I. INTRODUCTION

Perhaps the most intriguing open question in present-day fundamental physics concerns a quantum theory of all fundamental interactions including gravitation. Experimental research in this field is challenging because quantum-gravitational effects are expected to be suppressed by the Planck scale $M_{Pl} \simeq 10^{19}$ GeV. However, Lorentz violation is a promising candidate signature of fundamental physics that lies within the sensitivity range of experiments with current or near-future technology.$^1$

At presently attainable energy scales, the effects of Lorentz violation can be described within an effective-field-theory framework called the Standard-Model Extension (SME).$^{2-4}$ At the classical level, the action of the SME incorporates, e.g., all leading-order contributions to the Lagrangian that are formed by combining Standard-Model and gravitational fields with Lorentz-breaking parameters such that coordinate independence is maintained. Nonzero parameters for Lorentz violation can arise in a variety approaches to underlying physics including strings,$^5$ various non-string models of quantum gravity,$^6$ noncommutative field theories,$^7$ varying couplings,$^8$ random dynamics,$^9$ multiverses,$^{10}$ and brane-world scenarios.$^{11}$ The flat-spacetime limit of the SME has provided the basis for numerous investigations of Lorentz violation involving mesons,$^{12-14}$ baryons,$^{15-18}$ electrons,$^{19-22}$ photons,$^{23-26}$ muons,$^{27}$ and neutrinos.$^{28,29}$

The extraction of the physical content of the SME requires an initial investigation of the quadratic sectors of its Lagrangian paralleling the conventional case. More specifically, the source-free equations of motion, the associated Hamiltonians, the dispersion relations, and, in the fermion case, the eigenspinors form a cornerstone on which further theoretical studies and comparisons with experiment rest. For example, the majority of the aforementioned analyses of Lorentz tests involve the theory of free massive fermions of the SME. Some basics of this theory have been made plausible or have been derived in certain limits as needed, but a comprehensive treatment has been lacking. The present work is intended to fill this gap.$^{30}$ More specifically, we give a more rigorous and detailed study of the general free Dirac equation and its solutions in the context of the SME. These results provide important tools for further studies of the physical implications of Lorentz violation.

The paper is organized as follows. Section II sets up the notation, reviews the basics of the modified Dirac equation, and comments on its structure. The construction of a hermitian Hamiltonian to arbitrary order in the Lorentz-violating parameters is discussed in Sec. III. In Sec. IV, we perform a systematic analysis of discrete dispersion-relation symmetries. Explicit leading-order approximations of the fermion eigenenergies are obtained in Sec. V. Sections VI and VII contain an investigation of the eigenspinors including a discussion of their symmetries and a derivation of the generalized spinor projectors. A brief summary is presented in Sec. VIII.

II. BASICS

The general Lorentz-violating Lagrangian for a single spin-$\frac{1}{2}$ fermion$^2$ can be cast into a variety of forms. One such form reminiscent of the ordinary Dirac Lagrangian and emphasizing the derivative structure is$^{16}$

$$\mathcal{L} = \frac{1}{2} \gamma^\mu \Gamma^\nu \partial_\nu \psi - \bar{\psi} M \psi ,$$

where

$$\Gamma^\nu \equiv \gamma^\nu + c^{\mu \nu} \gamma_\mu + d^{\mu \nu} \gamma_5 \gamma_\mu + e^{\nu} + if^{\nu \rho} \gamma_5 + \frac{i}{2} g^{\lambda \mu \nu} \sigma_{\lambda \mu} ,$$

$$M \equiv m + a_\mu \gamma^\mu + b_\mu \gamma_5 \gamma^\mu + \frac{1}{2} H^{\mu \nu} \sigma_{\mu \nu} .$$

The gamma matrices $\{1, \gamma_5, \gamma^\mu, \gamma_5 \gamma^\mu, \sigma^{\mu \nu}\}$ have conventional properties, and the signature of the Minkowski metric $\eta_{\mu \nu}$ is $-2$. The Lorentz-violating parameters $a_\mu, b_\mu, c_{\mu \nu}, d_{\mu \nu}, e_\mu, f_\mu, g_{\mu \nu \lambda}$, and $H_{\mu \nu}$ are taken as real with $c_{\mu \nu}$ and $d_{\mu \nu}$ traceless, $g_{\mu \nu \lambda}$ antisymmetric in its first two indices, and $H_{\mu \nu}$ antisymmetric. Note that $a_\mu, b_\mu, e_\mu, f_\mu,$ and $g_{\mu \nu \lambda}$
In the above expressions, the dependence of the various quantities on the quantum-mechanical 4-momentum operator has been displayed for clarity. The Lagrangians (1) and (3) differ only by a total divergence, so that they are physically equivalent. Note that the replacement $i\not{\partial} \rightarrow \frac{1}{i}i\not{\partial}$ in Def. (4) renders both Lagrangians identical.

The associated modified Dirac equation is given by
\[
(i\Gamma^{\mu}\partial_{\mu} - M)\psi(x) = 0 .
\] (5)

A Klein-Gordon-type equation can be obtained employing the following squaring procedure. Consider the modified Dirac operator $(i\Gamma^{\mu}\partial_{\mu} - M)$ and change the signs of the parameters $b^{\mu}$, $d^{\mu\nu}$, $g^{\mu\nu}$, and $H^{\mu\nu}$. Application of the resulting operator to Eq. (5) yields the desired second-order equation
\[
(\tilde{S} + \tilde{P}\gamma^{5} + V^{\mu}\gamma_{\mu})\psi(x) = 0 ,
\] (6)

where we have defined
\[
\tilde{S} \equiv S^{2} - P^{2} + V^{2} + A^{2} - 2T^{2} , \quad \tilde{P} \equiv 2(iPS - V \cdot A - iT\tilde{T}) , \quad \tilde{V}^{\mu} \equiv 2(SV^{\mu} + iPA^{\mu} - 2iT^{\mu\nu}V_{\nu} + 2\tilde{T}^{\mu\nu}A_{\nu}) .
\] (7)

In the above expressions, the dependence of the various quantities on $i\not{\partial}$ is understood. The tensor $\tilde{T}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}T_{\alpha\beta}$ denotes the dual and $\epsilon^{\mu\nu\alpha\beta}$ is the totally antisymmetric symbol with $\epsilon^{0123} = +1$, as usual. The unconventional off-diagonal pieces in Eq. (6) can be eliminated with a second squaring procedure involving the application of $(\tilde{S} - \tilde{P}\gamma^{5} - \tilde{V}^{\mu}\gamma_{\mu})$ to the Klein-Gordon-type equation (6). As expected, the resulting fourth-order operator can be expressed as the determinant of the modified Dirac operator, paralleling the conventional case:
\[
det(i\Gamma^{\mu}\partial_{\mu} - M)\psi(x) = 0 .
\] (8)

Thus, each individual component of a spinor solving the modified Dirac equation (5) satisfies Eq. (8).

