A CH-type Inequality For Real Experiments

A. Shafiee*

1)Department of Chemistry, Kashan University,
Kashan, 87317-51167, Iran.
2)Institute for Studies in Theoretical Physics & Mathematics,
P.O.Box 19395-5531, Tehran, Iran.

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Abstract

We derive an efficient CH-type inequality. Quantum mechanics violates our proposed inequality independent of the detection-efficiency problem.

In photonic Bell-type experiments [1], when photon pairs with parallel linear polarizations are emitted, one can consider a Clauser-Horne (CH) inequality [2], at the level of hidden variables, in the form

$$-1 \leq S_{rq,HV}(\hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda) \leq 01$$ (1)

where

$$S_{rq,HV}(\hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda) = p_r^{(1)}(\hat{a}, \lambda) \left[ p_q^{(2)}(\hat{b}, \lambda) - p_q^{(2)}(\hat{b}', \lambda) \right] + p_r^{(1)}(\hat{a}', \lambda) \left[ p_q^{(2)}(\hat{b}, \lambda) + p_q^{(2)}(\hat{b}', \lambda) \right] - p_r^{(1)}(\hat{a}', \lambda) - p_q^{(2)}(\hat{b}, \lambda)2$$ (2)

In (2), we are considering four sub-ensemble of photon pairs with linear polarizations along (\(\hat{a}, \hat{b}\), (\(\hat{a}, \hat{b}'\)), (\(\hat{a}', \hat{b}\)), and (\(\hat{a}', \hat{b}'\)) in which one registers.

*E-mail: shafiee@theory.ipm.ac.ir
the result \( r \) for the first photon with an appropriate probability \( p_r^{(1)} \) and the result \( q \) for the second one with probability \( p_q^{(2)} \) where \( r, q = \pm 1 \).

To extend the CH inequality to a more efficient one, we propose a function \( S'_{rq} \) in the form:

\[
S'_{rq,HV}(\hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda) = p_r^{(1)}(\hat{a}, \lambda) \left[ p_q^{(2)}(\hat{b}, \lambda) - p_q^{(2)}(\hat{b}', \lambda) \right] \\
+ p_r^{(1)}(\hat{a}', \lambda) \left[ p_q^{(2)}(\hat{b}, \lambda) + p_q^{(2)}(\hat{b}', \lambda) \right] \\
- p_r^{(1)}(\hat{a}, \lambda) p_q^{(2)}(\hat{a}', \lambda) - p_r^{(1)}(\hat{b}, \lambda) p_q^{(2)}(\hat{b}', \lambda) \tag{3}
\]

In contrast to the CH inequality, the upper limit of relation (3) should not necessarily be equal to zero in non ideal experiments. To avoid this difficulty, we first consider the following inequality for the single-particle probabilities in an actual experiment:

\[
0 \leq p_j^{(k)}(\hat{x}_k, \lambda) \leq 1 - p_0^{(k)}(\hat{x}_k, \lambda) \tag{4}
\]

where \( \hat{x}_1 = \hat{a}, \hat{a}' \) or \( \hat{b}, \hat{b}' \) or \( \hat{d} \) and \( j = \pm 1 \). The function \( p_0^{(k)}(\hat{x}_k, \lambda) \) denotes non-detection probability for the \( k \)th photon with the polarization along \( \hat{x}_k \). Then, we define the following relation:

\[
\sum_{j=\pm 1} p_j^{(k)}(\hat{x}_k, \lambda) = \alpha^{(k)}(\hat{x}_k, \lambda) = 1 - p_0^{(k)}(\hat{x}_k, \lambda) \tag{5}
\]

where \( \alpha^{(k)}(\hat{x}_k, \lambda) \) is a measure of inefficiencies at the level of hidden-variables. For more convenience, we call \( \alpha^{(1)}(\hat{a}, \lambda) \equiv \alpha_1, \alpha^{(1)}(\hat{a}', \lambda) \equiv \alpha'_1, \alpha^{(2)}(\hat{a}', \lambda) \equiv \alpha_2, \alpha^{(2)}(\hat{b}, \lambda) \equiv \beta_2, \alpha^{(2)}(\hat{b}', \lambda) \equiv \beta'_2 \) and \( \alpha^{(1)}(\hat{b}, \lambda) \equiv \beta_1 \). According to the definition of the inefficiency measures in (5), we have:

