Ponzano-Regge model revisited I: Gauge fixing, observables and interacting spinning particles

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Abstract

We show how to properly gauge fix all the symmetries of the Ponzano-Regge model for 3D quantum gravity. This amounts to do explicit finite computations for transition amplitudes. We give the construction of the transition amplitudes in the presence of interacting quantum spinning particles. We introduce a notion of operators whose expectation value gives rise to either gauge fixing, introduction of time, or insertion of particles, according to the choice. We give the link between the spin foam quantization and the hamiltonian quantization. We finally show the link between Ponzano-Regge model and the quantization of Chern-Simons theory based on the double quantum group of SU(2).
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I. INTRODUCTION

It is well known that three dimensional gravity is an integrable system carrying only a finite number of degrees of freedom \([1, 2]\). Despite its apparent simplicity, this theory has not been yet fully solved, especially at the quantum level. Such a solution would be relevant to people interested in quantum gravity since 3 dimensional quantum gravity raises many of the key issues involved in the quantization of gravity such as: the problem of time, the problem of the construction of physically relevant observables, the coupling of quantum gravity to particles and matter fields, the emergence of semi-classical space-time geometry, the fate of black holes in a quantum geometry, the effect of cosmological constant, the holography principle, the quantum causal structure, the role of diffeomorphisms, the sum over topologies... The notable exception is the absence of gravitons in three dimensions.

The first problem one faces when dealing with the quantum theory is the variety of techniques and strategies one can use in order to quantize the theory [3]. The most prominent ones being the ADM quantization, the Chern-Simons quantization and the spin foam approach. ADM formulation has the virtue to be expressed in geometrical variables but this quantization scheme has not been very successful except in a few simple cases [4]. The Chern-Simons quantization is the most successful quantization scheme so far, which has allowed the inclusion of a cosmological constant and particles [5–7]. The drawback of this approach is that there is a clear lack of geometrical and physical understanding when working with the Chern-Simons observables. Moreover, this quantization scheme is tailored to three dimensions and the techniques developed there have no chances to be exported to the higher dimensional case. Eventually, the spin foam approach is the most promising approach in a direct quantization of higher dimensional gravity [8, 9]. Being a space time version of loop quantum gravity it has an interpretation in terms of quantum geometry [10]. Indeed, the first model of quantum gravity to be ever proposed by Ponzano and Regge in 1968 was a spin foam model for Euclidean quantum gravity without cosmological constant [11, 12].

Among several strategies, coupling gravity to particles is the simplest and most effective way to understand the physics and dynamics of quantum spacetime and the construction of observables. At the classical level, there is an extensive literature on this subject using a diversity of related descriptions: the polygonal t’Hooft approach [13, 14], the Waelbroeck discrete phase space approach [15–17], the conformal gauge approach [18], the Chern-Simons formalism [19, 20] and the continuum first order formalism [21, 22]. Almost all these works concentrate on the case of spinless particles. At the quantum level, most of the explicit results concerning the scattering of particles have been obtained using semi-classical techniques.
In our paper, we consider the spin foam quantization of three dimensional gravity coupled to quantum interacting spinning particles. We revisit the original Ponzano-Regge model in the light of recent developments and we propose the first key steps toward a full understanding of 3d quantum gravity in this context, especially concerning the issue of symmetries and the inclusion of interacting spinning particles. The first motivation is to propose a quantization scheme and develop techniques that could be exported to the quantization of higher dimensional gravity. As we will see, the inclusion of spinless particles is remarkably simple and natural in this context and allows us to compute quantum scattering amplitudes. This approach goes far beyond what was previously done in this context by allowing us to deal with the interaction of particles. The inclusion of spinning particles is also achieved. The structure is more complicated but the operators needed to introduce spinning particles show a clear and beautiful link with the theory of Feynman diagrams [26].

Another motivation of our work is to give a unified picture of possible quantization schemes. First, we show that the spin foam quantization is related to the discrete Waelbroeck hamiltonian approach which we generalize to include spinning particles. This approach is known to be related to ’t Hooft polygonal approach [17] and this gives a first step towards the understanding of scattering amplitude in the ’t Hooft approach as a spin foam model. We are aware of an independent work to come [27] in the context of loop quantum gravity leading to results similar to ours. We also completely unravel in our work, and in a companion paper [28] the link of the Ponzano-Regge model with the Chern-Simons quantization. We give a complete treatment of the gauge symmetries of the Ponzano-Regge model generalizing the work done in [29]. This opens the way to a finite quantization of Lorentzian gravity in the spin foam approach [30–32]. We introduce the notion of Ponzano-Regge observables and show that the computation of the expectation values of these PR-observables leads to gauge fixing, introduction of time or inclusion of particles, depending on the choice.

We will proceed as follows: In section (II), we review three dimensional gravity in the first order formalism, the description of spinning particles in this context and some facts about path integral quantization and transition amplitudes. In section (III), we review the Ponzano-Regge model, its gauge symmetries and their gauge fixing. In section(IV), we describe the notion of Ponzano-Regge observables, their application to gauge fixing, or insertion of particles and some duality properties as well as the link of Ponzano-Regge quantization with Chern-Simons quantization. In section (V) we present a hamiltonian description of three dimensional gravity allowing the inclusion of spinning particles and its quantization. In section (VI), we present the insertion of interacting spinless particles. In section(VII), we describe the inclusion of interacting spinning particles and the notion of particle graph functional. In section (VIII), we compute explicitly, starting from our general definition the physical scalar product and the action of the braid group. We show that we recover what is expected from canonical quantization and that the braid group action is governed by the quantum group $D(SU(2))$, known as the kappa deformation of the Poincaré group.
II. 3D CLASSICAL AND QUANTUM GRAVITY AND PARTICLE COUPLING

A. 3D classical gravity

In this part we recall briefly the properties of 3-dimensional Euclidean gravity in the first order formalism [9].

We consider the first order formalism for 3d gravity. The field variables are the triad frame field $e^i_\mu$ ($i = 1, 2, 3$) and the spin connection $\omega^i_\mu$. The metric is reconstructed as usual from the triad $g_{\mu\nu} = e^i_\mu \eta_{ij} e^j_\nu$ where $\eta = (+, +, +)$ for Euclidean gravity. In the following, we will denote by $e^i, \omega^i$ the one-forms $e^i_\mu dx^\mu, \omega^i_\mu dx^\mu$. We also introduce the SU(2) Lie algebra generator $J_i$, taken to be $i$ times the Pauli matrices, satisfying $[J_i, J_j] = -2\epsilon_{ijk} \eta^{kl} J_l$, where $\epsilon_{ijk}$ is the antisymmetric tensor. The trace is such that tr$(J_i J_j) = -2\delta_{ij}$. One defines the Lie algebra valued one-forms $e = e^i J_i$ and $\omega = \omega^i J_i$. The action is

$$ S[e, \omega] = -\frac{1}{8\pi G} \int_M \epsilon_{ijk} e^i \wedge F^{jk}(\omega) = \frac{1}{16\pi G} \int_M \text{tr}(e \wedge F(\omega)), $$

where $\wedge$ is the antisymmetric product of forms and $F(\omega) = d\omega + \omega \wedge \omega$ is the curvature of $\omega$. The equations of motion of this theory are

$$ d\omega e = 0, \quad F(\omega) = 0, $$

where $d_\omega = d + [\omega, \cdot]$ denotes the covariant derivative. If $M$ possess some boundaries $\partial M$, the variation of the action is not zero on on-shell configurations but contains a boundary contribution

$$ \delta S = \frac{1}{16\pi G} \int_{\partial M} \text{tr}(e \wedge \delta \omega). $$

This boundary term vanishes if one fixes the value of the connection on $\partial M$. The gauge symmetries of the continuum action (1) are the Lorentz gauge symmetry

$$ \omega \rightarrow g^{-1} dg + g^{-1} \omega g, \quad e \rightarrow g^{-1} e g, $$

locally parameterized by a group element $g$, and the translational symmetry locally parameterized by a Lie algebra element $\phi$

$$ \omega \rightarrow \omega, \quad e \rightarrow e + d_\omega \phi. $$

and which holds due to the Bianchi identity $d_\omega F = 0$. This supposes that $\phi = 0$ on $\partial M$. The infinitesimal diffeomorphism symmetry is equivalent on-shell to these symmetry when we restrict to non-degenerate configurations $\text{det}(e) \neq 0$. The action of an infinitesimal diffeomorphism generated by a vector field $\xi^\mu$ can be expressed as the combination of an infinitesimal Lorentz transformation with parameter $\omega_\mu \xi^\mu$ and a translational symmetry with parameter $e_\mu \xi^\mu$. The conjugate phase space variables are the pull-back of $(\omega, e)$ on a two dimensional spacelike surface, their Poisson brackets being

$$ \{\omega^i_\mu, e^j_\nu\} = \delta^j i \epsilon_{\mu\nu}. $$

The generator of the translational gauge symmetry is given by the pull-back of the curvature on the two dimensional slice, whereas the pull-back of the torsion generates the Lorentz gauge symmetry.
B. Classical particles

We will be interested in the coupling of gravity to particles. For the reader interested in a more detailed and complete treatment of this problem at the classical level we recommend the reference [24] which contains a comprehensive analysis of this problem. It is well know [2] that the metric associated with a spinning particle of mass \( m \) and spin \( s \) coupled to 3 dimensional Euclidean gravity is a spinning cone

\[
ds^2 = (dt + 4Gs d\varphi)^2 + dr^2 + (1 - 4Gm)^2 r^2 d\varphi^2,
\]

where \( m \) is the mass of the particle and \( s \) is its spin\(^1\). This is a locally flat space, \( t \) is the Euclidean time coordinate, \( r \) the radial coordinate measuring at fixed time, the geodesic distance from the location of the particle along constant \( \varphi \) geodesics and \( \varphi \) is an angular coordinate with the identification \( \varphi \rightarrow \varphi + 2\pi \). When \( 4Gm < 1 \) this space can be identified with a portion of Minkowski space. Lets consider the wedge \( 0 < \varphi < 2\pi(1 - 4Gm) \), the spinning cone is obtained after an identification of the two faces of the wedge by a translation along the \( t \) axis of length \( 8\pi Gs \). Around \( r = 0 \), which is the location of the particle, there is a deficit angle of \( 8\pi Gm \) and a time offset of length \( 8\pi Gs \). The mass of the particle is necessarily bounded by \( 1/4G \). A frame field for this geometry can be given by

\[
e = J_0 dt + (\cos \varphi J_1 + \sin \varphi J_2) dr + ((1 - 4Gm)r(\cos \varphi J_2 - \sin \varphi J_1) + 4GsJ_0) d\varphi,
\]

and the spin connection by

\[
\omega = 2GmJ_0 d\varphi.
\]

The torsion and the curvature have a distributional contribution at the location of the particle\(^2\)

\[
d_\omega e = 8\pi GsJ_0 \delta^2(x) dx^2,
\]

\[
F(\omega) = 4\pi GmJ_0 \delta^2(x) dx^2,
\]

where the delta function is along the plane \( t = cste \). Since the torsion is the generator of Lorentz gauge symmetry we see that having a spin means that this symmetry is broken at the location of the particle, also the mass is breaking the translational symmetry at the location of the particle. We can explicitly see that this is the case if we perform a Lorentz transformation labelled by \( g^{-1} \) and then a translational transformation labelled by \( -\phi \), the equations (12) then become

\[
d_\omega e = 4\pi GJ_0 \delta^2(x) dx^2,
\]

\[
F(\omega) = 4\pi Gp \delta^2(x) dx^2,
\]

where \( j, p \) are the following Lie algebra elements

\[
p = mgJ_0 g^{-1},
\]

\[
j = 2sgJ_0 g^{-1} - m[gJ_0 g^{-1}, \phi].
\]

\(^1\) in \( \hbar \) unit so it has dimension of an angular momenta.

\(^2\) we use the distributional identity \( dd\varphi = 2\pi \delta^2(\vec{x}) dxdy \).
\( p \) is the momenta of the particle and \( j \) the total angular momentum, they satisfy the constraints

\[
-\frac{1}{2} \text{tr} p^2 = m^2; \quad -\frac{1}{2} \text{tr}(pj) = 2ms. \tag{18}
\]

From the canonical point of view these constraints are first class \[24\], the mass constraint generates time reparameterization and the spin constraint generates \( U(1) \) gauge transformation \( g \to gh \). Due to the breaking of symmetry at the location of the particle the gauge degrees of freedom \( g, \phi \) become dynamical (modulo the remnant reparameterization plus \( U(1) \) gauge symmetry): \( g \) describes the Lorentz frame of the particle, \( \phi \) describes the position of the particle. Moreover the knowledge of \( p, j \) is enough to reconstruct \( g, \phi \) modulo the remnant gauge symmetries. Indeed \( gH \) is determined by (16). If we denote by \( x_\perp \) the position of the particle perpendicular to the momenta, \( \phi = \frac{(p \cdot \phi)}{m^2} + x_\perp \), then \( x_\perp = \frac{1}{m^2} [j, p] \), also \( j^2 = s^2 + m^2 x_\perp^2 \).

We can easily understand the canonical commutation relations of \( p, j \) from the equations (14,15). Since the LHS of (14) is the generator of Lorentz transformations and the LHS of (15) is the generator of translational symmetry, these constraints are first class and from their canonical algebra we can easily deduce that the Poisson algebra of \( p, j \) is given by

\[
\{j^a, j^b\} = -2\epsilon^{abc} j_c, \quad \{j^a, p^b\} = -2\epsilon^{abc} p_c, \quad \{p^a, p^b\} = 0. \tag{19}
\]

This analysis shows that, instead of treating the gravity degrees of freedom and the particle degrees of freedom as separate entities, we can reverse the logic and consider that the equations (12,13) are defining equations for a spinning particle. This allows us to describe a particle as a singular configuration of the gravitational field giving a realization of matter from geometry. The ‘would-be gauge’ degree of freedom \[33\] are promoted to dynamical degrees of freedom at the location of the particle. This is the point of view we are going to take in this paper. In order to get the equations (12,13) from an action principle we have to add to the gravity action (1) the following terms

\[
\bar{S}_{Pm,s}(e, \omega) = -\frac{1}{2} \int dt \text{tr}[(me_t + 2s\omega_t)J_0], \tag{20}
\]

where the integral is along the worldline of the particle. This action describes a ‘frozen’ particle without degrees of freedom. We have seen in (14,15) that the particle degrees of freedom are encoded in the former gauge degrees of freedom. To incorporate the dynamics of the particle we perform the transformation

\[
\omega \to \tilde{\omega} = g^{-1}\omega g + g^{-1}dg, \quad e \to \tilde{e} = g^{-1}(e + d\omega)g \tag{21}
\]

the action (20) becomes

\[
\bar{S}_{Pm,s}(\tilde{e}, \tilde{\omega}) = -\frac{1}{2} \int dt \text{tr}[\tilde{e}_t p + \omega_t j] + S_{Pm,s}(g, \phi) \tag{22}
\]

where the first term describes the interaction between the particle degree of freedom and gravity. The second term

\[
S_{Pm,s}(g, \phi) = -\frac{1}{2} m \int dt \text{tr}(g^{-1}\dot{\phi}gJ_0) - s \int dt \text{tr}(g^{-1}\dot{g}J_0), \tag{23}
\]

\[6\]
is the action for a relativistic spinning particle in a form first describe by Sousa Gerbert [24]. One sees again that the original gauge degree of freedom are now promoted to dynamical degree of freedom describing the propagation of a particle.