A plane-wave ansatz $\psi(x) = \exp(-i\lambda^{\mu}x_{\mu})W(\lambda)$ for solutions to the modified Dirac equation (5) yields
\[
(\Gamma^{\mu}\lambda_{\mu} - M)W(\lambda) = 0
\] (9)

determining the four-component spinor $W(\lambda)$, where the 4-momentum $\lambda^{\mu}$ must solve the dispersion relation
\[
det(\Gamma^{\mu}\lambda_{\mu} - M) = 0 .
\] (10)

With our earlier assumption, dispersion relation (10) has two positive-valued roots $\lambda_{+(1,2)}^{\mu}(\lambda)$ and two negative-valued ones $\lambda_{-(1,2)}^{\mu}(\lambda)$. The corresponding 4-momenta and eigenspinors are $\lambda_{\pm(\alpha)}^{\mu}$ and $W_{\pm(\alpha)}(\lambda)$, respectively. Throughout this work, indices in parentheses can take the values 1 and 2. After the usual reinterpretation of the negative-energy solutions, the 4-momenta are denoted by
\[
p_{u}^{(\alpha)} \equiv \lambda_{+(\alpha)} , \quad p_{v}^{(\alpha)} \equiv -\lambda_{-(\alpha)} ,
\] (11)

where we have omitted the Minkowski indices for brevity. The notation for the reinterpreted eigenspinors is
\[
U^{(\alpha)}(\vec{p}) \equiv W_{+(\alpha)}(\vec{\lambda}) , \quad V^{(\alpha)}(\vec{p}) \equiv W_{-(\alpha)}(\vec{\lambda}) .
\] (12)

The spinors and the dispersion relation are discussed in more detail in subsequent sections.
For a 4-momentum $\lambda_\mu$ that fails to satisfy dispersion relation (10), the cofactor matrix of the modified Dirac operator $(\Gamma^\mu \lambda_\mu - M)$ in $\lambda$-momentum space is given by
\[
\text{cof}(\Gamma^\mu \lambda_\mu - M) = \text{det}(\Gamma^\mu \lambda_\mu - M)(\Gamma^\mu \lambda_\mu - M)^{-1}.
\] (13)

This matrix appears in many applications of our model (1), such as the anticommutator function. A more explicit expression is therefore desirable. As a corollary of the above discussion of the equations of motion, we obtain
\[
\text{cof}(\Gamma^\mu \lambda_\mu - M) = (S - \tilde{P}\gamma_5 - \tilde{V}^\mu \gamma_\mu)(S + iP\gamma_5 + V^\mu \gamma_\mu - A^\mu \gamma_5 \gamma_\mu - T^{\mu\nu} \sigma_{\mu\nu}) .
\] (14)

As we are now working in momentum space, the according change from $i\partial$ to $\lambda$ in Defs. (4) and (7) is implied. Replacing $\lambda \to i\partial$ everywhere in (14), yields the associated relation in position space, as usual.

III. HAMILTONIAN

In concordant frames, $\Gamma^0$ is invertible, so that the modified Dirac equation (5) can be cast into Schrödinger form:
\[
i\partial_\tau \psi(x) = (\Gamma^0)^{-1}(i\vec{\nabla} \cdot \vec{\nabla} + M)\psi(x) .
\] (15)

Although the operator $\hat{H}[i\vec{\nabla}] \equiv (\Gamma^0)^{-1}(i\vec{\nabla} \cdot \vec{\nabla} + M)$ appearing above is reminiscent of a Hamiltonian, it fails to be hermitian in general. This results in such undesirable features like a non-unitary time evolution. This issue can be resolved by a spinor redefinition $A\chi \equiv \psi$ chosen to eliminate the unconventional time-derivative couplings. Here, $A$ is a non-singular spacetime-independent $4\times4$ matrix, which exists in concordant frames. This field redefinition leaves unaffected the physics because it is a canonical transformation. It amounts to a change of basis in spinor space.

The existence of $A$ is equivalent to the positive definiteness of $\gamma^0\Gamma^0$. This permits the definition of a unique, positive-definite, invertible matrix $(\gamma^0\Gamma^0)^{1/2}$. The matrix $A$ can then be expressed as
\[
A = (\gamma^0\Gamma^0)^{-1/2} .
\] (16)

Note that the hermiticity of $\gamma^0\Gamma^0$ yields $A = A^\dagger$. It can now be verified that the Hamiltonian $H$ given by
\[
H[i\vec{\nabla}] = (\gamma^0\Gamma^0)^{-1/2}\gamma^0(i\vec{\nabla} \cdot \vec{\nabla} + M)(\gamma^0\Gamma^0)^{-1/2} ,
\] (17)

which governs the time evolution of the redefined field $\chi$, is hermitian, as desired. We also point out that $H$ and $\hat{H}$ are related by the similarity transformation
\[
H = (\gamma^0\Gamma^0)^{1/2}\hat{H}(\gamma^0\Gamma^0)^{-1/2} .
\] (18)

The determination of the explicit form of the matrix $A$ is challenging in the general. In practice, however, it usually suffices to determine $A$ up to a given order in the Lorentz-violating coefficients. Notice that $\gamma^0\Gamma^0$ can be split into the $4\times4$ identity $I$ plus a Lorentz-breaking correction: $\gamma^0\Gamma^0 = I + \gamma^0(\Gamma^0 - \gamma^0)$. This suggests the following expansion:
\[
A = I + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} (I - \gamma^0\Gamma^0)^n .
\] (19)

In a basis in which $\gamma^0\Gamma^0$ is diagonal, it is straightforward to verify that the expansion (19) indeed converges and is consistent with the definition of $(\cdot)^{1/2}$. It can therefore be used to determine $A$ to arbitrary order.

