scheme.

In the next section we show how the formalism works for the simple example of Bohmian mechanics with only one beable and also present a visualization of Bohmian mechanics that allows us to dispense with the auxiliary $\lambda_\ell$ variables altogether.
Bohmian mechanics is integrable for the case in which there is only one operator promoted to beable status. For in that case

\[ <0|d\hat{L}(\lambda(t), t)|0> = d <0|\dot{\hat{L}}(\lambda(t), t)|0> = 0. \tag{4.1} \]

so the solution is

\[ <0|\dot{\hat{L}}(\lambda(t), t)|0> = <t|\dot{\hat{L}}(\lambda(t))|t> = L_0. \tag{4.2} \]

The integration constant, \( L_0 \) is uniformly distributed between zero and one. This equation has the following visual interpretation. Consider a line that goes from zero to one. The integration constant, \( L_0 \) sits immovable on the line. Associate the portion of the line from zero to \(< t|\hat{P}_a|t>\) with beable value \( a \), the portion of the line from \(< t|\hat{P}_a|t>\) to \(< t|\hat{P}_b|t>\) with beable value \( b \), the portion of the line from \(< t|\hat{P}_a|t>\) to \(< t|\hat{P}_b|t>\) with beable value \( c \). These boundaries change with time. The value of the beable at any time is the value associated with the portion of the line that \( L_0 \) sits on at that time. As an example consider a two state case in which an operator \( \xi \), with eigenvalues \pm 1 and projectors \( \hat{P}_\pm \) is the beable (we have chosen a slightly different parameterization of the eigenvalues than we did in the previous section to take advantage of the symmetry in the two state case). The Bohmian dynamics dictate that \( \xi = -1 \) for \(< t|\hat{P}_-|t> > L_0 \) and \( \xi = +1 \) for \(< t|\hat{P}_-|t> < L_0 \). These results can be combined into the equation of motion

\[ \xi(t) = \text{sign}(L_0 - <t|\hat{P}_-|t>) \tag{4.3} \]

or using \(< t|\hat{\xi}|t> = 1 - 2 <t|\hat{P}_-|t> \)

\[ \xi(t) = \text{sign}(<t|\hat{\xi}|t> - \xi_0). \tag{4.4} \]

where \( \xi_0 = 1 - 2L_0 \) which is uniformly distributed from -1 to 1. Note that \( \xi_0 \) encodes observable information about the times \( t_j \) that \( \xi \) changes its state via \(< t_j|\hat{\xi}|t_j> = \xi_0 \) so it is not "hidden", although it is uncontrollable. Also, the average value of \( \xi(t) \) over all \( \xi_0 \) agrees with the quantum expectation value

\[ \int_{-1}^{+1} \xi(t)P(\xi_0)\,d\xi_0 = <t|\hat{\xi}|t> \tag{4.5} \]

as is required for Bohmian mechanics to be consistent with quantum mechanics.

The exact solution for one beable suggests a visual interpretation of Bohmian Mechanics for any number of beables that allows us to dispense with \( \lambda \) as we were able to do for the one beable case. For \( n \) beables consider an \( n \) dimensional space of area 1 with fixed boundaries. The space is divided into several \( n \)-dimensional bubbles. Each bubble corresponds to a particular \( \Xi \) configuration and the volume of each bubble is the quantum probability of that configuration. Since the probabilities change with time, the bubbles are continuously contracting and expanding against each other. An immovable point is chosen at random in the \( n \)-dimensional space. The physical \( \Xi \) configuration at time \( t \) is the \( \Xi \) configuration corresponding to the bubble enclosing the immovable point at time \( t \).

**V. SUMMARY**

In this paper we have generalized Bohmian mechanics so that it can incorporate beables associated with an arbitrary set of commuting continuous and discrete operators. The equations are deterministic and time reversible and agree with Bohm’s original formulation for the case of continuous position operators. The simple case of only one beable is presented and the solution suggests an intriguing visualization of Bohmian mechanics.

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