Note that \( S_{P_{m,-s}}(g, \phi) = S_{P_{m,s}}(g \epsilon, -\phi) \) where

\[
\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

so this action describes a particle carrying both spin \( s \) and \(-s\), which is necessary in order to have \( P, T \) invariance.

### C. Quantum amplitudes

So far we have described the classical features of 3d Euclidean gravity in the first order formalism. We want now to present some general features of partition functions, transition amplitudes and particle insertions which are necessary in order to deal with the explicit quantification of the theory that we will present in the next sections. Before going on, one should keep in mind that the theory we are studying is classically equivalent to usual gravity as long as we restrict to non degenerate configurations of the frame field, \( \text{det}(\epsilon) > 0 \). This condition of non degeneracy is not easy to implement at the quantum level and we will pursue the quantification without worrying about this issue. We have however to keep in mind that the resulting quantum theory is expected to depend on this issue as shown by the study of symmetry [34] or by the study of spacetime volume expectation value [35].

Let us also emphasize that we are dealing in this paper with the quantification of Euclidean gravity. Euclidean gravity is a theory which admits an hamiltonian formulation and its quantification is a well posed problem as shown for instance by the Chern-Simons formulation. Since this is often a source of confusion let us stress that the quantification of Euclidean gravity has nothing to do with a Wick rotation of Lorentzian gravity, which has never been proven to make sense. This is why in the following we are dealing with quantum amplitudes and not ill-defined, Wick-rotated, statistical amplitudes.

Let us eventually emphasize that the discussion in this section is a formal discussion which will help us to introduce all the relevant notions and notations that we will be properly defined in the subsequent sections.

### 1. Partition function

At the quantum level, the prime object of interest is the partition function. Given a closed manifold \( M \) we consider

\[
Z_M = \int D\omega D\epsilon \exp \left[ \frac{i}{16\pi G} \int_M \text{tr} (\epsilon \wedge F(\omega)) \right].
\]

(24)

In order to have a proper definition of \( Z_M \), we should not overcount the configurations which are equivalent by gauge symmetries and one should restrict the integral over gauge equivalent classes of fields \((\epsilon, \omega)\) by gauge fixing. We will call ‘kinematical observable’ a general functional of the field \( \mathcal{O}(\epsilon, \omega) \). One of the main object of interest at the quantum
level is its expectation value

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\omega \mathcal{D}e \exp \left[ \frac{i}{16\pi G} \int_M \text{tr} (e \wedge F(\omega)) \right] \mathcal{O}(e, \omega). \quad (25)$$

A ‘physical observable’ is a gauge invariant (under Lorentz symmetries and diffeomorphisms) observable. Of course, the final interest is in physical observables, but the kinematical observables will also be of interest to us. For instance, a gauge fixing procedure or the introduction of a particle is realized by the insertion of a kinematical observable.

2. Transition amplitudes for pure gravity

In the case of a manifold $M$ with boundary we need to specify boundary data in order to have a well defined path integral. The unconstrained phase space of $2 + 1$ gravity is given by conjugate pairs $(\bar{e}, \bar{\omega})$, which are the pull-back of $(e, \omega)$ on $\partial M$. As we have seen in eq. (3) the natural choice is to take the boundary connection $\bar{\omega}$ to be fixed in $\partial M$. This amounts to a choice of polarization where we consider wave functionals to be functionals of the connection $\Psi(\bar{\omega})$. Another natural choice, is to fix $\bar{e}$, hence the geometry, on the boundary. In this case we have to add a boundary term to the action

$$S_b = -\frac{1}{16\pi G} \int_{\partial M} \text{tr} (e \wedge \omega). \quad (26)$$

The path integral, with the set of boundary data, defines for every manifold $M$ with a boundary, a wave functional

$$G_M(\bar{\omega}) = \int_{\omega|\partial M = \bar{\omega}} \mathcal{D}\omega \mathcal{D}e \ e^{iS[e,\omega]}. \quad (27)$$

We can choose to split the boundary into initial and final boundary components $\partial M = \Sigma_i \coprod \Sigma_f$, where $\Sigma_f$ is equipped with the orientation coming from $M$ and $\Sigma_i$ is equipped with the opposite orientation. For instance when $M = \Sigma \times I$, we have $\Sigma_i = \Sigma \times \{0\}$, $\Sigma_f = \Sigma \times \{1\}$. In this case $\bar{\omega} = (\bar{\omega}_f, \bar{\omega}_i)$ and $G_M(\bar{\omega}_f, \bar{\omega}_i)$ is the kernel of the propagator allowing us to compute transition amplitudes between two states

$$<\Phi_f|\Phi_i> = \int \mathcal{D}\omega \mathcal{D}e \ e^{iS[e,\omega]} \Phi_f^*(\bar{\omega}_f) \Phi_i(\bar{\omega}_i) = \int \mathcal{D}\omega \ \Phi_f^*(\bar{\omega}_f) G_M(\bar{\omega}_f, \bar{\omega}_i) \Phi_i(\bar{\omega}_i). \quad (28)$$

This scalar product is interpreted to be the physical scalar product, hence it should be positive. It is not, however, a definite positive product: it is expected to have a kernel characterized by the hamiltonian constraint.

A natural basis for gauge invariant functionals of $\bar{\omega}$ is given by spin network functionals [36, 37]. Such functionals are constructed from a closed trivalent graph $\Gamma$ whose edges $\bar{e}$ are colored by representation $j_{\bar{e}}$ of SU(2). We denote such states by $\Phi(\Gamma, j_{\bar{e}})(\bar{\omega})$. Such states are constructed by first taking the holonomy of the connection along the oriented edges $\bar{e} g_e(\omega) = \bar{e} \exp \left( \int_{\bar{e}} \bar{\omega} \right)$, then taking the matrix elements of this object $D_{j_{\bar{e}}} g_e(\omega))$ in the spin $j_{\bar{e}}$ representation and finally by contracting $\otimes_{\bar{e}} D_{j_{\bar{e}}} g_e(\omega)) \in \otimes_{\bar{e}} V_{j_{\bar{e}}} \otimes \bar{V}_{j_{\bar{e}}}$ using invariant and
normalized tensors $C : V_1 \otimes V_2 \otimes V_3 \rightarrow \mathbb{C}$ at the vertices $\bar{v}$ of $\Gamma^3$. So
\[
\Phi_{(\Gamma, j_e)}(\bar{\omega}) = \langle \otimes_{\bar{v}} C_{\bar{v}} \otimes_{\bar{e}} D_{\bar{e}}(g_{\bar{e}}(\bar{\omega})) \rangle.
\] (29)

The construction needs $\Gamma$ to be an oriented graph, however the results are independent of this orientation in the case of SU(2). Given a graph $\Gamma$, we can consider the vector space generated by all spin networks with support $\Gamma$, i.e. the space of all cylindrical functions of the connection having the form $\Phi = \sum_{j_e} c_{j_e} \Phi_{(\Gamma, j_e)}$ where the sum is a finite sum over admissible coloring. We denote this vector space by $H_{\Gamma}$. This vector space can be promoted to an Hilbert space with the norm
\[
||\Phi||^2 = \int \prod_{\bar{e}} dg_{\bar{e}} |\Phi(g_{\bar{e}})|^2,
\] (30)
d$g$ being the normalized SU(2) Haar measure. This Hilbert space admits a basis independent description
\[
H_{\Gamma} = L^2(G^{\bar{v}}/G^{\bar{e}}),
\] (31)
where $|\bar{e}|$ denotes the number of edges of $\Gamma$, $|\bar{v}|$ the number of vertices and the action of $G^{\bar{v}}$ on $G^{\bar{e}}$ is given by the structure of the graph: $g_{\bar{e}} \rightarrow h_{i(\bar{e})}^{-1}g_{\bar{e}}h_{s(\bar{e})}$, $t(\bar{e})$ and $s(\bar{e})$ being the target and the source of the edge $\bar{e}$.

3. Transition amplitudes with massive spinning particles

In this paper, we also want to describe the case where particles are present. In this case, as we have seen in the previous section, the gauge symmetries are broken at the location of the particle, moreover the previously gauge degrees of freedom are now promoted to dynamical degrees of freedom of the particle. In order to describe the insertion of particles, we have to add the particle action (20) to the gravity action. This breaks the initial gauge symmetry. The propagator is given by
\[
G_M(\bar{\omega}) = \int_{\omega|\partial M=\bar{\omega}} D\omega D_e e^{iS[\omega, e]+\sum_n S_{P_n}(\omega,e)}.
\] (32)

where $n \equiv (m_n, s_n)$ denotes a mass and spin. We can decompose the path integral into an integral over the gravitational fields and an integral over the gauge degrees of freedom at the location of the particles
\[
G_{M,P_n}(\bar{\omega}, \bar{g}_{n,i}, \bar{g}_{n,f}) = \int_{\omega|\partial M=\bar{\omega}} D\omega D_e e^{iS[\omega, e]} \prod_n G_P(\omega_{P_n}, e_{P_n}, \bar{g}_{n,i}, \bar{g}_{n,f}),
\] (33)

\[
G_{P_{m,s}}(\omega_P, e_P, \bar{g}_i, \bar{g}_f) = \int_{g|\partial M=(\bar{g}_i, \bar{g}_f)} Dg D\phi \exp(\int dt - \frac{1}{2} \text{tr}[e_P + \omega_{tj}]) + S_{P_{m,s}}(g, \phi)).
\] (34)

$(\omega_P, e_P)$ denotes the value of $(\omega, e)$ at the location of the particle. In the integral (33) over $(\omega, e)$, the action is gauge invariant, contrary to the one in (32), and the integration

\[\text{the space of trivalent intertwiners is one dimensional in the case of SU(2), since we consider only trivalent graph we do not need to specify an intertwiner label at vertices.}\]
is understood to be over gauge equivalence classes. If we concentrate first on the particle integral \( G_P(\omega, e) \) and suppose for the sake of the argument that \( \omega = e = 0 \) along the particle worldline, then the interaction term disappears and we are left with \( S_{P_m,s}(g, \phi) \). The propagator \( G_P \) is then the propagator of a relativistic particle and as such an operator acting on the Hilbert space obtained by the quantification of (19.18). Such a quantification can be easily described as follows. We take our kinematical Hilbert space to be \( L^2(G) \), such an Hilbert space is spanned by the Wigner functions \( D_{nk}^I(g) \) where the spin \( I \) is an half integer and \( n, k \) are the representation indices. We define the operators \( \hat{p} \) to act by multiplication, and \( \hat{j} \) to act as the right invariant derivative

\[
\hat{p}^a D_{nk}^I(g) = m \text{tr}(g J_0 g^{-1} J^a) D_{nk}^I(g); \quad \hat{j}_a D_{nk}^I(g) = -D_{nk}^I(J_a g).
\] (35)

The mass constraint \( p^2 = m^2 \) is trivially satisfied, the spin constraint implies that \( k = -s \)

\[
\hat{p}^a \hat{j}_a D_{n-s}^I(g) = -m \text{tr}(g J_0 g^{-1} J^a) D_{n-s}^I(J_a g) = m D_{n-s}^I(g J_0) = 2ms D_{n-s}^I(g).
\] (36)

The physical Hilbert space associated with the particle is the usual Poincaré representation,

\[
\mathcal{H}_{m,s} = \bigoplus_{I |I - s| \in \mathbb{N}} V_I = \{ D_{n-s}^I(g) | I - s \in \mathbb{N}, |n| \leq I \}.
\] (37)

Instead of labelling the particle propagator by couples \( \bar{g}_i, \bar{g}_f \) we can label it by a pair of Lorentz indices \( I_i, I_f \). This means that the particle propagator \( G_{P_{m,s}, I_i, I_f}^\omega, e \) is viewed as an operator in \( \text{Hom}(V_{I_i}, V_{I_f}) \). The restriction we made on the value of \( (\omega, e) \) along the particle world-line can be relaxed, it doesn’t change our argument but just the value of \( G_{P_{m,s}, I_i, I_f}^\omega, e \). This shows, in the case of particle insertions, that the propagator depends not only on a given interpolating manifold \( M \) but also on additional data characterizing the evolution of the particles. The labels \( I_i, I_f \) are labels of the total angular momenta, as we have seen, we can also understand them as labelling the position of the particle in the direction transverse to the momenta since \( I_f(I_f + 1) - I_i(I_i + 1) = m^2[(x_i^2)_{I_f} - (x_i^2)_{I_i}] \).

We will consider the general case of interacting particles of arbitrary spin. In this case the data we need are encoded into what we will call a decorated particle graph and will denote by \( \Upsilon_D \). \( \Upsilon \) is graph embedded in \( M \) such that its open ends \( v \) are all lying in \( \partial M \).

\( D = (m_e, S_e, I_{se}, I_{te}, t_e) \) is a decoration of \( \Upsilon \) where each edge \( e \) of \( \Upsilon \) is labelled by a mass and a spin \( (m_e, S_e) \); each starting point of an edge \( e \) is denoted by \( s_e \) and labelled by a SU(2) representation \( V_{se} \), each terminal point of an edge \( e \) is denoted by \( t_e \) and labelled by a SU(2) representation \( V_{te} \); finally each internal vertex \( \bar{v} \) is labelled by an intertwiner \( \tau_{\bar{v}} \) contracting the Lorentz representations \( I \) (see figure 1).

Given a decorated manifold with boundary \( M, \Upsilon_D \) we now want to describe the construction of the kinematical spin network states and their transition amplitudes. We have to consider states that are not gauge invariant at the location of the particle. Such kinematical states are described by spin networks with open ends [38]. We denote by \( \bar{v} \) the internal vertices of \( \Gamma \) and by \( v \) the open ends of \( \Gamma \), they are identified with the open ends of the particle graph \( \Upsilon \). We also denote by \( \bar{e} \) the internal edges of \( \Gamma \) and by \( e \) the edges of \( \Gamma \) ending

\[\text{footnote 4 We could equivalently call these graphs Feynman graphs since they label all possible Feynman graphs of 3d field theories, however this terminology usually suppose that there is a specific field theory behind the construction of this graphs and that we should sum over them. It is an interesting problem to describe such a theory but this is not the goal of this paper.}\]
on an open end \( v \). Given a graph \( \Gamma \) whose internal edges are colored by \( j_\ell \) and external edges colored by \( I_v \), we can construct in the same spirit as before a spin network function \( \Phi(\Gamma, j_\ell, I_v)(\bar{\omega}) \). The difference with the previous case is that now \( \Phi(\Gamma, j_\ell, I_v)(\bar{\omega}) \) is not a scalar but take value in \( \otimes_v V_{I_v} \). The construction is similar to the previous one, let us denote by \( g_\ell(\bar{\omega}) \) the holonomy along internal edges and \( g_v(\bar{\omega}) \) the holonomy along external edges. The spin network functional is given by

\[
\Phi(\Gamma, j_\ell, I_v)(\bar{\omega}) = \langle \otimes_v C_\bar{v} \otimes_\ell D_{j_\ell}(g_\ell(\bar{\omega})) \otimes_v D_{I_v}(g_v(\bar{\omega})) \rangle \in \otimes_v V_{I_v}. \tag{38}
\]

The intertwiners \( C_\bar{v} \) live only at internal vertices where a pairing occur, there is no contraction at external vertices. As before, we can consider the linear combination of all spin networks supported on a graph and define the basis independent spin network Hilbert space

\[
\mathcal{H}_{\Gamma, I_v} = L^2\left((G^{[|\ell|]+|v|} \rightarrow \otimes_v V_{I_v})/G^{[|v|]}\right), \tag{39}
\]

where \( |\ell| \) denotes the number of internal edges of \( \Gamma \), \( |v| \) the number of open ends of \( \Gamma \), \( |\bar{v}| \) the number of internal vertices and the action of \( G^{[|v|]} \) on \( G^{[|\ell|]} \) is given by the structure of the graph. This space is our kinematical Hilbert space.