IV. SYMMETRIES OF THE DISPERSION RELATION

An explicit expression for single-particle dispersion relation (10) can be found by expanding the determinant:
\[
\text{det}(\Gamma^\mu \lambda_\mu - M) = 4[V_\mu A_\nu] - V_\mu V_\nu + A_\mu A_\nu + PT_{\mu\nu} - S^\mu S_{\mu\nu} + T^\mu T_\mu + \tilde{T}_\mu \tilde{T}_\mu ,
\] (20)

Here, $S, P, V_\mu, A_\mu$, and $T^{\mu\nu}$ are given in $\lambda$-momentum space, and $V_\mu A_\nu = V_\mu A_\nu - A_\mu V_\nu$ denotes the antisymmetric part. Relation (20) also follows directly from Eqs. (13) and (14). The position-space version of Eq. (20) can be used to cast Eq. (8) into a more explicit form.
Further insight about the structure of dispersion relation (10) can be gained by analysing its properties under charge conjugation C, parity inversion P, and time reversal T. For example, in the absence of explicit expressions for the eigenenergies, these investigations can be used to obtain information about their degeneracy. One such transformation, charge conjugation C, has been considered previously. It was shown that the two sets of roots

$$\left\{ \chi_{-(\alpha)}^0(\tilde{\lambda}, m, a^\mu, b^\mu, e^{\mu\nu}, d^{\mu\nu}, e^\mu, f^\mu, g^{\mu\nu\rho}, H^{\mu\nu}) \right\} = \left\{ -\chi_{+(\beta)}^0(-\tilde{\lambda}, m, -a^\mu, b^\mu, e^{\mu\nu}, -d^{\mu\nu}, -e^\mu, -f^\mu, g^{\mu\nu\rho}, -H^{\mu\nu}) \right\} \quad (21)$$

are identical. In this section, we employ the same methodology to find additional symmetries of the dispersion relation (10) and point out some subtleties regarding the labeling of the roots.

The idea is as follows. Multiplication of the modified Dirac operator ($\Gamma^\mu \lambda_\mu - M$) with a nonsingular, $\lambda$-independent matrix contributes only a nonzero multiplicative factor to the determinant in Eq. (10) leaving this equation, and thus its roots, unchanged. The determinant remains also invariant under transposition or complex conjugation of the modified Dirac operator. The latter is true because det($\Gamma^\mu \lambda_\mu - M$) is real, which follows from Eq. (20) and Def. (4).

We first consider

$$\left( \Gamma^\mu \lambda_\mu - M \right) \rightarrow \gamma^0 \left( \Gamma^\mu \lambda_\mu - M \right) \gamma^0,$$

which corresponds to parity inversion P in spinor space. We remark in passing that det($\Gamma^\mu \lambda_\mu - M$) remains unchanged, since the overall factor induced is det($\gamma^0 \gamma^0$) = 1. It follows that the dispersion relation (10) is invariant under

$$\left\{ \lambda^\mu, m, a^\mu, b^\mu, e^{\mu\nu}, d^{\mu\nu}, e^\mu, f^\mu, g^{\mu\nu\rho}, H^{\mu\nu} \right\} \rightarrow \left\{ \lambda^\mu, m, a_\mu, -b_\mu, e_{\mu\nu}, -d_{\mu\nu}, e_\mu, -f_\mu, g_{\mu\nu\rho}, H_{\mu\nu} \right\}. \quad (23)$$

Hence, the two sets

$$\left\{ \lambda^0_{-(\alpha)}(\tilde{\lambda}, m, a^\mu, b^\mu, e^{\mu\nu}, d^{\mu\nu}, e^\mu, f^\mu, g^{\mu\nu\rho}, H^{\mu\nu}) \right\} = \left\{ \lambda^0_{+(\beta)}(-\tilde{\lambda}, m, a_\mu, b_\mu, e_{\mu\nu}, d_{\mu\nu}, e_\mu, f_\mu, g_{\mu\nu\rho}, H_{\mu\nu}) \right\}, \quad (24)$$

each containing the two positive-valued solutions of Eq. (10), must be identical. The result for the remaining sets of the two negative-valued roots is obtained by replacing the subscript + by − in Eq. (24).

Next, we investigate spinor-space time reversal given by

$$\left( \Gamma^\mu \lambda_\mu - M \right) \rightarrow i \gamma^5 C (\Gamma^\mu \lambda_\mu - M)^* i C \gamma^5,$$

where $\ast$ denotes complex conjugation and C is the usual charge-conjugation matrix. Again, det($\Gamma^\mu \lambda_\mu - M$) is left unchanged. The resulting symmetry between the positive-valued roots of the dispersion relation (10) takes the form

$$\left\{ \lambda^0_{-(\alpha)}(\tilde{\lambda}, m, a^\mu, b^\mu, e^{\mu\nu}, d^{\mu\nu}, e^\mu, f^\mu, g^{\mu\nu\rho}, H^{\mu\nu}) \right\} = \left\{ \lambda^0_{+(\beta)}(-\tilde{\lambda}, m, a_\mu, b_\mu, e_{\mu\nu}, d_{\mu\nu}, e_\mu, f_\mu, -g_{\mu\nu\rho}, -H_{\mu\nu}) \right\}. \quad (26)$$

For the corresponding expression involving the negative-valued solutions, the subscript + must be changed to −.

Note that the above arguments, which generated (21), (24) and (26), provide only equalities between sets of roots. In order to find relations between the individual roots, additional considerations are needed. Charge conjugation on one hand, and parity inversion and time reversal on the other hand have to be treated separately.