\[
\int_{\Lambda} \alpha^{(1)}(\hat{x}_1, \lambda) \alpha^{(2)}(\hat{x}_2, \lambda) \rho(\lambda) d\lambda = \int_{\Lambda} \sum_{r,q=\pm 1} p_r^{(1)}(\hat{x}_1, \lambda) p_q^{(2)}(\hat{x}_2, \lambda) \rho(\lambda) d\lambda \\
= \sum_{r,q} P^{(12)}_{rq}(\hat{x}_1, \hat{x}_2) \equiv M(\hat{x}_1, \hat{x}_2) \tag{6}
\]

where \( P^{(12)}_{rq}(\hat{x}_1, \hat{x}_2) \) is the joint probability for getting the results \( r \) and \( q \) for the first and second photons along \( \hat{x}_1 \) and \( \hat{x}_2 \), respectively, at the experimental level and \( \rho(\lambda) \) is a probability density in space \( \Lambda \). One can easily show that \( M(\hat{x}_1, \hat{x}_2) = 1 - P_0^{(1)}(\hat{x}_1) - P_0^{(2)}(\hat{x}_2) - P_{00}^{(12)}(\hat{x}_1, \hat{x}_2) \) which is a measure of
non-detection probabilities in real experiments. Now, we make the following assumption:

**A** - The experimental probabilities of non-detection are independent of the polarization directions.

It is important to notice that we are suggesting **A** only at the observational level. This indicates that \( M(\hat{x}_1, \hat{x}_2) \) should be independent of any direction. Using **A** and multiplying the limits of \( S'_{rq,HV} \) in (3) through \( \rho(\lambda) \) and integrating over the space \( \Lambda \), we get the following inequality at the experimental level:

\[
-1 \leq S'_{rq,exp}(\hat{a}, \hat{b}, \hat{a}', \hat{b}') \leq 0.7 \quad (7)
\]

This is our extended CH inequality where \( S'_{rq,exp} \) is defined as follows:

\[
S'_{rq,exp}(\hat{a}, \hat{b}, \hat{a}', \hat{b}') = P^{(12)}_{rq}(\hat{a}, \hat{b}) - P^{(12)}_{rq}(\hat{a}, \hat{b}') + P^{(12)}_{rq}(\hat{a}', \hat{b}) + P^{(12)}_{rq}(\hat{a}', \hat{b}') - P^{(12)}_{rr}(\hat{a}', \hat{a}') - P^{(12)}_{qq}(\hat{b}, \hat{b}) \quad (8)
\]

In deriving (7), we have used Bell’s locality (factorizability) assumption [3]. The inequality (7) is violated by quantum mechanical predictions in non-ideal regime. To show this one can define the joint probability \( P^{(12)}_{+,QM}(\hat{a}, \hat{b}) \) in a real experiment as [4]

\[
P^{(12)}_{+,QM}(\hat{a}, \hat{b}) \approx \frac{1}{4} \eta_1 \eta_2 f \left[ 1 + F \cos 2(\hat{a} - \hat{b}) \right] \quad (9)
\]

where \( \eta_k \) (\( k = 1, 2 \)) and \( f \) are respectively the efficiencies of the detectors and the collimators, and \( F \) is a measure of the correlation of the two photons. The efficiency parameters in (9) are usually believed to be independent of the polarization directions in the literature. So, the assumption **A** is naturally honored in quantum mechanical calculations.

Now, we consider the case \( | \hat{a} - \hat{b} | = | \hat{a}' - \hat{b} | = | \hat{a}' - \hat{b}' | = \frac{\varphi}{2} \) and \( | \hat{a} - \hat{b}' | = \frac{3\varphi}{2} \). Substituting (9) and relations similar to it in (8) and choosing \( r = q = +1 \), we get

\[
S'_{++,QM}(\varphi) \approx \frac{1}{4} \eta_1 \eta_2 F \left[ 3 \cos \varphi - \cos 3\varphi - 2 \right] \quad (10)
\]

Considering the upper limit in (7), we get
\[ (3 \cos \varphi - \cos 3\varphi) \leq 211 \]  
(11)

This is violated for certain ranges of \( \varphi \). We note that none of the efficiency parameters appear in this inequality. Thus, quantum mechanics violates (7) independent of the efficiencies of the apparatuses.

References