It should be clear, from the discussion of the classical particle, that the spin label \( I_v \) on open edges have the meaning of the total angular momenta carried by the particle. The individual vectors in the representation \( V_{I_v} \) label the direction of the angular momenta. \( I_v \) doesn’t have the meaning of the spin of the particle, which is the component of the angular momenta along the direction of the particle momenta, this is important to stress since this is often a source of confusion. The spin and the mass appear dynamically in the choice of the propagator determining the physical scalar product.

Due to Lorentz gauge invariance not all choices of \( I_v \) are admissible and when a choice is admissible not all vectors in \( \otimes_v V_{I_v} \) appear in the image of \( \Phi(\Gamma, j_\ell, I_v) \). The image of \( \Phi(\Gamma, j_\ell, I_v) \) lies in the invariant subspace \( (\otimes_v V_{I_v})^G \) which is the space of intertwiners \( \mathbb{C} \rightarrow \otimes_v V_{I_v} \), where \( \mathbb{C} \) denotes the trivial representation of SU(2).

The full propagator is not a scalar function but an operator depending on the decorated particle graph:

\[
G_{M, \tau_D}(\bar{\omega}_f, \bar{\omega}_i) : (\otimes_v V_{I_v})^G \rightarrow \left(\otimes_v V_{I_{v,f}}\right)^G. \tag{40}
\]
The amplitude is given by

$$\langle \Phi_f | \Phi_i \rangle_M, \gamma_D = \int \mathcal{D} \tilde{\omega} \langle \Phi_f (\tilde{\omega}_f) | G_M, \gamma_D (\tilde{\omega}_f, \tilde{\omega}_i) | \Phi_i (\tilde{\omega}_i) \rangle.$$  \hfill (41)

The measure $\mathcal{D} \tilde{\omega}$ is the Ashtekar-Lewandowski diffeomorphism invariant measure [37]. Due to diffeomorphism invariance the propagator depends only on the diffeomorphism class of the embedding of $\gamma$ in $M$. In the case where $M = \Sigma \times I$, $\Sigma$ being a 2d surface with $n$ punctures, the physical scalar product on $\mathcal{H}_{\Gamma, I_v}$ is given by the following decorated graph:

III. THE PONZANO-REGGE MODEL AND GAUGE FIXING

The Ponzano-Regge model is constructed from the continuum partition function

$$Z_M = \int \mathcal{D} \omega \mathcal{D} e \exp \left[ \frac{i}{16\pi G} \int_M \text{tr} (e \wedge F(\omega)) \right], \hfill (42)$$

by considering a triangulation $\Delta$ (and its dual $\Delta^*$) of $M$, and replacing the set of configurations variables by discrete analogs in the spirit of lattice gauge theory. The connection field is replaced by group elements $g_{e^*}$ associated to the dual edges $e^*$ of $\Delta^*$, and representing the holonomy of the connection field along these edges. The frame field is replaced by Lie algebra elements $X_e$ associated to the edges $e$ of $\Delta$ and representing the integration of $e$ along these edges. The curvature 2-form is now represented as group elements $G_e$ living on the edges $e$ (or dual faces $f^*$), and obtained as the ordered product of the group elements $g_{e^*}$ for dual edges $e^* \subset f^*$, upon the choice of a starting dual vertex on the dual face.

- The discretized partition function is thus

$$Z_{PR}(\Delta) = \left( \prod_{e^*} \int_{SU(2)} dg_{e^*} \right) \left( \prod_e \int_{su(2)} dX_e \right) \exp \left[ i \sum_e \text{tr} (X_e G_e) \right]. \hfill (43)$$

One can then integrate over the $X_e$ variables (see appendix (B)),

$$Z_{PR}(\Delta) = \left( \prod_{e^*} \int_{SU(2)} dg_{e^*} \right) \left( \prod_e \delta (G_e) \right) \hfill (44)$$

where $\delta (g)$ is the delta function over the group\(^5\). The Ponzano-Regge partition function is recovered from (44) by expanding the $\delta$ function using the Plancherel decomposition, and then integrating over $g_{e^*}$ using recoupling identities for SU(2) \([8]\). The result is a realization of (24) and can be expressed as a summation over coloring of a product of 6j symbols.

$$Z_{PR}(\Delta) = \sum_{\{j_e\}} \prod_{\{j_e\}} \prod_{\{j_e^*\}} \left\{ j_{e_1} \ j_{e_2} \ j_{e_3} \ j_{e_4} \ j_{e_5} \ j_{e_6} \right\}, \hfill (45)$$

\(^5\) As it is explained in the appendix (B) it is the delta function on SO(3) and not SU(2). It is also explained how we can modify (43) in order to get the delta function on SU(2).
where the summation is over all edges of $\Delta$ and the product of 6j symbols is over all tetrahedra. $d_j = 2j + 1$ denotes the dimension of the spin $j$ representation and $e_{t_e}$ denotes the six edges belonging to the tetrahedra $t$. $j_e$ plays the role of a discrete $X_e$, it is therefore interpreted as a length and the Ponzano-Regge sum is a sum over all geometries supported by a triangulation.

- In the case of manifold with boundaries but no particles, we can define the physical scalar product between spin network states $\Phi_{\Gamma_1,\bar{e}_{i_1}} \in \mathcal{H}_{\Gamma_1}$, $\Phi_{\Gamma_2,\bar{e}_{i_2}} \in \mathcal{H}_{\Gamma_2}$ in a similar way

\[
\langle \Phi_{\Gamma_1,\bar{e}_{i_1}} | \Phi_{\Gamma_2,\bar{e}_{i_2}} \rangle_{PR} \approx \left( \prod_{e \in \Gamma} d_{e^*} \right) \int_{\text{SU}(2)} \delta(G_e) \Phi_{\Gamma_1,\bar{e}_{i_1}}(g_{\bar{e}_{i_1}}) \Phi_{\Gamma_2,\bar{e}_{i_2}}(g_{\bar{e}_{i_2}}),
\]

the integral being over all group elements associated with dual edges including the ones $\bar{e}_f, \bar{e}_i$ lying in the boundary. As before we can recover Ponzano-Regge like expressions giving the physical scalar product (28) as a state sum model

\[
\langle \Phi_{\Gamma_1,\bar{e}_{i_1}} | \Phi_{\Gamma_2,\bar{e}_{i_2}} \rangle_{PR} = \sum_{\{j_e\}} \prod_{e \in \Gamma} d_{j_e} \prod_t \left\{ j_{e_{t_1}}, j_{e_{t_2}}, j_{e_{t_3}} \right\},
\]

where the summation is over all edges of $\Delta$ which do not belong to the boundary, the coloring of the boundary edges $\bar{e}_f, \bar{e}_i$ is fixed to be $j_{e_f}, j_{e_i}$.

- In the general case of manifold with boundaries and with particles we can also define a transition amplitude which gives an explicit realisation of (41). This one has never been written before\(^6\) and the explicit description of it is one of the main result of this paper. We will present its construction in detail in section VI. In the case of interacting spinless particles the result is very simple. The particle graph $\gamma$ is living on the edges of the triangulation $\Delta$, and we choose a decoration $D$ where all spins and angular momenta are 0. In this case the amplitude including boundaries and particles is given by

\[
\langle \Phi_{\Gamma_1,\bar{e}_{i_1}} | \Phi_{\Gamma_2,\bar{e}_{i_2}} \rangle_{M,\gamma} = \sum_{\{j_e\}} \prod_{e \in \gamma} d_{j_e} \left( \prod_{e \in \Gamma} \chi_{j_e}(h_{\theta_e}) \right) \prod_t \left\{ j_{e_{t_1}}, j_{e_{t_2}}, j_{e_{t_3}} \right\},
\]

the summation is over all edges of $\Delta$ which do not belong to the boundary, the coloring of the boundary edges $\bar{e}_f, \bar{e}_i$ is fixed to be $j_{e_f}, j_{e_i}$. The edges in $\gamma$ carry a factor $\chi_{j_e}(h_{\theta_e})$, which is the trace in the representation of spin $j$ of the group element $h_{\theta_e} = \exp(\theta J_0)$ (A1). $\theta$ is half the deficit angle created by the presence of a mass $m$, $\theta = 4\pi G m$.

The general case involving interacting spinning particles is similar but more involved and described in detail in section VII. For all these amplitudes we can recover Ponzano-Regge like expressions as state sum models. However the naive definitions of these models is plagued with divergences. We need to do a careful analysis of the gauge symmetries of the discrete model in order to understand these divergences and to take care of them. This is what we are presenting now.

### A. The gauge symmetries of the Ponzano-Regge model

In this part we present the identification of the gauge symmetries of the Ponzano-Regge model, and their gauge fixing. We discuss also the case of a manifold with boundaries and

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\(^6\) some hints concerning the insertion of spinless particles were already given in [39]
insertion of particles. This work completes the work begun in [29] where we only gauge fixed one of the symmetries. Note that we corrected here the exact form of the action (43) and as a consequence, the formula are slightly different from [29].

1. Discrete symmetries of the Ponzano-Regge model

The gauge symmetries of the continuum action (Lorentz and translation gauge symmetries) have been described in the first section. The discrete analog of the Lorentz symmetry is parameterized by group elements \( k_{e^*} \) living at the dual vertices of the triangulation. It acts as

\[

g_{e^*} \rightarrow k_{e^*}^{-1} g_{e^*} k_{e^*} \quad (49)
\]

\[
G_e \rightarrow k_{st_e}^{-1} G_e k_{st_e} \quad (50)
\]

\[
X_e \rightarrow k_{st_e}^{-1} X_e k_{st_e} \quad (51)
\]

where \( s_{e^*} \) and \( t_{e^*} \) denote the dual vertices source and target of \( e^* \), \( st_e \) denotes the dual vertex which is the starting point for computing the curvature on the dual face \( f^* \sim e \).

To express the discrete analog of the translational symmetry, we observe that the discrete action appearing in (43) can be written as

\[
S = \text{tr} \left( \sum_e X_e P_e \right) \quad (52)
\]

where \( P_e = P_e^a J_a \) is the projection of \( G_e \) on the Lie algebra and is defined by

\[
G_e = u_e \text{Id} + P_e^a J_a, \quad (53)
\]

where \( u_e^2 + P_e^a P_e a = 1 \). In the continuum we have seen that the translational symmetry is due to the existence of Bianchi identity. At the discrete level we also have a Bianchi identity for each vertex of the triangulation: [40]

\[
\prod_{e \supset v} (k^e_v)^{-1} G^{(e,v)}_e k^e_v = \text{Id}, \quad (54)
\]

where the product is over all edges meeting at \( v \), \( k^e_v \) are group elements dependent on \( g_{e^*} \) characterizing the parallel transport from a fixed vertex to \( st_e \) and \( \epsilon(e,v) = \pm 1 \) depending on whether \( e \) is pointing toward \( v \) or in the opposite direction. We can write this identity in terms of the \( P_e \). One finds that there exist Lie algebra elements \( \Omega^v_e \) and scalars \( U^v_e \) which can be expressed in terms of the \( P_e \)’s for edges meeting at \( v \), such that

\[
\sum_{e \supset v} \epsilon(e,v) (k^e_v)^{-1} (U^v_e P_e + [\Omega^v_e, P_e]) k^e_v = 0. \quad (55)
\]

It is easy to check that the discrete analog of the translational symmetry is then parameterized by Lie algebra elements \( \Phi_v \) living at the vertices of the triangulation and acts as

\[
X_e \rightarrow X_e + \epsilon(e,v) \left( U^v_e \left( k^e_v \Phi_v (k^e_v)^{-1} \right) - [\Omega^v_e, \left( k^e_v \Phi_v (k^e_v)^{-1} \right)] \right) \quad (56)
\]

for \( v \subset e \).
2. Gauge fixing of the symmetries

Having identify the discrete gauge symmetries of the discrete action (43), it has been observed [29] that the infinite gauge volume of the translational symmetry was actually responsible for the divergencies of the Ponzano-Regge model. We then proposed a procedure to gauge fix this symmetry. The Lorentz gauge group being compact in euclidian case, we ignored the gauge fixing of the Lorentz symmetry and concentrated on the translational symmetry. Having in mind the possible applications for the Lorentzian case where the Lorentz group is no longer compact, we now need to be able to perform both gauge fixing at the same time in a consistent way. This is the purpose of this part.

Both discrete gauge symmetries act at the vertices of a graph (1-skeleton and dual 1-skeleton of \(\Delta\)), on variables living on the edges of this graph. The usual method to gauge fix such a symmetry is to choose a maximal tree, i.e. a connected subgraph touching every vertex without forming a loop, then to use the gauge symmetries at the vertices to gauge fix all the variables living on the edges of the tree. This method has been applied to the Lorentz symmetry as a gauge fixing for the non-compact spin-networks [41], and to the translational symmetry for the Ponzano-Regge model [29]. Recall that this procedure uses the action of the gauge symmetries at every vertex except one, taken as the root of the tree. This gives rise to a remaining global symmetry that has to be studied in a second time.

- We want to perform both gauge fixing. We thus choose a maximal tree \(T\) in the 1-skeleton of the triangulation \(\Delta\) and a maximal tree \(T^*\) in its dual 1-skeleton. If \(|v|\) and \(|t|\) are the number of vertices in the 1-skeleton and dual 1-skeleton, these trees contain respectively \(|v| - 1\) and \(|t| - 1\) edges. We use the gauge symmetries to fix

\[
\begin{align*}
g_{e^*} &= 1, \quad \forall e^* \in T^*, \\
X_e &= 0, \quad \forall e \in T.
\end{align*}
\]

We now need to identify the corresponding Fadeev-Popov determinant. The jacobian for fixing the translational symmetry has been computed in [29]. Using the fact that \(T^*\) is a tree, one can show that the jacobian associated to the gauge fixing of the Lorentz symmetry is 1. Finally we have for the Fadeev-Popov determinant

\[
\Delta_{FP} = \prod_{e \in T^*} ((U^t_e)^2 + |\Omega^t_e|^2) \left|U^t_e\right|.
\]

Using the argument given in [29] we can prove that this determinant is actually 1 while included in the partition function. This is due to the fact that the product of \(\delta(G_e)\) for \(e \not\in T\) is enough to impose \(P_e = 0\) on every edge as long as \(T\) does not touch the boundary or the particle. The net result is that there is no Fadeev-Popov appearing in the partition function after the double gauge fixing. After this gauge fixing procedure we have

\[
Z[\Delta \setminus \{T, T^*\}] = \left( \prod_{e^* \not\in T^*} \int_{SU(2)} dg_{e^*} \right) \prod_{e \not\in T} \delta(G_e).
\]

- After this gauge fixing procedure, we are left with a triangulation which possesses only one vertex and one dual vertex. However, there are still some invariances associated. The remaining Lorentz invariance acting at the unique dual vertex is a diagonal \(AdG\) invariance.
$g_e \rightarrow k^{-1} g_e k$. It has been shown in [41] that this remaining invariance can be gauge fixed using the measure

$$d\mu(g_1, ..., g_N) = d\mu(g_1, g_2) dg_3 ... dg_N,$$

where $dg$ is the Haar measure on $G$ and $d\mu(g_1, g_2)$ is defined by

$$\int_{G^2} d\mu(g_1, g_2)f(g_1, g_2) = \int_{H \times G/H} dhdx, f(h, s(x))$$

where $H$ is the Cartan subgroup of SU(2) and $s : G/H \rightarrow G$ is a given section. This choice of measure allows to gauge fix the remaining Lorentz invariance.