We begin by discussing charge conjugation. Equality (21) relates positive- and negative-valued eigenenergies. It therefore follows already at this point that in principle relation (21) and the knowledge of one root suffices to construct another solution. The remaining less important question is how the subscript (α) behaves under C. If an additional conserved quantity commuting with $\tilde{H}$ is known (such as a spin component or helicity in the conventional case), the subscript (α) may be used to label its eigenvalues. A definite correspondence can then be determined by investigating the behavior of this conserved quantity under C. In the present case without the knowledge of such an additional conserved quantity, the label (α) becomes essentially arbitrary and can therefore be chosen freely. Our conventions agree with the conventional ones in the following sense: the labels of eigenvalues and eigenspinors match and change under charge conjugation. This produces after reinterpretation

$$E^{(1,2)}_u(p, m, a^\mu, b^\mu, e^{\mu\nu}, d^{\mu\nu}, e^\mu, f^\mu, g^{\mu\nu\rho}, H^{\mu\nu}) = E^{(2,1)}_u(p, m, -a^\mu, b^\mu, e^{\mu\nu}, -d^{\mu\nu}, -e^\mu, -f^\mu, g^{\mu\nu\rho}, -H^{\mu\nu}), \quad (27)$$

where $E^{(1,2)}_{u,v}$ denotes the zero-components of $p_{u,v}^{(1,2)}$ defined by (11). We remark that this labeling agrees with our previous choice.

We now turn to parity inversion and time reversal. The equalities (24) and (26) provide a correspondence between roots of the same sign. As opposed to the previous case for C, it is therefore unclear a priori whether P and T individually can be used to construct additional eigenvalues from a known one. However, the combined transformation PT should give a different root: it connects different eigenspinors, and according to our above conventions, this
reversing the sign of label (projection onto a fixed direction or helicity behave differently under both P and T. For definiteness, we choose the knowledge of an additional conserved operator this becomes a matter of choice constrained only by Eq. (28). Contrary

We mention that from any other combination of C, P, and T additional correspondences can be constructed straightforwardly. However, if one eigenenergy (with functional dependence agreeing with our choice of labeling) is straightforwardly. However, if one eigenenergy (with functional dependence agreeing with our choice of labeling) is known explicitly, the symmetries (27) and (28) suffice to determine the remaining three. As an illustrative example, consider the following case: all Lorentz-violating parameters except $a$ are known explicitly, the symmetries (27) and (28) suffice to determine the remaining three. As an illustrative example,

The above method for the construction of additional eigenvalues from a known root has to be taken with a grain of salt. It is required that the functional dependence of the given eigenenergy on the Lorentz-violating parameters is consistent with our above choice of labeling. The following example illustrates this issue. Consider a model in which Lorentz breaking can be described by $a^{\mu}$ and $b$ only. The following expressions for the reinterpreted eigenenergies satisfy its dispersion relation:

\[ E_{\mu}^{(a)} = \sqrt{m^2 + (|\vec{p} - \vec{a}| - (-1)^{\alpha}b_0)^2} + a_0 , \quad E_{\nu}^{(a)} = \sqrt{m^2 + (|\vec{p} + \vec{a}| - (-1)^{\alpha}b_0)^2} - a_0 . \]  

(29)

Suppose only one of the above energies, say $E_{\mu}^{(1)} = [m^2 + (|\vec{p} - \vec{a}| - b_0)^2]^{1/2} + a_0$, is known. According to Eq. (28), the eigenvalue $E_{\mu}^{(2)}$ for the other positive-energy solution can then be obtained by keeping the sign of $a_\mu$ the same, but reversing the sign of $b_0$ in complete agreement with Eq. (29). Similarly, the symmetry (27) permits the determination of the antiparticle energy $E_{\nu}^{(2)}$ by keeping all signs unchanged except for the one of $a_\mu$, which has to be reversed. Again, the result is identical to the one given by (29). The remaining antiparticle energy $E_{\nu}^{(1)}$ can be found in the same way as $E_{\mu}^{(2)}$, but starting from $E_{\mu}^{(2)}$ instead of $E_{\mu}^{(1)}$.

The above method for the construction of additional eigenvalues from a known root has to be taken with a grain of salt. It is required that the functional dependence of the given eigenenergy on the Lorentz-violating parameters is consistent with our above choice of labeling. The following example illustrates this issue. Consider a model in which Lorentz breaking can be described by $a^{\mu}$ and $b$ only. The following expressions for the reinterpreted eigenenergies satisfy its dispersion relation:

\[ E_{\mu}^{\pm} = \left[ m^2 + (|\vec{p} - \vec{a}|) \pm 2\sqrt{m^2b^2 + (\vec{b} \cdot (\vec{p} - \vec{a}))^2 + b^2} \right]^{1/2} + a_0 , \]

\[ E_{\nu}^{\pm} = \left[ m^2 + (|\vec{p} + \vec{a}|) \pm 2\sqrt{m^2b^2 + (\vec{b} \cdot (\vec{p} + \vec{a}))^2 + b^2} \right]^{1/2} - a_0 . \]  

(30)

Suppose again that only one of these eigenenergies, say $E_{\mu}^{\pm}$, were known. Employing symmetry (28), i.e., reversing the sign of $\vec{b}$, does not yield any of the other eigenenergies. This can be traced to the fact that the functional dependence in Eq. (30) does not agree with our convention that the labels should change under PT. In Eq. (30), the labels reflect the sign of a square root in the expression rather than being related to the PT transformation. We remark that a suitable labeling can be obtained by multiplying the inner square roots in Eq. (30) by a quantity $D$ that can take the values $+1$ and $-1$ and changes sign when $\vec{b}$ is reversed (i.e., a possible choice would be $D = \vec{b} \cdot \vec{B}/|\vec{b} \cdot \vec{B}|$, where $\vec{B}$ is arbitrary but fixed and satisfies $\vec{b} \cdot \vec{B} \neq 0$).

The symmetries (27) and (28) can also be used to find parameter combinations yielding degenerate roots. It follows from Eq. (28) that for

\[ b^{\mu} = d^{\mu\nu} = g^{\mu\nu\rho} = H^{\mu\nu} = 0 , \]  

(31)

roots of the same sign are equal. Thus, Eq. (31) provides a sufficient condition for energy equality of two (distinct) particle states of a given 3-momentum. The same holds true for the antiparticles. Suppose the Lorentz-violating parameters obey

\[ a^{\mu} = d^{\mu\nu} = e^{\mu} = f^{\mu} = H^{\mu\nu} = 0 , \]  

(32)

or

\[ a^{\mu} = b^{\mu} = e^{\mu} = f^{\mu} = g^{\mu\nu\rho} = 0 . \]  

(33)

Either one of the conditions (32) and (33) is sufficient for an energy degeneracy such that for each particle there exists an antiparticle of equal 4-momentum. This can be verified by using the symmetries (27) and (28). As a corollary of the above discussion we remark that if $c^{\mu\nu}$ is the only non-vanishing Lorentz-violating parameter, then all four eigenenergies become degenerate after reinterpretation.
V. FIRST-ORDER APPROXIMATION OF THE EIGENENERGIES

In principle, the dispersion relation (20) and Eq. (4) allow the determination of the exact eigenenergies at a given 3-momentum in the presence of Lorentz violation. In many circumstances, however, only leading-order corrections to the conventional eigenenergies are of interest. They can be obtained with the method described below.