Let us now consider the case of the remaining translational symmetry. This symmetry is supposed to arise from the Bianchi identity around the last vertex. As this vertex is the last one, every edge starts and ends on it. The corresponding Bianchi identity (54) thus involves both $G_e$ and $G_e^{-1}$. The key point is to understand the order of the elements in (54). This order is related to the topology of the neighborhood of the vertex. Since we started from a triangulation of a manifold, the neighborhood of each original vertex was a three-sphere. Now we remove the edges corresponding to a tree, which by definition has no loops. Hence its tubular neighborhood has the topology of a sphere and we are left with a last vertex whose neighborhood is still a 3-sphere. The Bianchi identity is trivial which means that there is no residual action of the translational symmetry. This just means that our original parameterization of the gauge group was redundant, and that the gauge group was actually $su(2)|v|^{-1}$.

### 3. Gauge fixing: a mathematical argument

There is a beautiful and simple mathematical argument which allows to understand the necessity of gauge fixing which goes as follows. We know that in the theory of integration what we should integrate over a manifold of dimension $n$ is not a function but a density of weight $n$. A delta function on the group SU(2) is a distribution, that is a density of weight 3, i.e., the only thing that make sense is the quantity $dg\delta(g)$ and not $\delta(g)$. Before gauge fixing, the integration manifold is a manifold of dimension $|f|$, the number of faces of $\Delta$ and the integrand is a density of weight $|e|$, the number of edges of $\Delta$. The dimension of the space does not match the density weight of the integrand and the integration does not really make sense. After gauge fixing the density weight of the integrant (60) is $3(|e| - |v| + 1)$ and the dimension of the integration manifold is $3(|f| - |t| + 1)$, where $|v|$ (resp. $|t|$) denotes the number of vertices (resp. tetrahedra) of $\Delta$. The difference between the two is given by

$$-3\chi = 3(|t| - |f| + |e| - |v|),$$

where $\chi$ is the Euler characteristic of the 3d manifold $M$. If $M$ is a closed manifold its Euler characteristic is zero, therefore the integral (60) is a priori well defined. If $M$ is not closed $\chi(M) = 1/2 \chi(\partial M) = (1 - g)$, where $g$ is the genus of the boundary if $\partial M$ is connected. Therefore if the boundary does not contain any sphere or torus $-3\chi(M) = 3(g - 1)$ is strictly

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7 We present here only the measure in the case of a unique Cartan subgroup, relevant for SU(2). The case of many Cartan subgroup relevant for SL(2, $\mathbb{R}$) can be treated in the same way, see [41].
positive. The result of integration in this case is a density of positive weight \(3(g - 1)\) which is exactly half the dimension of the moduli space of flat SU(2) connection on \(\partial M\). This is the correct result since we will see that in the case of manifold with boundaries, the quantity \(\rho\) should be interpreted as the integration kernel for quantum transition amplitudes. One should be careful however that the counting argument we have just given is naive and does not always guarantee the definiteness of the integral \(\rho\).

In terms of the mathematical argument given at the beginning of this part, the result of this integration can be evaluated to be

\[
\frac{\wedge^* g \int_{SU(2)} dg^*}{\wedge e \delta(G_e) \bar{\Phi}_{\Gamma_f,j_f}(g_{\bar{e}_f}) \Phi_{\Gamma_i,j_i}(g_{\bar{e}_i})},
\]

where \(\wedge\) denotes the wedge product of forms. When the manifold is closed this is just a jacobian, when the manifold admits boundaries this is a form of positive degree by the previous argument. This argument does not guarantee the definiteness of this jacobian since there still could be situations in which \((g_e^*)^e \in \Delta^*\) does not depend on some component of \((g_e)^e \in \Delta\). In this case the jacobian will be infinite. We will analyze more precisely the finiteness issues in section VIII D.

### 4. Inclusion of boundaries and particles

We now discuss the generalization of the gauge-fixing procedure to the case of a manifold with boundaries, when we try to compute transition amplitudes between spin-network states. We have seen that at the classical level the Lorentz symmetry also acts on the boundary, while the translational symmetry is such that the parameter of transformation has to vanish on the boundary. This is translated into the fact that the physical scalar product between spin-network states

\[
\langle \Phi_{\Gamma_f,j_f} \Phi_{\Gamma_i,j_i} \rangle_{PR} = \left( \prod_{e^*} \int_{SU(2)} dg^* \right) \prod_e \delta(G_e) \bar{\Phi}_{\Gamma_f,j_f}(g_{\bar{e}_f}) \Phi_{\Gamma_i,j_i}(g_{\bar{e}_i}),
\]

has a Lorentz symmetry acting at all the dual vertices of the triangulation (including the dual vertices lying on the boundary) while the translational symmetry is restricted to the bulk vertices. The gauge fixing of such amplitudes involves then a maximal tree \(T\) of the triangulation, touching at most one vertex on the boundary (recall that there is one vertex in the maximal tree that we actually don’t use for the gauge fixing). The gauge fixing of the Lorentz symmetry involves a dual maximal tree \(T^*\) touching every dual vertex, including those on the boundary.

As discussed in the classical case, it is expected that the introduction of a particle will break a part of the gauge symmetries of gravity, turning them into particle degrees of freedom and modify the discussion of the gauge fixing. Actually, the inclusion of a particle on an edge \(e\) breaks the translational gauge symmetry as does the inclusion of mass classically. As we will discuss explicitly later, the discrete transformation \(\delta(56)\) is not a symmetry of the discrete action anymore. A maximal tree \(T\) for the gauge fixing of the translational symmetry has to be chosen outside the vertices lying on the trajectory of the particle.

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**Footnote:** In a similar way as having a negative naive superficial divergences degree of a Feynman integral does not always guarantee its convergence.
B. BRST invariance

We have computed the gauge fixed partition function in terms of a gauge fixing procedure involving two maximal trees. In a general gauge theory the gauge symmetry is obviously no longer visible after gauge fixing. However there is an in-print of this symmetry in the gauge fixed theory which is BRST invariance assuring us that the gauge fixed partition function is independent of the gauge choice. In our case we have to check if the gauge fixed partition function
\[ Z[\Delta \{T, T^*\}] = \prod_{e^* \not\in T^*} \int_{SU(2)} dg_{e^*} \prod_{e \not\in T} \delta(G_e), \]
depends on the choice of \((T, T^*)\). It is shown in a companion paper [28] that the gauge fixing procedure is independent of the choice of maximal trees i.e.
\[ Z[\Delta \{T_1, T^*_1\}] = Z[\Delta \{T_2, T^*_2\}]. \]
This property is a remnant of the original gauge + translational symmetry. It follows from the invariance properties of the path-integral but is proved more easily using the graphical tools introduced in [28]. It is also proved in this paper that the partition function does not depend on the choice of the triangulation matching the boundary data so it is a topological invariant
\[ Z[\Delta_1 \{T_1, T^*_1\}] = Z[\Delta_2 \{T_2, T^*_2\}]. \]

IV. PONZANO-REGGE OBSERVABLES

In this section we introduce the notion of Ponzano-Regge observables that are discrete analog of ‘kinematical observables’ presented in section II.C.1. We show that the gauge fixing procedure can be written as the evaluation of the expectation value of a gauge fixing operator. We also present the Ponzano-Regge observables realizing the insertion of particles. We describe the link between Ponzano-Regge model and Chern-Simons theory and show a remarkable duality property between gauge fixing and insertion of particles.

First, let us remark that if one uses the Plancherel identity
\[ \delta(g) = \sum_j \frac{d_j}{V_G} \chi_j(g), \]
where \(j\) is a spin, \(\chi_j(g) = \text{tr}_{V_j}(D_j(g))\) is the character of the spin \(j\) representation, \(d_j\) is the dimension of the representation and \(V_G\) is the volume of the group, then the partition function \((65)\) associated with a triangulation \(\Delta\) can be written as
\[ \sum_j \frac{d_j}{V_G} \int \prod_{e^*} dg_{e^*} Z_j(e, g_{e^*}), \]
where
\[ Z_{PR}(\Delta) = Z_j(e, g_{e^*}) = \prod_{e^* \not\in T^*} \chi_{j_e}(G_e), G_e = \prod_{e^* \subset e} g^{(e, e^*)}_{e^*}, \]
where \( \epsilon(e, e^*) = \pm 1 \) is the relative orientation of \( e^* \) with respect to \( e \). We will call PR-observable 9 a function \( \mathcal{O}(j_e, g_e^*) \) and define its expectation value as

\[
\langle \mathcal{O} \rangle_{\Delta} = \sum_{j_e} \frac{d_{j_e}}{V_G} \prod_{e^* \in T^*} \int dg_{e^*} Z_{\Delta}(j_e, g_{e^*}) \mathcal{O}(j_e, g_{e^*}).
\]  

(71)

A. Gauge fixing observables

In particular, an important class of PR-observables are those fixing the lengths along a graph \( \Gamma \) of \( \Delta \)

\[
\mathcal{O}_{\Gamma,j_{\Gamma}^e}(j_e, g_{e^*}) = \prod_{e \in \Gamma} \delta_{j_e, j_{\Gamma}^e},
\]  

(72)

where \( \delta_{j,j'} \) is the Kronecker delta function. And those fixing the holonomy along a dual graph \( \Gamma^* \)

\[
\mathcal{O}_{\Gamma^*, \theta_{\Gamma^*}^e}(j_e, g_{e^*}) = \prod_{e^* \in \Gamma^*} \delta_{\theta_{\Gamma^*}^e}(g_{e^*}),
\]  

(73)

where \( \theta \in [0, 2\pi] \) and \( \delta_{\theta}(g) \) is a delta function fixing \( g \) to be in the same conjugacy class as \( h_\theta = \exp(2\theta J_0) \) 10. More generally, we can define operators fixing edges length along a graph \( \Gamma \) of \( \Delta \) and the conjugacy class of \( g_{e^*} \) along a dual graph \( \Gamma^* \) of \( \Delta^* \).

\[
\mathcal{O}_{(\Gamma,j_{\Gamma}^e)(\Gamma^*, \theta_{\Gamma^*}^e)}(j_e, g_{e^*}) = \prod_{e \in \Gamma} \frac{\delta_{j_e, j_{\Gamma}^e}}{d_{j_{\Gamma}^e}} \prod_{e^* \in \Gamma^*} \delta_{\theta_{\Gamma^*}^e}(g_{e^*}).
\]  

(74)

In general the evaluation of the expectation value of \( \mathcal{O}_{(\Gamma,j_{\Gamma}^e)(\Gamma^*, \theta_{\Gamma^*}^e)} \) are quite involved; however if \( \Gamma = T \) and \( \Gamma^* = T^* \) are trees in \( \Delta \) and \( \Delta^* \) then the evaluation simplifies drastically, we have the following [28]

\[
\langle \mathcal{O}_{(T,j_T^e)(T^*, \theta_{T^*}^{e^*})} \rangle_{\Delta} = \left( \prod_{e \in T} d_{j_T^e} \right) \left( \prod_{e^* \in T^*} V_G \right) \frac{V_G}{V_H} Z[\Delta \{ T, T^* \}],
\]  

(75)

where \( V_G, V_H \) denote the volume of \( G \) and its Cartan subgroup \( H \). This shows that the degrees of freedom living along trees or dual trees decouple from the bulk degrees of freedom. We have seen in the precedent section that indeed, such degrees of freedom are pure gauge and therefore the expectation value of PR-observables depending only on such degrees of freedom is expected to factorize. Equation (75) show that such expectation value does not depend for instance on the topology of the manifold \( M \) triangulated by \( \Delta \) or the decoration on the boundary of \( M \). Such PR-observables are therefore not physical observables, they couple only to the gauge degrees of freedom and not to the physical degrees of freedom. If we consider the observable \( \mathcal{O}_{\Gamma,j_{\Gamma}^e} \) where \( \Gamma \) is not a tree then this PR-observable will couple to the dynamical degree of freedom of the theory and it is a physical observable giving us details about the geometry of our space-time. Due to gauge invariance two PR-observables \( \mathcal{O}_{\Gamma_1,j_{\Gamma_1}^e} \)

9 PR stands for Ponzano-Regge of course, these observables are not in general gauge invariant and in this case they are 'kinematical observables'

10 the normalization and other properties of this function are described in the appendix A

19
\( \mathcal{O}_{\Gamma_2, j_2} \) will carry the same physical information if \((\Gamma_1, j_1)\) and \((\Gamma_2, j_2)\) are diffeomorphic colored graphs or if they can be collapsed using a tree to diffeomorphic colored graphs.

An other interesting evaluation of the expectation value of this operator is when \(\Gamma\) is the one-skeleton of \(\Delta\) (in this case we denote it \(\Delta\), by an abuse of notation) and when \(\Gamma^* = \Delta^*\) is the one skeleton of \(\Delta^*\). In this case we introduce a special notation for this object

\[
\mathcal{Z}_{\Delta}(j_e, \theta_{e^*}) = \langle \mathcal{O}_{(\Delta, j_e)}(\Delta^*, \theta_{e^*}) \rangle_{\Delta}. \tag{76}
\]

This object does not contain anymore any summation over the length of the edges and integration over the conjugacy class of \(g_e\), its evaluation is therefore trivially finite since

\[
\mathcal{Z}_{\Delta}(j_e, \theta_{e^*}) = \int_{G/H} \prod_{e \in \Delta^*} dx_e \prod_{e \in \Delta} \chi_{j_e}(\prod_{e^* \subseteq e} x_{e^*} h_{\theta_e}^{g(e, e^*)} x_{e^*}^{-1}) \tag{77}
\]

where \(dx\) is the induced measure on \(G/H\) and \(h_{\theta}\) is defined in (A1). This formula follows from the Weyl integration formula (see appendix A).

**B. Chern-Simons formulation**

The remarkable property of this object, which is proven in detail in [28], is that \(\mathcal{Z}_{\Delta}(j_e, \theta_{e^*})\) can be interpreted as a quantum group evaluation of a chain mail link \(L_{\Delta}\). The link in question \(L_{\Delta}\) is described in detail in [28] but can be sketchily described as follows: One can consider the handlebodies \(H\) and \(H^*\) respectively built as the thickening of the 1-skeleton and dual 1-skeleton of \(\Delta\). The gluing of \(H\) and \(H^*\) along their boundaries gives a Heegard splitting of the manifold \(M\) [42]. One can now consider the meridian circles of \(H\) and \(H^*\), and by slightly pushing the meridians of \(H\) into \(H^*\), one obtains the link \(L_{\Delta}\). It should be noted that this link is made from two different kinds of components: the ones coming from \(H\) and the others coming from \(H^*\). The quantum group in question here is a kappa deformation of the Euclidean Poincaré group \(ISO(3)\), it is denoted \(D_\kappa(SU(2))\) \(^{11}\) and can constructed as a Drinfeld double of \(SU(2)\) [25]. Its relevance in the context of 2+1 gravity as been recently analyzed in [43]. The link is colored by spins \(j\) for the components coming from \(H\) and angles \(\theta\) for the \(H^*\) components. \(j\) label pure spin representations of the kappa Poincaré group whereas \(\theta\) is interpreted as the mass of a spin zero representation.