With the aid of generalized Foldy–Wouthuysen techniques one can find a (momentum-dependent) unitary matrix $U$ transforming the Hamiltonian (17) into the following block-diagonal form:

$$U^\dagger H U = \begin{pmatrix} h_{\text{rel}} & 0 \\ 0 & \tilde{h}_{\text{rel}} \end{pmatrix},$$

(34)

where the two $2 \times 2$ matrices $h_{\text{rel}}$ and $\tilde{h}_{\text{rel}}$ are the respective Hamiltonians for the fermion and the antifermion. From the procedure it is obvious that the eigenvalues of the matrices $h_{\text{rel}}$ and $\tilde{h}_{\text{rel}}$ are the respective particle and antiparticle energies. To make further progress, consider the expansion

$$h_{\text{rel}} = h_0 + \sum_{j=1}^{3} h_j \sigma_j$$

(35)

of $h_{\text{rel}}$ with respect to the basis $\{I, \sigma_j\}$. Here, $I$ is the $2 \times 2$ identity and $\sigma_j$ are the usual Pauli matrices. The components $h_0, \ldots, h_3$ depend on the 3-momentum, the mass, and the parameters for Lorentz breaking as determined by the Foldy–Wouthuysen transformation. They yield the fermion eigenenergies by means of the equation

$$E^{(1,2)}_u = h_0 \pm \sqrt{\sum_{j=1}^{3} h_j h_j}.$$  

(36)

Note that the correspondence between the energy superscript (・) and the sign of the square root in (36) is only constrained by the symmetry (28).

This method is suitable for extracting the leading-order approximation of the eigenenergies because the components $h_0, \ldots, h_3$ are known to first order in the Lorentz-violating parameters:

$$h_0 = \gamma m + (a_0 - \frac{mc_0}{\gamma} - mc_0) + \left[ a_j - \gamma m (c_{0j} + c_{j0}) - mc_j \right] \frac{\gamma_j}{\gamma m} - (c_{jk} - \eta_{jk} c_{00}) \frac{p_i p_k}{\gamma m},$$

$$h_j = -\frac{1}{\gamma} b_j + md_{j0} + \frac{1}{2} \varepsilon^{klj} H_{kl} - \frac{1}{2\gamma} m \varepsilon^{klj} g_{kl0}$$

$$+ \left[ \eta_{jk} b_0 + m (d_{jk} - \eta_{jk} d_{00}) + \varepsilon^j_{kl} H_{0l} - \gamma m \varepsilon^j_{lm} \left( \frac{1}{2} g_{lmk} - \eta_{km} g_{lo0} \right) \right] \frac{p^l}{\gamma m}$$

$$+ \left[ \frac{(\gamma - 1)m}{\bar{p}^2} (b_k + md_{k0} + \frac{1}{2} \varepsilon^{mn}_{\ k} H_{mn} + \frac{1}{2} \varepsilon^{mn}_{\ km} g_{ln0}) \eta_{jl} - (d_{0k} + d_{k0}) \eta_{jl} + \varepsilon^j_{lm} (g_{m0k} + g_{mk0}) \right] \frac{p^l p^m}{\gamma m}$$

$$+ \frac{(\gamma - 1)}{\bar{p}^2} \left[ - (d_{kl} - \eta_{kl} d_{00}) - \frac{1}{2} \varepsilon^{nm}_{\ \ k} g_{n0l} \right] \eta_{jm} p^k p^l p^m \frac{\eta_{jm}}{\gamma},$$

(37)

where $\gamma \equiv \sqrt{1 + \bar{p}^2/m^2}$ is the usual relativistic gamma factor, and the totally antisymmetric rotation tensor $\varepsilon^{klj}$ satisfies $\varepsilon_{123} = +1$ and $\varepsilon^{jkl} = -\varepsilon_{jkl}$. Note that the parameter $f^\mu$ does not contribute to the eigenenergies at leading order. We remark that the symmetries (27) and (28) permit the construction of the antifermion energies.

As an illustration, consider the previously considered $(a_0, b_0)$ model with the eigenenergies (29). For this model, we have $h_0 = \gamma m + a_0 - \bar{p} \cdot \bar{a}/\gamma m$ and $h_j = b_0 p_j/\gamma m$. For this case, Eq. (36) yields

$$E^{(1,2)}_u = \sqrt{m^2 + \bar{p}^2 + a_0 - \frac{\bar{p} \cdot \bar{a} \pm b_0 \bar{p}}{\sqrt{m^2 + \bar{p}^2}}}.$$  

(38)

One can verify that Eqs. (29) and (38) agree to leading order in the Lorentz-violating coefficients provided the upper and lower signs in Eq. (38) are identified with the labels $\alpha = 1$ and $\alpha = 2$, respectively. The corresponding antiparticle energies to first order can now be obtained with the aid of symmetry (27), as discussed previously.
It is necessary to begin this section by introducing our conventions and some more notation. The four eigenspinors \( w^{(\alpha)}_{\pm}(\vec{\lambda}) \) of \( H(\vec{\lambda}) \) are determined by

\[
(H(\vec{\lambda}) - \lambda_{\pm(\alpha)}^0) w^{(\alpha)}_{\pm}(\vec{\lambda}) = 0 .
\]  

(39)

We have used that the dispersion relation (10), and thus its roots \( \lambda_{\pm(\alpha)}^0 \), remain unchanged under the field redefinition.