It is well known that, at the classical level, three dimensional gravity can be expressed as a Chern-Simons theory where the gauge group is the Poincaré group. The Chern-Simons connection \(A\) can be written in terms of the spin connection \(\omega\) and the frame field \(e\), \(A = \omega_i J^i + e_i P^i\) where \(J^i\) are rotation generators and \(P^i\) translations. Moreover, since the work of Witten [44], it is also well known that quantum group evaluation of colored link gives a computation of expectation value of Wilson loops in Chern-Simons theory. Our result therefore gives an exact relation, at the quantum level, between expectation value in the Ponzano-Regge version of three dimensional gravity and the Chern-Simons formulation. More precisely, given a link \(L_{\Delta}\) whose components are colored by \(j_e, \theta_{e^*}\) we can construct a Wilson loop functional \(W_{L_{\Delta}, j_e, \theta_{e^*}}(A)\) by taking the ordered exponential of the Chern-Simons connection along the components of \(L_{\Delta}\) and tracing the result in the Poincaré representations.

\(^{11}\) \(\kappa = 1/4G\) is the Planck mass in three dimensional gravity
labelled by \( j_e, \theta_e \). If we denote by \( \langle \cdot \rangle_{CS} \) the expectation value in the Poincaré Chern-Simons theory the result proved in [28] reads

\[
\langle O(\Delta, j_e, \Delta^*, \theta_e) \rangle_\Delta = Z_\Delta(j_e, \theta_e) = \langle W_{L, \Delta, \theta_e, \theta_e^*}(A) \rangle_{CS}.
\] (78)

Note that the full partition function can be recovered as a sum

\[
Z_{PR}(\Delta) = \sum_{j_e} \frac{d_{j_e}}{V_G} \int_{H/W} \prod_e d\theta_e^* \Delta^2(\theta_e^*) Z_\Delta(j_e, \theta_e^*).
\] (79)

From the Chern-Simons point of view this gives a surgery presentation of the partition function.

C. Particle observables

Let us define another type of observables

\[
\tilde{O}_{\gamma, \theta_e}(j_e, g_e^*) = \prod_{e \in \Gamma} \delta_{j_e, 0} \delta_{\theta_e}(G_e),
\] (80)

where \( \theta_e \in [0, 2\pi] \). The expectation value of this PR-observable removes (with \( \delta_{j_e, 0} \)) the flatness condition around the edges of \( \Gamma \) imposed by the partition function, and fixes of the holonomies around the edges of \( \Gamma \) to be in the conjugacy class of \( h_{\theta_e} \) (A5). We will justify in greater detail in section VI that the insertion of this operator can be interpreted as the presence of a spinless\(^{12}\) particle of mass \( m_e = \theta_e/4\pi G \) on the edge \( e \). For instance, one can prove that the mass is conserved at bivalent vertices of \( \Gamma \). One can also check that the expectation value of (80) is given by the expression (48).

Spinning particles can also be introduced by the insertion of a PR observable described precisely in section VII, The structure of the observable is as follows. First we consider a decorated particle graph \( \Gamma_D \) (see section II C 3) living along the edges of the triangulation \( \Delta \), we introduce the operator defined for each edge of the particle graph

\[
\tilde{O}_{\theta_e, s_e, t_e, I_{se}, I_{te}}(j_e, g_e^*) = \delta_{j_e, 0} \int_{G/H} du_e \delta(G_e u_e h_{\theta_e} u_e^{-1}) D_t^I(g_e^* u_e) P_{I_{te}, I_{se}}^S(u_e) \] (81)

where \( \theta_e \) is the deficit angle associated with the mass \( m_e = \theta_e/8\pi G \), \( S_e \) is the spin, \( s_e, t_e \) are the source and target of the edge \( e \), \( I_{se}, I_{te} \) are Lorentz indices, \( D^I(g) \) is the representation matrix of the group element \( g \) in the representation of spin \( I \), \( P_{I_{te}, I_{se}}^S(u_e) = |I_{te}, S_e \rangle \langle I_{se}, S_e| \) where \( |I, S \rangle \) is the vector of magnetic moment \( S \) in the representation \( I \) and \( g_{I_{te}}^* \) is the group element associated with a path going from \( s_I(e) \) to \( t_I^* \) (see section II C 3). Associated with each internal vertex of \( \Gamma_D \) we have an intertwiner \( t_{\tilde{\theta}} \) that can be used to contract the open indices of \( \tilde{O}_{\theta_e, s_e, t_e, I_{se}} \) to get

\[
\tilde{O}_{\Gamma_D} = \prod_{e \in \Gamma} \langle \otimes s_{I_{te}} |_{\otimes e} \tilde{O}_{\theta_e, s_e, t_e, I_{se}} \rangle.
\] (82)

\(^{12}\) more precisely this correspond to a particle which do not carry any angular momenta.
The expectation of this PR-observable removes (with $\delta_{j_e,0}$) the flatness condition imposed by the partition function around the edges of $\Gamma$,fixes of the holonomies around the edges of $\Gamma$ to be in the conjugacy class of $h_{\theta_e}$ (A5) and insert a projector forcing the particle to be of spin $S_e$. On top of a mass conservation rule there is also a spin conservation rule. In the case were all $I$’s and $S$’s are zero we recover the operator (80).

D. Wilson lines and time observable

Before presenting a detailed justification of the introduction of these massive spinning observables corresponding to the inclusion of massive spinning interacting particles it is of interest to describe other PR-observables.

The first one we would like to consider is what we call the identity PR-observable. Let $\Gamma$ be a graph living in $\Delta$ we denote

$$I_\Gamma = \prod_{e \in \Gamma} \delta_{j_e,0} \delta(G_e).$$

Such an observable removes (with $\delta_{j_e,0}$) the flatness condition around the edges and insert back this condition (with $\delta(G_e)$). It clearly does not change anything and does not depend on $\Gamma$. It is therefore neither physical nor observable, its only interest lies in the fact that it appears naturally when we consider the particle observables associated with a zero mass and zero spin propagating along $\Gamma$, it is therefore not a surprise that it is trivial.

More interesting PR-observables are the Wilson observables. Given a graph $\Gamma^* \in \Delta^*$ whose edges are colored by spins $j_{e^*}$, we can define as in section II C the spin network functional $\Phi_{(\Gamma^*,j_e^{*})}$ which depends only on group elements $g_e^*$, $e^* \in \Gamma^*$. The Wilson PR-observable is given by

$$W_{\Gamma^*,j_e^{*}}(j_e, g_e^*) = \prod_{e^* \in \Gamma^*} \Phi_{(\Gamma^*,j_e^{*})}(g_e^*)$$

Since this observable depends both on the spin connection and the spin on the graph it is sensitive to the curvature and metric degrees of freedom.

The last observable we want to describe here, is by far the most interesting since it allows to recover the notion of time at the quantum level. One should however say that since we are working in the Euclidean context there is no notion of time orientability and we cannot distinguish $T$ and $-T$, this will be different in the Lorentzian context which will be the subject of another work [32].

First, we can define the time operator only in the case where the manifold $M$ possesses boundaries $\partial M = \Sigma_i \coprod \Sigma_f$. In this case we choose $L_n \in \Delta$ to be a succession of edges of $\Delta$; $L = (e_1, e_2, \cdots e_n)$ such that $s(e_{i+1}) = t(e_i)$ and $s(e_1) = v_i \in \Sigma_i$, $t(e_n) = v_f \in \Sigma_f$. We also suppose that $L_n$ is not knotted, it is a simple line going from $v_i$ to $v_f$ which are two vertices of the boundary triangulation. We consider the PR-operator

$$T_{L_n}(j) = \delta_{j_{e_1},j} \prod_{i=2}^n \delta_{j_{e_i},0}.$$  

One can check that the expectation value of this operator does not depend on the number of edges composing $L_n$ or which edge we choose to be of length $j$. Ideally we would like to define the proper time as the distance between the slices $\Sigma_i$ and $\Sigma_f$ along a minimal curve.
with a fixed initial direction. The insertion of the operator \( T(j) \) in the amplitude amounts to fix the distance between the two slices but ignoring the information about the direction of the curve. It computes a superposition over all initial directions of amplitudes for which the distance between the two slices is \( j \). This will be clear in our hamiltonian analysis.

E. Duality properties

In this section we would like to present a remarkable duality property between the particle PR observables and the gauge fixing PR observables.

**Proposition 1** Consider a graph \( \Gamma \) in \( \Delta \). The observables \( \mathcal{O}_{\Gamma,j_e^\Gamma} \) and \( \tilde{\mathcal{O}}_{\Gamma,\varphi_e} \) are dual by Fourier transform in the following sense

\[
< \mathcal{O}_{\Gamma,j_e^\Gamma} > = \int \prod_{e \in \Gamma} d\varphi_e \sin \left[ (2j_e^\Gamma + 1)\varphi_e \right] \Delta(\varphi_e) < \tilde{\mathcal{O}}_{\Gamma,\varphi_e} >, \tag{86}
\]

\[
\Delta(\varphi_e) < \tilde{\mathcal{O}}_{\Gamma,\varphi_e} > = \frac{2}{\pi} \sum_{\{j_f^\Gamma\} \in \Gamma} \prod_{e \in \Gamma} \sin \left[ (2j_f^\Gamma + 1)\varphi_e \right] < \mathcal{O}_{\Gamma,j_e^\Gamma} >, \tag{87}
\]

where \( \Delta(\theta) = \sin \theta \).

**Proof:**

Let us prove the first formula. The RHS is written

\[
\text{RHS} = \sum_{\{j_e, e^* \notin \Gamma\}} \int \prod_{e^*} dg_{e^*} \left( \int \prod_{e \in \Gamma} d\varphi_e \sin \left[ (2j_e^\Gamma + 1)\varphi_e \right] \frac{\Delta(\varphi_e)}{V_G} \delta_{\varphi_e}(G_e) \right) \prod_{e \in \Gamma} d_{j_e} V_G \chi_{j_e}(G_e), \tag{88}
\]

where \( G_e = \prod_{e^* \supset e} g_{e^*} \). In the following it is always understood that \( e \in \Delta, e^* \in \Delta^* \) unless otherwise specified. We first write

\[
\sin \left[ (2j_e^\Gamma + 1)\varphi_e \right] = \frac{\sin \left[ (2j_e^\Gamma + 1)\varphi_e \right]}{\sin \varphi_e} \times \sin \varphi_e = \chi_{j_e^\Gamma}(h_{\varphi_e}) \times \Delta(\varphi_e), \tag{89}
\]

and then integrate \( \delta_{\varphi_e}(g_e) \) over \( \varphi_e \) using the identity (A8). One obtains

\[
\text{RHS} = \sum_{j_e, e^* \notin \Gamma} \int \prod_{e^*} dg_{e^*} \left( \prod_{e \in \Gamma} \frac{\chi_{j_e^\Gamma}(G_e)}{V_G} \right) \left( \prod_{e \in \Gamma} \frac{d_{j_e}}{V_G} \chi_{j_e}(G_e) \right), \tag{90}
\]

which is

\[
\text{RHS} = \sum_{j_e} \int \prod_{e^*} dg_{e^*} \left( \prod_{e \in \Gamma} \frac{\delta_{j_e^\Gamma}(j_e)}{d_{j_e^\Gamma}} \right) \prod_{e \in \Gamma} d_{j_e} V_G \chi_{j_e}(G_e) \tag{91}
\]

and prove the first duality formula. The other one follows from the first one and the orthogonality relation

\[
\sum_{d_j=1}^{\infty} \sin(d_j \theta) \sin(d_j \phi) = \frac{\pi}{2} (\delta(\theta - \phi) - \delta(\theta + \phi)). \tag{93}
\]
\textbf{Example:} A simple application of this duality formula is given by the case where $M = S^3$, $\Delta$ being its simplest triangulation with two tetrahedra, one tetrahedra being the interior of a 3-ball in $S^3$ the other one the exterior. We take $\Gamma$ to be the 1-skeleton of the triangulation, this is a tetrahedral graph. In that case, the gauge fixing PR-observable is the square of the $6j$-symbol,

\[
< \mathcal{O}_\Gamma \mathcal{J}_e > \Delta = \left\{ \frac{J_{e_1}}{J_{e_2}} \frac{J_{e_3}}{J_{e_4}} \frac{J_{e_5}}{J_{e_6}} \right\}^2,
\]

while the particle PR-observable is related to the Gram determinant

\[
< \mathcal{O}_\Gamma \varphi_e > \Delta = \frac{1}{\sqrt{\det(\cos \varphi_e)}}.
\]

This is the measure for $G^4/(G \times \text{Ad}G)$, the duality between these observables was first presented in [45]. The above proposition is a generalization of the structure which was discovered there. It is also a generalization to the classical case of the duality relation proposed by Barrett [46].

\vspace{1cm}

\section*{V. HAMILTONIAN FORMULATION AND TIME EVOLUTION}

In this section we present a hamiltonian formulation of three dimensional gravity which generalizes the formalism of Waelbroeck [15] and allows the inclusion of spinning particles (more details will be given in [48]). We also briefly discuss its quantization and give an Hamiltonian interpretation to the PR amplitude with time.

\subsection*{A. Classical analysis}

We consider a 2dimensional cauchy surface with punctures and denote $\Delta$ one of its triangulation. The graph dual to $\Delta$ is an oriented trivalent graph with open ends ending at the punctures. We can construct the phase space of 3d gravity by assigning to each oriented edge $e$ of $\Gamma$ a set of variables $(X_e, g_e)$. We call $\mathcal{e}$ the internal edges and $e_v$ the edges starting from an open end $v$. $g_e \in \text{SU}(2)$ denotes the parallel transport of the spin connection along the edge and $X_e = X^a e_a$ is a Lie algebra element characterizing the direction and length of the edge of the triangulation dual to $e$. More precisely, we can obtain $X_e$ by the integration of the frame field along edge of the triangulation dual to $\Gamma$. $X^a_e = \int_{e} dx^\mu g^x_{s_e} e^a_\mu (g_{s_e}^x)^{-1}$ where $g^x_{s_e}$ denotes the parallel transport from the starting vertex $s_e$ of $e$ to $x$. The phase space algebra is given by

\[
\{X^a_e, g_e\} = g_e J^a; \{X^a_e, X^b_e\} = -2\varepsilon^{ab}_c X^c_e,
\]

\begin{equation}
\text{together with the relations}
\end{equation}

\[
g_{-e} = g_e^{-1}; \quad X_{-e} = -g_e X_e g_e^{-1},
\]

\begin{equation}
\text{where } e \text{ denotes any edge and } -e \text{ denotes the reversed orientation.}
\end{equation}

\textsuperscript{13} $J^a = J_a = i\sigma_a$, where $\sigma_a$ are the Pauli matrices, $\sigma_a \sigma_b = \delta_{ab} + i\varepsilon_{abc}\sigma_c$, is a basis of the Lie algebra
Given a face \( f \) of \( \Gamma \) and a vertex \( s_f \) of \( f \) we define the holonomy around \( f \) starting from \( s_f \), 
\[ G(f) = \prod_{e \in f} g_e. \]
We include only internal edges in the definition of \( G(f) \), and if \( f \) contains an open end \( v \) we chose \( s_f \) to be the terminal point of \( e_v \). We denote \( G(f) = U(f)Id + P^a(f)J_a \), note that \( U(f)^2 + P^a(f)P_a(f) = 1 \) since we are in \( SU(2) \). We call \( P(f) \) the momentum of \( f \). Given an internal edge \( \bar{e} \subset f \) we can define the 'position' \( Q_\bar{e}(f) \) and 'orbital momenta' \( L_\bar{e}(f) \) of \( \bar{e} \) with respect to \( f \) as follows. First, lets introduce \( X_\bar{e}(f) = G^\bar{e}_f(f)X_e(G^\bar{e}_f(f))^{-1} \) where \( G^\bar{e}_f(f) \) is the parallel transport from \( s_f \) to \( s_\bar{e} \) along \( f \), then
\[
Q_\bar{e}(f) \equiv U(f)\tilde{X}_\bar{e}(f) + \tilde{X}_\bar{e}(f) \land P(f) + \frac{\tilde{X}_\bar{e}(f) \cdot P(f)}{U(f)}P(f),
\]
\[
L_\bar{e}(f) \equiv -(P(f))^2\tilde{X}_\bar{e}(f) + U(f)\left(\tilde{X}_\bar{e}(f) \land P(f) + \frac{\tilde{X}_\bar{e}(f) \cdot P(f)}{U(f)}P(f)\right),
\]
where \( (X \land P)^a = e^a_{bc}X^bP^c \) and \( X \cdot X = X^aX_a \). We can check that
\[
\begin{align*}
\{Q_\bar{e}(f), P_b(f)\} &= \delta^b_\bar{e}; & \{L_\bar{e}(f), P_b(f)\} &= e^a_{bc}P^c(f), \\
Q_\bar{e}(f) \cdot P(f) &= L_\bar{e}(f).
\end{align*}
\]
Given any internal vertex \( \bar{v} \) of \( \Gamma \) we define \( J(\bar{v}) = \sum_{v : s_v = 0} X_v \).

The constraints of the theory are the flatness constraints and the Gauss law
\[ G(f) = 1; \quad J(\bar{v}) = 0, \]
for all internal vertices \( \bar{v} \) and all faces which do not contain open ends. These constraints are first class. For faces \( f_v \) which contains open ends \( v \) we impose particle constraints
\[ g_{e_v}^{-1}G(f_v)g_{e_v} = h_{\theta_v}; \quad X_{e_v} = 2S_vJ_0. \]
where \( \theta_v = 4\pi m_v \) is half the deficit angle created by the presence of a particle of mass \( m_v \), \( S_v \) is the spin of the particle and \( h_{\theta_v} = \exp(\theta_vJ_0) \). Among these six constraints, four are second class and two are first class, they generate time reparametrisation and \( U(1) \) gauge symmetry. It is easy to check that \( P(f_v) \) and \( J(\bar{v}) \equiv -1/2X_{-e_v} + L_\bar{e}(f) \) commute with these constraints. The Dirac bracket involving these observables is therefore equal to the original bracket and reads
\[
\{J^a(v), J^b(v)\} = \epsilon^{ab}_{\ c}J^c(v), \quad \{J^a(v), P^b(f_v)\} = \epsilon^{ab}_{\ c}P^c(f_v).
\]
In term of these observables the constraints are
\[ P^a(f_v)P_a(f_v) = \sin^2\theta_v; \quad J_a(v)P^a(f_v) = S_v \sin \theta_v. \]

The Gauss law generates gauge transformations
\[ \{J_A, X_e\} = [X_e, \Lambda(s_e)]; \quad \{J_A, g_e\} = g_e\Lambda(s_e) - \Lambda(t_e)g_e, \]
where \( \Lambda(\bar{v}) \) is a collection of Lie algebra elements associated with internal vertices and \( J_A = \sum_\bar{v} J^a(\bar{v})\Lambda_a(\bar{v}) \). The momentum generates translational symmetry. In order to understand its action we choose a rooted maximal tree \( T_R \) of \( \Gamma \) passing through all internal vertices,
with a distinguished vertex $R$ (the root). We denote by $G^R_s(T)$ the parallel transport along the tree from any internal vertex $\bar{v}$ to the root. We choose a collection of Lie algebra element $\phi(f)$ for all faces of $\Gamma$. These elements are all sitting at the root from the gauge transformation points of view. We need to transport them to the faces using $G^R_s(T)$. Finally, we define $P_\phi \equiv \sum_f \Phi^s(f) [G^R_s(T)^{-1}\phi(f)G^R_s(T)]_a$. $P_\phi$ commutes with all group elements $g_e$, its commutation with $X_e$ is given by

$$\{P_\phi, X_e\} = -\sum_{f\supset e} \epsilon(f,e) \left( U(f)\bar{\phi}_{se}(f) + [P_{se}(f), \bar{\phi}_{se}(f)] \right).$$

where $\epsilon(f,e)$ denotes the relative orientations of $f$ and $e$ and

$$P_{se}(f) = (G^{st}_e(f))^{-1}P(f)G^{st}_e(f); \quad \bar{\phi}_{se}(f) = \left( G^{R}_s(T)G^{st}_e(f) \right)^{-1}\phi(f)G^{R}_s(T)G^{st}_e(f),$$

$G(T), G(f)$ being the parallel transport along $T,f$. This is the hamiltonian version of the translational gauge symmetry defined in (56). We can now introduce a parametric time by taking $P_\phi$ to be our hamiltonian. With respect to this time evolution all $J(\vec{v})$'s $X_e$, and $P(f)$'s are conserved. However the $J(v)$'s or $X_e, Q_e(f)$ are not conserved and we could use one of them as a clock in order to define a relational time [16, 49].

**B. Quantization**

The quantization of this system is implemented by taking the spin network representation presented in section II C 3 and more precisely in section VII. $\Phi_{(\Gamma,j;I_e,v)}(g_e;g_{e_\gamma}) \in \otimes_v V_{I_e}$ is a gauge invariant functional of all $g_e$'s valued in the tensor product of representations associated with open ends. The dependance on the open end group elements is explicitly given by

$$\Phi_{(\Gamma,j;I_e,v)}(g_e;g_{e_\gamma}) = \otimes_v D_{I_e}(g_{e_\gamma}^{-1})\Phi_{(\Gamma,j;I_e,v)}(g_e;1),$$

where $D_I(g)$ is the matrix element of $g$ in the representation $V_I$.

In the spin network representation the $g_e$'s are acting by multiplication and the $X_e$'s are acting by left invariant derivative

$$g_{e}\Phi_{(\Gamma)}(g_e) = g_e \Phi_{(\Gamma)}(g_e); \quad X_{e}^a \Phi_{(\Gamma)}(g_e) = i\Phi_{(\Gamma)}(g_e;J^a).$$

The physical states are obtained by ‘projecting’ the spin network states on the kernel of the constraints. This ‘projector’ is rigourously defined as a rigging map [50] mapping spin network states onto distributional spin network states (Linear functionals on the space of spin networks). This rigging map is easily obtained: We have to multiply the spin network states by a factor $\delta(G(f))$ for each face which does not contain open ends. We multiply by a factor $\delta(G(f)v_{g_e}\hbar_{\theta_e}g_{e_\gamma}^{-1})$ for each face with open ends. Eventually we have to insert a spin projector $P_{I_e,I_v}^{S_e}$ for each open end where $P_{I_e,I_v}^{S} = |S,I\rangle\langle I,S|$, $|S,I\rangle$ being the normalized vector of spin $S$ in $V_I$, $|I,S\rangle$ being its conjugate. This is clearly a projector, moreover it satisfies $P_{I_e,I_v}^{S_e}D_{I_v}(J_0) = -2iS_eP_{I_e,I_v}^{S_e}$. By an analysis similar to the one performed in eq. (36), one can check that

$$\hat{P}(f)^a J(v) P_{I_e,I_v}^{S_e} \Phi_{(\Gamma,j;I_e,v)}(g_e;g_{e_\gamma}) = S_e \sin \theta_e P_{I_e,I_v}^{S_e} \Phi_{(\Gamma,j;I_e,v)}(g_e;g_{e_\gamma}),$$

(107)
and leads to the imposition of the spin constraint (105) at the quantum level. The physical scalar product between two spin network states is obtained by integrating the product of the spin network states with the rigging map over all group elements. It is explicitly written in section (VIII).

One of the main goals is to be able to compute the matrix elements of the unitary operator \( \exp P_\phi \). We are not there yet, but the Ponzano-Regge time observable we propose is one step in this direction. More precisely, let us consider \( \Sigma_{g,n} \) the genus \( g \) surface with \( n \) punctures. Using the gauge fixing we can collapse the triangulation of \( \Sigma_{g,n} \) to a triangulation with one vertex.

In this case the dual graph possess only one face and \( P_\phi = \text{tr} \phi G_T \) where \( G_T = \prod_{i=1}^{g} [a_i, b_i] \prod_{p=1}^{n} u_p h_{\theta_p} u_p^{-1} \). Let us define the operator

\[
U_{|\phi|} = \int d\varphi R \exp(iP_{g_{\varphi}^{-1} \phi g_R}).
\]

In the appendix A we prove that

\[
\int \frac{d\varphi}{4\pi} e^{i\text{tr}(\varphi)} = \int \frac{d|\varphi|}{4\pi} |\varphi|^2 U_{|\varphi|} = \sum_{j} d_{j}^{2} \frac{\chi_{j}(g)}{d_{j}}.
\]

Therefore at the level of distributions we have

\[
U_{|\varphi|} = \delta(|\varphi| - d_{j}) \frac{\chi_{j}(G_T)}{d_{j}}.
\]

The inclusion of the time operator had as an effect to replace the expectation value of \( \delta(G_T) \) by \( \chi_{j}(G_T)d_{j} \) (see the definition of the time operator and also section VIII), therefore it is computing the expectation value of the operator \( U_{d_{j}} \).

VI. MASSIVE SPINLESS PARTICLES

In this section we are going to describe the computation of transition amplitudes for massive but non spinning particles.

A. Definition of the amplitude

We are considering a manifold \( M \) with boundaries \( \partial M = \Sigma_{i} \coprod \Sigma_{f} \), \( \Delta \) a triangulation of \( M \) and \( \Delta^* \) the dual triangulation. We denote by \( \Gamma_{i} = \Delta^* \cap \Sigma_{i} \) and \( \Gamma_{f} = \Delta^* \cap \Sigma_{f} \) the closed trivalent oriented graph obtained by the intersection of the 2-skeleton of \( \Delta^* \) with \( \partial M \). We label the edges \( \bar{e}_{i}, \bar{e}_{f} \) of \( \Gamma_{i}, \Gamma_{f} \) by spins \( j_i, j_f \) and consider spin network functionals \( \Phi_{\Gamma_{i,j_i}}(g_{\bar{e}_{i}}), \Phi_{\Gamma_{f,j_f}}(g_{\bar{e}_{f}}) \) defined in section (II C 1).

In our model, we consider that the particles propagate along the edges of the triangulation. In order to describe their propagation and interaction we consider \( \mathcal{T} \) an oriented graph with open ends whose edges are edges of the triangulation \( \Delta \), the open ends of \( \mathcal{T} \) all lie in the boundary and are identified with vertices of the boundary triangulation \( \Delta \cap \Sigma_{i,f} \). The

\footnote{the boundaries are possibly empty}
internal vertices of $\Gamma$ describe the interaction of particles. We label each edge $e$ of $\Gamma$ by an angle $\theta_e$, it is related to the mass of the particle by $\theta_e = 4\pi Gm_e$. This correspond to a decoration $D = (m_e, S_e; I_{se}, I_{te}, \tau_e)$ where $S_e = I_{se} = I_{te} = 0$. According to our gauge fixing procedure we have to choose a maximal tree $T^*$ of $\Delta^*$ and a maximal tree $T$ of $\Delta - \Gamma$.

We define the transition amplitude to be

$$\langle \Gamma_f, j_f | \Psi_{\Gamma, j_i} \rangle_{M, \gamma} = \int \prod_{e^* \notin T^*} dg_{e^*} \prod_{e \notin T \cup \Gamma} \delta(G_e) \prod_{e \in \gamma} \delta_{\theta_e}(G_e) \Psi_{\Gamma_f, j_f}(g_{e^*}) \Psi_{\Gamma, j_i}(g_{e_i}).$$

This amplitude does not depend on the triangulation $\Delta$ and the choices of trees. It also depends only on the diffeomorphism class of $\Gamma$ [28]. The inclusion of particles amounts to insert a particle PR-observable

$$\bar{\delta}_{\gamma, \theta_e}(j_e, g_{e^*}) = \prod_{e \in \gamma} \delta_{\theta_{e^*}}(G_e).$$

Expanding the delta function and integrating over $g_{e^*}$ we can write this amplitude in a Ponzano-Regge like form\(^\text{15}\)

$$\langle \Gamma_f, j_f | \Psi_{\Gamma, j_i} \rangle_{M, \gamma} = \sum_{j_e} \left( \prod_{e \in T} \delta_{j_e, 0} \right) \left( \prod_{e \notin T \cup \Gamma} d_{j_e} \right) \left( \prod_{e \in \gamma} \chi_{j_e}(h_{\theta_e}) \right) \prod_t \left\{ j_{e_1} j_{e_2} j_{e_3} \right\},$$

(116) where the summation is over all spins belonging to internal edges of $\Delta$, the spins of the boundary edges $\bar{e}_i, \bar{e}_f$ given by $j_i, j_f$ are not summed over and $e_t$ denotes the six edges belonging to the tetrahedra $t$.

As we have seen in section II C, we can express the transition amplitude in the polarization were $e$ or $\omega$ is fixed. In order to change polarization we have to add a boundary term $S_b = - \int_{\partial M} tr(e \wedge \omega)$. The same is possible at the discrete level, we can express the transition amplitude as a propagator $G_{M, \gamma}(g_{e_i}, g_{e_f})$ depending on the boundary connection:

$$G_{M, \gamma}(g_{e_i}, g_{e_f}) = \sum_{j_i, j_f} \left( \prod_{e \in T} d_{j_e} d_{j_f} \right) \Psi_{\Gamma_f, j_f}(g_{e_f}) \Psi_{\Gamma, j_i}(g_{e_i}) \langle \Psi_{\Gamma_f, j_f} | \Psi_{\Gamma, j_i} \rangle_{M, \gamma},$$

(117) and

$$\langle \Psi_{\Gamma_f, j_f} | \Psi_{\Gamma, j_i} \rangle_{M, \gamma} = \int dg_{e_i} dg_{e_f} G_{M, \gamma}(g_{e_i}, g_{e_f}) \Psi_{\Gamma_f, j_f}(g_{e_f}) \Psi_{\Gamma, j_i}(g_{e_i}).$$

(118) The term $\Psi_{\Gamma, j_i}(g_{e_i})$ is the discrete analog of $\exp \int_{\partial M} tr(e \wedge \omega)$.