The eigenspinors \( w^{(\alpha)}_{\pm}(\vec{\lambda}) \) are related to the observer-covariant momentum-space spinors \( W^{(\alpha)}_{\pm}(\vec{\lambda}) \) obeying Eq. (9) by

\[
W^{(\alpha)}_{\pm}(\vec{\lambda}) = A w^{(\alpha)}_{\pm}(\vec{\lambda}) ,
\]

(40)

where \( A \) is the field-redefinition matrix discussed earlier. After reinterpretation of the negative-energy solutions we denote the eigenspinors of \( H \) by \( u^{(\alpha)}(\vec{p}) \) and \( v^{(\alpha)}(\vec{p}) \) in complete analogy to Def. (12). Thus, the transformation (40) remains valid even after the reinterpretation. The four eigenspinors for a given momentum, which can be taken as orthogonal, span spinor space. Our choice of normalization is

\[
u^{(\alpha)}(\vec{p}) u^{(\alpha')}(\vec{p}) = \delta^{\alpha \alpha'} \frac{E^{(\alpha)}_u}{m} , \quad v^{(\alpha)}(\vec{p}) v^{(\alpha')}(\vec{p}) = \delta^{\alpha \alpha'} \frac{E^{(\alpha)}_v}{m} , \quad u^{(\alpha)}(\vec{p}) v^{(\alpha')}(\vec{p}) = 0 .
\]

(41)

Note that the physical spinors \( w^{(\alpha)}_{\pm} \), and thus \( u^{(\alpha)} \) and \( v^{(\alpha)} \), fail to be observer Lorentz covariant due to the frame dependence of \( A \).

The discrete transformations \( C, P, \) and \( T \) determine correspondences between sets of spinors, paralleling the eigenenergy case. For the charge-conjugation transformation our previous result\(^3\)

\[
\left\{ W^{(\alpha)}_{-}(\vec{\lambda}, m, a^{\mu}, b^{\mu}, c^{\mu}, d^{\mu}, e^{\mu}, f^{\mu}, g^{\mu \nu \rho}, H^{\mu \nu}) \right\} \propto \left\{ W^{(\beta)}_{+}(-\vec{\lambda}, m, -a^{\mu}, b^{\mu}, c^{\mu}, -d^{\mu}, e^{\mu}, -f^{\mu}, g^{\mu \nu \rho}, H^{\mu \nu}) \right\}
\]

holds, which we provide here for completeness. The charge-conjugated spinor \( W^{\sigma} \equiv CW^{T} \) is defined with the conventional charge-conjugation matrix \( C \). A \( \neq \) sign, such as the one in relation (42), is to be understood as follows. For each spinor in one set there exists a spinor in the other set such that the two spinors are linearly dependent.

The parity transformation \( (22) \) induces the following relation

\[
\left\{ W^{(\alpha)}_{+}(\vec{\lambda}, m, a^{\mu}, b^{\mu}, c^{\mu}, d^{\mu}, e^{\mu}, f^{\mu}, g^{\mu \nu \rho}, H^{\mu \nu}) \right\} \propto \left\{ W^{(\beta)}_{+}(-\vec{\lambda}, m, a^{\mu}, b^{\mu}, c^{\mu}, d^{\mu}, e^{\mu}, -f^{\mu}, g^{\mu \nu \rho}, H^{\mu \nu}) \right\}
\]

(43)

where \( W^{P} \equiv \gamma^{0} W \) denotes the parity-inverted spinor as usual. The relation for the negative-eigenvalue spinors is obtained by replacing the subscripts \( + \) with \( - \). The result for time reversal \( (25) \) is given by

\[
\left\{ W^{(\alpha)}_{+}(\vec{\lambda}, m, a^{\mu}, b^{\mu}, c^{\mu}, d^{\mu}, e^{\mu}, f^{\mu}, g^{\mu \nu \rho}, H^{\mu \nu}) \right\} \propto \left\{ W^{(\beta)}_{+}(-\vec{\lambda}, m, a^{\mu}, b^{\mu}, c^{\mu}, d^{\mu}, e^{\mu}, -f^{\mu}, g^{\mu \nu \rho}, -H^{\mu \nu}) \right\} .
\]

(44)

Here, \( W^{T} \equiv -i \gamma^{5} C W^{*} \) is the conventional time-reversed spinor. Again, changing the subscripts from \( + \) to \( - \) yields the corresponding relation for the remaining eigenspinors.

To find correspondences between individual eigenspinors, a definite labeling scheme for the spinors must be selected. The associated subtleties are analogous to the eigenenergy case and do not require additional discussion. Our previous convention that the labeling of the roots and eigenspinors matches leads after reinterpretation to the symmetries

\[
V^{(1,2)}(\vec{p}, m, a^{\mu}, b^{\mu}, c^{\mu}, d^{\mu}, e^{\mu}, f^{\mu}, g^{\mu \nu \rho}, H^{\mu \nu}) = \zeta U^{(2,1)}(\vec{p}, m, -a^{\mu}, b^{\mu}, c^{\mu}, -d^{\mu}, e^{\mu}, -f^{\mu}, g^{\mu \nu \rho}, -H^{\mu \nu})
\]

(45)

and

\[
U^{(1)}(\vec{p}, m, a^{\mu}, b^{\mu}, c^{\mu}, d^{\mu}, e^{\mu}, f^{\mu}, g^{\mu \nu \rho}, H^{\mu \nu}) = \eta U^{(2)}(\vec{p}, m, -a^{\mu}, -b^{\mu}, c^{\mu}, -d^{\mu}, e^{\mu}, f^{\mu}, -g^{\mu \nu \rho}, -H^{\mu \nu})
\]

(46)

resulting from \( C \) and \( PT \), respectively. Here, \( \zeta \) and \( \eta \) are (possibly spinor-dependent) proportionality factors, and the superscript \( pt \) stands for the combined spinor transformations \( P \) and \( T \) defined above. The relation arising from \( PT \), but involving the spinors \( V^{(1,2)} \) can be obtained by replacing \( U \) with \( V \) in Eq. (46). If one eigenspinor is known explicitly (with functional dependence agreeing with our choice of labeling), the symmetries (45) and (46) can in principal be used to construct the remaining three.