We are now going to give justifications for the amplitude 114.

**B. Link with the discretized action**

At the classical level the particle is inserted by adding to the gravity action a particle term $S_P = m \int dt tr(e(t)J_0)$. The total action reads $S_T = \int d^2x tr(e(x) \wedge (F(\omega) + 4Gm J_0 \delta_P(x)))$, where $\delta_P(x) = \int dt \delta^{(3)}(x - x_P(t))$. At the discrete level, the curvature term is replaced by

\(^{15}\) using recoupling theory and the expansion of $\delta_{\theta_e}$ in terms of characters $\delta_{\theta_e}(G) = \sum_j \chi_j(h_{\theta_e}) \chi_j(G)$ (see appendix A)
the holonomy $G_e$ and the frame field by the Lie algebra element $X_e$, we choose the discrete action to be

$$S_T = \sum_{e \in \sqcup} \text{tr}(X_e G_e) + \sum_{e \in \sqcup} \text{tr}(X_e G_e h_{\theta_e}),$$

(119)

where $\theta_e = 4\pi G m_e$. In the limit where the loops around $e$ become infinitesimal and the mass is small compared to the Planck mass we recover the continuum action. If we integrate over $X_e$ we get for each edge of $\sqcup$ a factor $\delta(G_e h_{\theta_e})$. This is not yet the term we want, we have to remember that when we introduce a particle via the action $S_{P\mu}$ we are breaking the gauge symmetry at the location of the particle. In order to define $G_e$ we have to choose a dual vertex $st(e)$ along the face dual to $e$, the discrete Lorentz gauge group is acting at this vertex $G_e \to u^{-1}_e G_e u_e$. If we integrate over the gauge group action we get

$$\int du_e \delta(G_e u_e h_{\theta_e} u_e^{-1}) = \delta_{\theta_e}(G_e).$$

(120)

So the insertion of a particle forces $G_e$ to be in the same conjugacy class as $h_{\theta_e}$. Note that now the momentum of the particle is no longer labelled by a Lie algebra element as in (16), but by a group element $u_e h_{\theta_e} u_e^{-1}$. This property is characteristic of a theory exhibiting the symmetry of Doubly Special Relativity [43]. The fact that a spinless particle coupled to gravity can be inserted by imposing the holonomy around the particle to be in a given conjugacy class can be easily seen from our hamiltonian analysis. It was first recognized, at the classical level, in [15] and analyzed in great details in [21]. Our amplitudes extend these works to the quantum case.

C. Physical interpretation of the particle observables

The partition function contains for each edge $e$ of the triangulation a term $\delta(G_e)$ which ensures that the curvature around $e$ is zero. The insertion of a PR-observable $\hat{O}_{\tau e}$ removes the zero curvature condition and replaces it by the fact that the curvature around $e$ has to be in the conjugacy class of $h_{\theta_e}$. This can be interpreted as the presence of a particle of mass $m_e = \theta_e / 4\pi G$ moving along $e$, creating a line of conical singularity. For such an interpretation to be valid, we have to check that it satisfies a kind of ‘mass conservation property’.

![FIG. 2: particle vertex](image)

Consider a bivalent vertex of the graph. The two edges of this graph carry group elements $g_1$ and $g_2$ which are imposed to lie in the conjugacy class labelled by, say, $\theta_1$ and $\theta_2$. In
the triangulation this vertex is surrounded by other edges, with group elements $g_e$ which are imposed to be 1 due to the flatness condition. Now the overall still satisfies the Bianchi identity, which in this case reduces on-shell to $g_1g_2^{-1} = 1$ (provided we choose an ingoing orientation for the first edge and an outgoing for the second one). This proves that the expectation value of the particle PR-observable is zero unless $\theta_1 = \theta_2$. If fact, if there is a bivalent vertex, this analysis shows that the amplitude (114) will be proportional to $\delta_{\theta_1}(h_{\theta_2})$ in this case the physical amplitude is the proportionality coefficient. If we naively put equal masses at bivalent vertices in (114) we get an infinite result, this is due to the fact that there is an extra gauge symmetry associated with the propagation of a particle which is time reparametrisation.

Consider now a trivalent vertex. This time the Bianchi identity around the vertex reduces to

$$g_1 = g_2g_3 \text{ or } u_1h_{\theta_1}u_1^{-1} = u_2h_{\theta_2}u_2^{-1}u_3h_{\theta_3}u_3^{-1}. \quad (121)$$

This can only holds if $m_1, m_2$ and $m_3$ satisfy the inequality

$$\cos\left(\frac{\theta_2 + \theta_3}{2}\right) \leq \cos \frac{\theta_1}{2} \leq \cos\left(\frac{\theta_2 - \theta_3}{2}\right). \quad (122)$$

This inequality is the analog of the kinematical inequalities of relativistic particles. It is shown in the companion paper [28] that this condition actually corresponds to the existence of the intertwiner between the associated three representations of $DSU(2)$.

VII. MASSIVE SPINNING PARTICLES

In this section we are going to describe the inclusion of quantum massive spinning particles coupled to quantum gravity.

We are considering a manifold $M$ with boundaries\textsuperscript{16} $\partial M = \Sigma_i \sqcup \Sigma_f$, $\Delta$ a triangulation of $M$ and $\Delta^*$ the dual triangulation. We denote by $\Delta_i^* = \Delta^* \cap \Sigma_i$ and $\Delta_f^* = \Delta^* \cap \Sigma_f$ the closed trivalent oriented graph obtained by the intersection of the 2-skeleton of $\Delta^*$ with $\partial M$. We denote by $v_i$ (resp. $v_f$) the vertices of $\Delta \cap \Sigma_i$ (resp. $\Delta \cap \Sigma_f$). Each $v_i$ is at the center of a face of $\Delta_i^*$, for each $v_i$ we choose a vertex $\bar{v}_i$ lying in the middle of one of the dual edges surrounding $v_i$ and draw an edge called $e_{v_i}$ from $v_i$ to $\bar{v}_i$, see figure (3). We denote

![FIG. 3: particle vertex](image)

\textsuperscript{16} the boundaries are possibly empty
by $\Gamma_i = \Delta_i^\ast \coprod \{e_{v_i}\}_i$ the union of $\Delta_i^\ast$ with these edges, this is an oriented graph with open ends. We do the same construction for $\Sigma_f$ and denote the resulting graph $\Gamma_f$. We color the internal edges $\tilde{e}_i$ of $\Gamma_i$ by $SU(2)$ spins $j_{\tilde{e}_i}$ and the open edges $e_{v_i}$ of $\Gamma_i$ by $SU(2)$ spins $I_{v_i}$. The resulting spin network with open ends is denoted $(\Gamma_i, j_{\tilde{e}_i}, I_{v_i})$ and similarly for $\Gamma_i^\ast$.

Given such data we construct a spin network functional denoted by $\Phi(\Gamma_i^*, j_{\tilde{e}_i}, I_{v_i})$ which belongs to

$$\mathcal{H}_{\Gamma, I_v} = L^2 \left( (G^{|e|} \to \otimes_v V_{I_v}) / G^{|\bar{v}|} \right),$$

where $|e|$ denotes the number of edges of $\Gamma$, $|\bar{v}|$ the number of internal vertices of $\Gamma$, $v$ the open ends of $\Gamma^\ast$, and the action of $G^{|\bar{v}|}$ on $G^{|e|}$ is given by the structure of the graph. The functional is explicitly defined as

$$\Phi(\Gamma_i^*, j_{\tilde{e}_i}, I_{v_i})(g_e; g_{e_v}) = \langle \otimes_v C_v \otimes \bar{e} D^{ie}(g_{\tilde{e}}) \otimes_v D^{I_v}(g_{e_v}) \rangle \in \otimes_v (V_{I_v})^G,$$

where $C_v$ are normalized intertwiners for internal vertices. This is a construction similar to the one presented in section II C, but adapted to a triangulation.

These spin network states arise naturally in the hamiltonian quantization of 3D gravity with spinning particles as kinematical states satisfying the Gauss law. The operators $g_e$ act by multiplication and the operators $X_e$ as left invariant derivatives, see (109).

### A. Particle graph functional

The purpose of this section is to describe the construction of a particle graph functional which contains the data characterizing the propagation and interaction of particles in the Ponzano-Regge model. The data we need are the following: we consider $\mathfrak{D}$ an oriented graph with open ends whose edges are edges of the triangulation $\Delta$, the open ends of $\mathfrak{D}$ all lie in the boundary and are identified with the boundary vertices $v_i, v_f$.

Each edge $e$ of $\mathfrak{D}$ is labelled by a mass and a spin $(m_e, S_e)$; also each starting point of an edge $e$ is denoted by $s_e$ and labelled by a $SU(2)$ representation $V_{I_e}$, each terminal point of an edge $e$ is denoted by $t_e$ and labelled by a $SU(2)$ representation $V_{I_e}$, both $I_e - S_e, I_e$ are positive integers and each internal vertex $\bar{v}$ of $\mathfrak{D}$ is labelled by an intertwiner $\bar{v}$. The collection of data $(m_e, S_e; I_e, I_e, t_e)$ is called a decoration and denoted for short $D$, the decorated particle graph is denoted $\mathfrak{D}_D$.

#### 1. Spin projector

In order to construct the particle graph functional we need to introduce an important object: the spin projector $\Pi_{I,I'}^s : G \rightarrow V_I \otimes V_{I'}$ which plays a key role in the construction of the propagator of spinning particles coupled to gravity

$$\Pi_{I,I'}^s(u) \equiv D^I(u^{-1})P_{I,I'}^s D_{I'}(u),$$

where $D^I(u)$ is the matrix element of $u$ in the representation $V_I$ and $P_{I,I'}^s = |s, I\rangle \langle I', s|$, $|s, I\rangle$ being the normalized vector of spin $s$ in $V_I$, $\langle I, s|$ being its conjugate. It is clear that $I - s, I' - s' \in \mathbb{N}$ in order for this definition to make sense. The spin projector is characterized by the following properties

$$\Pi_{I,I'}^s(u)\Pi_{I',I''}^{s'}(u) = \delta_{s,s'}\Pi_{I,I''}^{s'}(u),$$

$$\Pi_{I,I'}^{s'}(u) = \Pi_{I,I'}^{s'}(u)$$

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We denote by \( \Pi \) the following properties:

- \( \Pi^s_{I,I'}(hu) = \Pi^s_{I,I'}(u) \), for all \( h \in H \), \( (127) \)
- \( \Pi^s_{I,I'}(gu) = D^I(g^{-1}) \Pi^s_{I,I'}(u) D^{I'}(g) \), for all \( g \in G \), \( (128) \)
- \( \Pi^s_{I,I'}(1) = P^s_{I,I'} \). \( (129) \)

The first property justifies the name projector for \( \Pi \), the second property shows that \( \Pi \) is a functional on \( H \setminus G \). The space \( H \setminus G \) is isomorphic to the momenta space of a particle: In the Lorentzian case, the momenta \( p \in \mathbb{R}^3 \) of a 3d particle lie in the upper sheet \( \mathcal{H} \cong SO(2,1)/SO(1,1) \) of the hyperboloid, due to the mass shell condition \( p^2 = m^2 \). Given \( p \) we can choose a section \( u : \mathcal{H} \to G \). If a functional \( f(u(p)) \) is invariant under the right action of \( H \) then \( f \) doesn’t depend on the choice of the section and is therefore a functional of the momenta. In our case the space of Euclidean momenta is the sphere, and \( u \) is interpreted as a particle momenta. The third property shows that \( \Pi \) intertwines the action of boosts on the momenta and on the external Lorentz index. The last property is a normalization condition showing that we are describing a spin \( s \) particle. Note that \( \Pi \) does not depend on the mass of the particle.

One of the main justifications for introducing \( \Pi \) is that it appears naturally in the computation of the Feynman propagator of spin \( s \) particle propagating in flat space. Let \( \phi^s(x) \) be a field of spin \( s \), which is a function valued in \( V_s \) and let us denote by \( \partial'^{(n)} \phi^s(x) \) the traceless \( n \)-th derivative of \( \phi^s \), it is a function valued in \( V_n \otimes V_s \) since \( \partial'^{(n)} \) is specified once we choose a traceless symmetric product of vectors in \( V_1 \). We can project \( V_n \otimes V_s \) onto \( V_{n+s} \) and denote by \( \partial'^{(n)} \phi^s(x) \) the corresponding field. The flat space Feynman propagator for this field is

\[
\langle \partial'^{(1-s)} \phi^s(\partial'^{1-s} \phi^s) \rangle(p) = \frac{\Pi^s_{I,I'}(u(p))}{p^2 - m^2 + i\epsilon}.
\] \( (130) \)

This is proven in [26] in the three dimensional case, following the analogous proof of [47] in the case of dimension 4.

2. Framed particle graphs

In order to construct the particle graph functional we need to give a framing to \( \Gamma \). We associate to \( e \in \Delta \) the set of edges of \( \Delta \) which touch \( \Gamma \) but are not in \( \Gamma \)

\[
T_e = \{e \in \Delta_1 \mid e \cap \Gamma \neq \emptyset \text{ and } e \notin \Gamma\},
\] \( (131) \)

where \( \Delta_1 \) refers to the one-skeleton of \( \Delta \). Each edge of \( e \in \Delta \) can be viewed as a dual face \( f_e^* \) of \( \Delta^* \). We define the tube of \( \Gamma \), denoted \( T^*_\Gamma \) to be the set of dual faces corresponding to edges in \( T_\Gamma \). So \( e \in T_\Gamma \) iff \( f_e^* \in T^*_\Gamma \). The neighborhood of \( \Gamma \) from the point of view of \( \Delta^* \) is given by \( T^*_\Gamma \) together with the set of dual faces \( f_e^* \), \( e \in \Gamma \). \( T^*_\Gamma \) is made up of tubes or cylinders surrounding the edges of \( \Gamma \) and the topology of \( T^*_\Gamma \) is a 2d surface with holes which can be viewed as the boundary of a thickening of \( \Gamma \), the holes being at the location of the particle (see figure 4). We denote by \( \Gamma^f \) and call it a framed particle graph, a graph in \( \Delta^* \) with the following properties: \( \Gamma^f \) is a graph entirely lying in \( T^*_\Gamma \), it is topologically equivalent to \( \Gamma \) and provides \( \Gamma \) with a framing \( f \). The end points of \( \Gamma^f \) are identified with the boundary dual vertices \( \bar{v}_i, \bar{v}_f \) introduced in the previous section. The framing being given by a choice of vectors on \( \Gamma \) pointing towards \( \Gamma^f \). We induce the decoration of \( \Gamma \) on \( \Gamma^f \) and denote the corresponding decorated graph by \( \Gamma^f_D \), see figure (5). For each vertex \( v \) of \( \Gamma \) we choose a
corresponding vertex $v^*$ of $\mathcal{V}$, however not all vertices of $\mathcal{V}$ are of these type there is for each edge of $\mathcal{V}$ a vertex $st(e)$ which comes from the intersection of $\mathcal{V}$ with $f^*_e$, (see figure 5). Given an edge $e$ of $\mathcal{V}_D$ we consider

\[
\Pi_{I_{te},I_{se}}^s (g^*_e, u_e) = D_{I_{te}}((g^*_e)^{-1})\Pi_{I_{te},I_{se}}^s (g^*_e) \in \text{Hom}(V_{I_{te}}, V_{I_{se}}),
\]

where $g^*_e$ is the group element corresponding to the holonomy along the dual edges of $\mathcal{V}$ going from $s^*_e$ to $st(e)$ and $g^*_e$ corresponds to the holonomy from $t^*_e$ to $st(e)$.