VI. SYMMETRIES OF THE EIGENSPINORS
As an illustration, we again consider the \((a_{\mu}, b_{0})\) model. Its eigenspinors in Pauli-Dirac representation are

\[
U^{(\alpha)}(\vec{p}, m, a^\mu, b^0) = \left( \frac{E_u^{(\alpha)}(E_v^{(\alpha)} - a_0 + m)}{2m(E_v^{(\alpha)} - a_0 + m)} \right)^{1/2} \phi^{(\alpha)}(\vec{p} - \vec{a})
\]

\[
V^{(\alpha)}(\vec{p}, m, a^\mu, b^0) = \left( \frac{E_v^{(\alpha)}(E_u^{(\alpha)} + a_0 + m)}{2m(E_u^{(\alpha)} + a_0 + m)} \right)^{1/2} \phi^{(\alpha)}(\vec{p} + \vec{a})
\]

where the two-component spinors \(\phi^{(\alpha)}(\vec{k})\) are given by

\[
\phi^{(1)}(\vec{k}) = \begin{pmatrix} \cos(\theta/2) \\ e^{i\varphi} \sin(\theta/2) \end{pmatrix}, \quad \phi^{(2)}(\vec{k}) = \begin{pmatrix} -e^{-i\varphi} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}.
\]

Here, \(\theta\) and \(\varphi\) are the spherical-polar angles subtended by \(\vec{k}\). Suppose that only one of the spinors in Eq. (47), say \(U^{(1)}(\vec{p}, m, a^\mu, b^0)\), is known. With the symmetry (46) one can now construct \(U^{(2)}(\vec{p}, m, a^\mu, b^0)\) up to a constant: in \(U^{(1)}(\vec{p}, m, a^\mu, b^0)\), the sign of \(b^0\) has to be reversed. Note that this entails changing \(E_u^{(1)}\) to \(E_u^{(2)}\) by virtue of Eq. (28). Complex conjugation resulting from time reversal affects only \(\phi^{(1)}(\vec{p} - \vec{a})\) because all other quantities in the expression are real for this specific model. The final step is multiplication with the matrix \(-i\gamma^3 \gamma^1 \gamma^0 = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\), where \(\sigma^2\) denotes the usual Pauli matrix associated with 2-direction. This is somewhat simplified by observing that \(\sigma^2 \phi^{(1)}(\vec{k}) = i \phi^{(2)}(\vec{k})\). Comparison with Eq. (47) shows that the resulting spinor is indeed \(U^{(2)}(\vec{p}, m, a^\mu, b^0)\) up to a factor of \(-i\). The symmetry (45) determines (up to constants) the remaining two spinors \(V^{(1)}(\vec{p}, m, a^\mu, b^0)\) and \(V^{(2)}(\vec{p}, m, a^\mu, b^0)\) from \(U^{(1)}(\vec{p}, m, a^\mu, b^0)\) and \(U^{(1)}(\vec{p}, m, a^\mu, b^0)\), respectively; reverse the sign of \(a^\mu\), complex conjugate and multiply by \(i\sigma^2\). This procedure yields \(V^{(2)}(\vec{p}, m, a^\mu, b^0)\) exactly and \(V^{(1)}(\vec{p}, m, a^\mu, b^0)\) up to a relative minus sign.

With an explicit labeling scheme like that selected for the eigenenergies, additional relations between the spinors can be determined using other combinations of Eqs. (42), (43), and (44). The corresponding symmetries for the physical spinors defined earlier can be obtained by replacing \(\mathbf{U}^{(1,2)}\) and \(\mathbf{V}^{(1,2)}\) in Eqs. (45) and (46) by \(u^{(1,2)}\) and \(v^{(1,2)}\), respectively. Note that the field-redefinition matrix \(\mathbf{A}\) depends on the Lorentz-violating parameters. For example, to construct \(u^{(\alpha)}(\vec{p}, m, -a^\mu, b^\mu, c^\nu, -d^\nu, -e^\rho, f^\mu, g^{\mu\nu}, -H^{\mu\nu})\) from \(U^{(\alpha)}(\vec{p}, m, a^\mu, b^\mu, c^\nu, d^\nu, e^\rho, f^\mu, g^{\mu\nu}, H^{\mu\nu})\) and \(A(c^\rho, d^\rho, e^\sigma, f^\sigma, g^{\mu\rho}, H^{\mu\rho})\), the appropriate sign changes have to be implemented both in \(U^{(\alpha)}\) and in \(\mathbf{A}\).

**VII. GENERALIZATION OF THE CONVENTIONAL SPINOR PROJECTORS**

In the ordinary Dirac case, the spinor matrices that project on the positive- and negative-energy eigenspaces

\[
\pm \sum_{\alpha=1}^{2} w^{(\alpha)}_\pm \otimes \pi^{(\alpha)}_\pm = \frac{\lambda_\pm + m}{2m}
\]

are an indispensable tool in numerous calculations. To obtain the Lorentz-violating analog, we fix an arbitrary 3-momentum \(\vec{\lambda}\) and express the left-hand side of Eq. (13) in terms of the Hamiltonian (17):

\[
\text{cof}(\Gamma_{\mu}^\lambda - M) = \det(\Gamma^0) \prod_{[j]} (\lambda^0_j - \lambda^0_{[j]}) (\gamma^0 \Gamma^0)^{-1/2} (\lambda^0 - H)^{-1} (\gamma^0 \Gamma^0)^{-1/2} \gamma_j^0.
\]

Here and in what follows, the dependence of the eigenvalues, the eigenspinors, and the Hamiltonian on the fixed \(\vec{\lambda}\) is omitted for brevity. The two positive and two negative eigenvalues \(\lambda^0_{+ (1,2)}\) and \(\lambda^0_{- (1,2)}\) of \(H\) are denoted collectively by \(\lambda^0_{[j]}\), where \([j] \in \{-2, -1, +1, +2\}\). The product in Eq. (50) runs over all four of these eigenvalues. Since the Hamiltonian (17) is hermitian, there exists a spinor basis in which \(H\) is diagonal. In this basis, we have explicitly

\[
\text{cof}(\Gamma_{\mu}^\lambda - M) = \det(\Gamma^0) (\gamma^0 \Gamma^0)^{-1/2} P_{\lambda^0} (\gamma^0 \Gamma^0)^{-1/2} \gamma_j^0,
\]

where the diagonal matrix \(P_{\lambda^0}\) is given by

\[8\]
For $\lambda^0 \to \lambda^0_{+(1)}$, the nondegenerate case $\lambda^0_{+(1)} \neq \lambda^0_{+(2)}$ and the degenerate case $\lambda^0_{+(1)} = \lambda^0_{+(2)}$ have to be distinguished. In the nondegenerate case, the matrix $P_{\lambda^0}$ simplifies to

$$
P_{\lambda^0} = 
\begin{pmatrix}
(\lambda^0 - \lambda^0_{+(2)})(\lambda^0 - \lambda^0_{+(1)})(\lambda^0 - \lambda^0_{-(2)}) & \cdots & 0 \\
(\lambda^0 - \lambda^0_{+(1)})(\lambda^0 - \lambda^0_{+(1)})(\lambda^0 - \lambda^0_{-(2)}) & \ddots & \vdots \\
\vdots & \ddots & (\lambda^0 - \lambda^0_{+(1)})(\lambda^0 - \lambda^0_{+(2)})(\lambda^0 - \lambda^0_{-(1)}) \\
0 & \cdots & (\lambda^0 - \lambda^0_{+(1)})(\lambda^0 - \lambda^0_{+(2)})(\lambda^0 - \lambda^0_{-(1)})
\end{pmatrix}
$$

(52)

For any square matrix $B$, the relation $U^\dagger \text{cof}(B) U = \text{cof}(U^\dagger B U)$ holds, where $U$ is unitary. Thus the expressions (56) and (59) for the generalized projectors are independent of the spinor-space basis. We also remark that the projectors (49) for the ordinary Dirac case can be recovered from these results, as expected. In this case, the eigenenergies are...
degenerate so that Eq. (59) must be employed. Moreover, no field redefinition is necessary, so that the covariant and physical spinors are identical. The matrix of cofactors is given by \((\lambda^2 - m^2)(\lambda + m)\), which can be verified directly or can be obtained from Eq. (14). Assembling everything yields Eq. (49).

As an immediate application, the generalized projector (56) permits the construction of a more explicit form of the eigenspinor in the case when there is a nondegenerate eigenvalue \(\lambda_0^r\):

\[
W_{[r]}(\vec{\lambda}) = N_{[r]}(\vec{\lambda})\text{cof}(\Gamma^\mu_\lambda\lambda^\mu_{[r]} - M)W^0_{[r]}(\vec{\lambda}) ,
\]

where \(N_{[r]}(\vec{\lambda})\) is a normalization factor and \(W^0_{[r]}(\vec{\lambda})\) any spinor only constrained by \(W^0_{[r]}(\vec{\lambda}) \not\in \text{Ker}[\text{cof}(\Gamma^\mu_\lambda\lambda^\mu_{[r]} - M)]\). The remaining spinors can be determined, e.g., with the symmetries discussed earlier. If both the negative- and the positive-valued roots are degenerate and an additional conserved quantity commuting with the Hamiltonian is unknown, no orthonormal spinor basis spanning the eigenspaces is preferred. One can then replace \(\text{cof}(\cdot)\) with \(\hat{\text{cof}}(\cdot)\) in Eq. (60) and in the associated requirement on \(W^0_{[r]}(\vec{\lambda})\). It is now possible to proceed as in the nondegenerate case.

VIII. SUMMARY

This work has discussed the theory of the Lorentz-violating Dirac equation in the SME. The main results include various symmetry properties of the solutions and generalizations of conventional relations. In particular, Eq. (19) permits the construction of a hermitian Hamiltonian to arbitrary order in the Lorentz-violating parameters. Symmetries of the eigenenergies and the eigenspinors arising from the discrete C, P, and T transformations are given by Eqs. (27) and (28), and by Eqs. (45) and (46), respectively. The analog of the conventional spinor projectors is provided by Eq. (56) in the non-degenerate case or Eq. (59) in the degenerate case. These latter two equations involve the matrix of cofactors of the modified Dirac operator, which is given explicitly by Eq. (14).

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1 For recent experimental progress and theoretical ideas in this subject, see, e.g., CPT and Lorentz Symmetry II, edited by V.A. Kostelecký (World Scientific, Singapore, 2002).


4 V.A. Kostelecký, hep-th/0312310.


Suppose this were not true. In the conventional case, the arguments of the spinors, $\lambda, m, b$, remain unchanged under the combined transformation of parity inversion and time reversal. This implies that there exist spinors $W$ that are proportional to their PT image $-iC\gamma^5\gamma^0W$. That this is impossible can be verified by an explicit calculation in Pauli-Dirac representation.


F. Canè et al., physics/0309070.


See also R. Lehner, UMI-30-5449.


Note, however, that the labeling should reflect in a consistent way the functional dependence of the roots on the 3-momentum, the mass, and Lorentz-violating parameters. To illustrate this issue, consider a $b_0$ model with dispersion-relation roots labeled $\lambda^0_+(\alpha, \lambda, m, b_0) = \sqrt{m^2 + (|\lambda| + (-1)^a b_0)^2}$ and $\lambda^0_-(\alpha, \lambda, m, b_0) = -\sqrt{m^2 + (|\lambda| + (-1)^a |b_0|)^2}$. This labeling entails the undesirable feature that the behavior of $(\alpha)$ under $C$ depends on the sign of $b_0$. A different choice of labels, such as $|b_0| \rightarrow b_0$ in $\lambda^0_-(\alpha)$, would be more natural. This is issue is particularly important for the PT transformation.

In the ordinary Dirac case, either helicity or spin projection with respect to a given direction in the rest frame of the particle is usually employed to characterize the degeneracy of the eigenenergies. Note that under charge conjugation both of these operators change sign. For the present purpose it is therefore irrelevant whether helicity or spin-projection labels are taken as the conventional ones.

Suppose this were not true. In the conventional case, the arguments of the spinors, $\vec{p}$ and $m$, remain unchanged under the combined transformation of parity inversion and time reversal. This implies that there exist spinors $W$ that are proportional to their PT image $-iC\gamma^5\gamma^0W$. That this is impossible can be verified by an explicit calculation in Pauli-Dirac representation. Thus the claim holds true in the conventional case for any labeling. Assuming that in the present context the spinors depend smoothly on the Lorentz-breaking parameters as they approach zero, the result can be extended to the case of small Lorentz violation.