We can finally define the particle graph functional to be

\[
\Pi_{\gamma_D} (g^*_e, u_e) = \langle \otimes_{\gamma \in \mathcal{V}_D} \otimes_{\gamma \in \mathcal{V}_S} \Pi_{I_{te},I_{se}}^s (g^*_e, u_e) \rangle \in \text{Hom} \left( \otimes_{\gamma} V_{I_{te}} ; \otimes_{\gamma} V_{I_{se}} \right).
\]

The PR-observable that corresponds to the inclusion of interacting spinning particles is given
by
\[ \Pi_{\gamma_0}(j_e, g_{e^*}) = \prod_{e \in T} \delta_{j_e, 0} \int \left( \prod_{e \in T} du_e \delta(G_e u_e h_{e} u_e^{-1}) \right) \Pi_{\gamma_0}(g_{e^*}, u_e). \] (134)

This operator is clearly a generalization of the operators we have already considered. For instance if we take the decoration such that all \( S_e \) and all \( I_e \) are zero we recover the operator (115) describing the propagation of a spinless particle without external angular momenta. If we take a decoration such that \( \theta_e = 0 \), we chose \( I_{s_e} = I_{t_e} \) and we sum over \( S_e \) we recover the Wilson line operator (84).

In the case of spinning particles, the spin network states are functional of the holonomies \( g_e \) along internal edges and of the holonomies (\( g_{e^*} \)) along the external edges. \( \Phi_{(r_1, j_{e_1}, I_{e_1})}(g_e; g_{e^*}) \) (resp. \( \Phi_{(r_f, j_{e_f}, I_{e_f})}(g_e; g_{e^*}) \)) are valued in \( \otimes_{v_i} V_{I_{e_i}} \) (resp \( \otimes_{v_f} V_{I_{e_f}} \)). We can therefore contract these functionals (we denote them \( \Phi_i, \Phi_f \) for short) with the particle graph functional (133) to get a scalar function
\[ \langle \Phi_f(g_e; g_{e^*}) | \Pi_{\gamma_0}(j_e, g_{e^*}) | \Phi_i(g_e; g_{e_i}) \rangle. \] (135)

The transition amplitude is given by the following integral
\[ \langle \Phi_f | \Phi_i \rangle_{\gamma_0} = \sum_{j_e | e \in T^*} \int \prod_{e^* \in T^* | e^* \neq e_v} dg_{e^*} \langle \Phi_f(g_e; 1) | \Pi_{\gamma_0}(j_e, g_{e^*}) | \Phi_i(g_e; 1) \rangle. \] (136)

Note that we integrate only over the internal boundary variables \( g_e \) and we fix the value of \( g_{e_v} \) to be unity. This amplitude depends on the choice of the framing. If we make a change of framing along the edge \( e \) by adding one more twist one sees that the amplitude is modified by a multiplicative factor \( e^{i \theta_e 2S_e} \).

### B. Spin conservation

It is now easy to check that the expectation value of the PR-observable (134) satisfies a mass and a spin conservation property. The argument is similar to the one presented in section VI C. Let's consider a bivalent vertex \( v = t_{e_1} = s_{e_2} \), this vertex is surrounded by edges whose group elements \( G_e \) are imposed to be unity by the flatness constraint. Therefore, the Bianchi identity reduces to
\[ (g_{e_1}^*)^{-1} G_{e_1} g_{e_1}^* = (g_{e_2}^*)^{-1} G_{e_2} g_{e_2}^*, \] (137)

where \( g_{e_1}^*; g_{e_2}^* \) corresponds to the holonomy from \( v \) to \( st(e_1), st(e_2) \). \( G_{e_1} \) is imposed to be equal to \( u_{e_1}^{-1} h_{-\theta_{e_1}} u_{e_1} \) and similarly for \( G_{e_2} \). This implies that \( h_{-\theta_{e_1}} \) is conjugated to \( h_{-\theta_{e_2}} \), so \( \theta_{e_1} = \pm \theta_{e_2} \). This is the mass conservation. If the sign is +, this means that \( u_{e_1} g_{t_{e_1}} = h u_{e_2} g_{s_{e_2}} \) where \( h \) is an arbitrary Cartan group element. The PR observable (134) contains the product
\[ D_{I_{e_2}}((g_{e_2}^*)^{-1}) \Pi_{I, I'}^{S_1}(u_{e_2}) D_{I_{e_2}}(g_{s_{e_2}}^*) D_{I_{e_1}}((g_{e_1}^*)^{-1}) \Pi_{I, I'}^{S_2}(u_{e_1}) D_{I_{e_1}}(g_{s_{e_1}}^*), \] (138)

which is equal to
\[ \delta_{S_1, S_2} D_{I_{e_2}}((hu_{e_2} g_{t_{e_2}}^*)^{-1}) \Pi_{I, I'}^{S_1}(u_{e_1}) D_{I_{e_1}}(g_{s_{e_1}}^*), \] (139)

and the spin is conserved.
If $\theta_1 = -\theta_2$ then $u_{e_1, g_{e_1}^*} = \epsilon h u_{e_2, g_{e_2}^*}$ where $h$ is an arbitrary Cartan group element and $\epsilon$ is the Weyl group element

$$\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

When acting on $|I, S\rangle$, $D_I(\epsilon)$ gives $|I, -S\rangle$. We see that in this case the expectation value of (134) contains a factor $\delta_{S_1, -S_2}$ and the absolute value of the spin is conserved. This is the quantum analog of the remark at the end of section II B, showing that we have to include at once both spins $S$ and $-S$. If we have two incoming particles and one outgoing at an interaction vertex the same reasoning implies that $|S_1| + |S_2| = |S_3|$

VIII. COMPUTATION OF TRANSITION AMPLITUDES

In this section we present explicit computations of transition amplitudes (114, 136) in the simple cases where we have non-interacting spinless and spinning particles on a genus $g$ surface. We show that our general definition is free of any infinities in this case and reproduces the physical scalar product one expects from the canonical quantization. We also compute explicitly the effect of the exchange of two particles and show that the result is characterized by the braiding matrix of the quantum group $D_\kappa(SU(2))$. We also discuss the finiteness of our amplitudes for closed manifold.

A. The physical scalar product and time propagation

We are now going to specialize to the case of non-interacting particles. We consider a simple triangulation of $\Sigma_{0,n}$, the sphere minus $n$ points, in terms of $2n$ triangles and $n + 2$ vertices. The triangulation and its dual are drawn in figure 7. The particles are sitting inside the small tubes.

![Figure 7](image-url)

FIG. 7: Triangulation and dual triangulation of $\Sigma_{0,n}$ and dual triangulation of $\Sigma_{0,n} \times I$

We consider a triangulation $\Delta$ of $\Sigma_{0,n} \times I - \nabla_n$, where $\nabla_n$ is the graph consisting of $n$ vertical unbraided segments $\{x_i\} \times I$. The triangulation we take contains $6n$ tetrahedra with all its vertices in the boundary, the dual triangulation is drawn in figure 7. We now chose a gauge fixing condition. Since all triangulation vertices are in the boundary we need to gauge fix only the Lorentz symmetry. In figure 8 we crossed the gauge fixed edges and show in the RHS the result of the gauge fixing. We can simplify the picture by using the flatness condition around dual faces where all edges but one are gauge fixed to eliminate the last edge of this face. After this procedure we get a simpler picture drawn in the RHS of fig. 8 which contains $4n + 2$ non gauge fixed edges and $4n + 2$ faces. We label as in the figure the group elements around a tube $p$ by $k_p^L, k_p^R, \tilde{g}_{p,i}, \tilde{g}_{p,f}$ where the $k$'s label internal edges and we
integrate over them. We also denote by $\bar{g}_{0,i}, \bar{g}_{0,f}$ the edges of the boundary going from the tube $n$ to the tube $1$. The two vertical faces lying on the tube give us two delta functions: $\delta(k_p^L(k_p^R)^{-1})\delta(k_p^L\bar{g}_{p,f}(k_p^R)^{-1}(\bar{g}_{p,i})^{-1})$, the horizontal face closing the tube gives us $\delta_{\theta_p}(\bar{g}_{p,f})$, the faces going from the tube $p$ to $p + 1$, $p = 1, \ldots, n - 1$ give $\delta(k_p^R(k_p^L)^{-1})$, the face going from the tube $n$ to $1$ gives $\delta(k_n^R\bar{g}_{0,f}(k_n^L)^{-1}(\bar{g}_{0,i})^{-1})$, eventually, there are two faces encircling all the tubes one gives $\delta(\bar{g}_{0,i}\bar{g}_{1,i} \cdots \bar{g}_{n,i})$, the other one gives $\delta(\bar{g}_{0,f})$. After integration over all $k$'s variables but one, we are left with the propagator

$$G_n(\bar{g}_{p,f}, \bar{g}_{p,i}) = \int dk \left( \prod_{p=1}^n \delta(k_p^L(k_p^R)^{-1}(\bar{g}_{p,i})^{-1})\delta_{\theta_p}(\bar{g}_{p,f}) \right) \delta(\bar{g}_{1,i} \cdots \bar{g}_{n,i})\delta(\bar{g}_{0,f})\delta(\bar{g}_{0,i}).$$

(140)

After gauge fixing and taking into account the flatness condition for $\bar{g}_0$, the boundary spin network states depend only on $(\bar{g}_p)_{p=1,\ldots,n}$. The physical scalar product between two spin network states is then simply

$$\langle \Phi_f | \Phi_i \rangle = \int \prod_{p=1}^n du_p \bar{\Phi}_f(u_p \theta_p u_p^{-1}) \Phi_i(u_p \theta_p u_p^{-1})\delta(u_1 \theta_1 u_1^{-1} \cdots u_n \theta_n u_n^{-1}).$$

(141)

This shows that the physical Hilbert space of $n$ particles on a sphere is isomorphic to $L^2(\text{SU}(2)/U(1))^{n-1}$ and a physical state $\Phi_P(u_1, \cdots, u_{n-1})$ can be viewed as a distributional functional $\bar{\Phi}$ on the kinematical Hilbert space using a version of the so called ‘rigging map’ [50]

$$\bar{\Phi}_P(\bar{g}_p) = \delta(\bar{g}_1 \cdots \bar{g}_n) \int \left( \prod_{p=1}^n du_p \delta(\bar{g}_p u_p \theta_p (u_p)^{-1}) \right) \Phi_P(u_1, \cdots, u_{n-1}).$$

(142)

We recover in this way the scalar product that we obtained in the Hamiltonian quantization in section V. Also the physical Hilbert for $n$ particles coupled to gravity is isomorphic to the Hilbert space of $n$ particles on the sphere without gravity and can be written as the tensor product of $n - 1$ Poincaré representations $\bigotimes_p \mathcal{H}_{\theta_p,0}$. However, even if the Hilbert spaces are isomorphic the representation of operators acting on it is very different, this is exemplified in the next section where we compute the action of the braiding operators.

If we insert the ‘time’ operator (85) in the computation of the amplitude its effect is to replace the $\delta$ function by a character $\chi_j$ and we get

$$\langle \Phi_f | \Phi_i \rangle_j = \int \left( \prod_{p=1}^n du_p \bar{\Phi}_f(u_p \theta_p u_p^{-1}) \Phi_i(u_p \theta_p u_p^{-1}) \right) \chi_j(u_1 \theta_1 u_1^{-1} \cdots u_n \theta_n u_n^{-1}).$$

(143)
B. Spinning amplitudes

We have seen that the amplitude including spinning particles is given by

$$\langle \Phi_f | \Phi_i \rangle_{\mathcal{D}} = \sum_{j_v \notin \mathcal{G}^T} \int_{e \notin \mathcal{G}^T \setminus e \notin \mathcal{D}_v} \prod_{e} \delta_{g_e} \langle \Phi_f (g_e; 1) \Pi \gamma_{i_v} (j_v, g_e) | \Phi_i (g_e; 1) \rangle. \quad (144)$$

We want to specialize to the case where we have \( n \) non-interacting spinning particles on the sphere using the triangulation described in the previous section. In this case the spin network states are functional of the holonomies \((\tilde{g}_p)_{p=1, \ldots, n}\) around the particle and the holonomies \((u_p)_{p=1, \ldots, n}\) along the external edges. \( \Phi_i (\tilde{g}_p; u_p) \) (resp. \( \Phi_f (\tilde{g}_p; u_p) \)) are valued in \( \otimes_v I_{I_v} \) (resp \( \otimes_v V_{I_v} \)). For non interacting particles the particle graph functional factorises as a product of projectors. It is therefore easy to contract the spin network functionals with the particle graph functional (133). The computation of the amplitude is similar to the one made in the previous section, if we restrict to the case where \( \Phi_i = \Phi_f \) it reads

$$\langle \Phi_f | \Phi_i \rangle_{\mathcal{D}} = \prod_v d u_v | \langle \Phi_f (u_v h_{v_1} u_v^{-1}; u_v) | \otimes_v | I_v, S_v \rangle |^2. \quad (145)$$

This shows that the physical Hilbert space of \( n \) particle on a sphere is isomorphic to \( \otimes_v \mathcal{H}_{S_v} \) where

$$\mathcal{H}_S = \{ F \in L^2(G) | F(h_\theta g) = e^{i2gs} F(g) \forall h_\theta \in H \}. \quad (146)$$

A physical state \( \Phi_P (u_1, \ldots, u_{n-1}) \) in \( \otimes_v \mathcal{H}_{S_v} \) can be viewed as a distributional functional \( \tilde{\Phi} \in \otimes_v V_{I_v} \) on the kinematical Hilbert space

$$\tilde{\Phi}_P (\tilde{g}_v; u_v) = \delta(\tilde{g}_1 \cdots \tilde{g}_n) \left( \prod_{v=1}^n d u_v \delta(\tilde{g}_v u_v h_{v_1} (u_v)^{-1}) \right) \langle \otimes_v | I_v, S_v \rangle \Phi_P (u_1, \ldots, u_{n-1}). \quad (147)$$

C. Braiding of particles and general surface

Using these techniques it is possible to compute the expectation value of a more general particle graph. If we take \( M \) to be the three sphere and consider a closed decorated particle graph \( \mathcal{G}_D \). It is shown in [28] that the PR evaluation \( Z(M, \mathcal{G}_D) \) is equal to a quantum group evaluation using \( D_\kappa (SU(2)) \) a kappa deformation of the Poincaré group. We are now going to show this explicitly by looking at the exchange two particles. The effect of gravity is to give a non trivial statistic to the particles which is governed by a deformation of the Poincaré group. We will explicitly show this now.

We consider a triangulation \( \Delta \) of \( \Sigma_{0,n} \times I - \mathcal{G}_n \), where \( \mathcal{G}_n \) is the graph consisting of \( n \) vertical segments. \( n - 2 \) of them are unbraided and the segment 1 is crossing the segment 2 so that the particles are exchanged. The triangulation we consider is the same as the previous one except around the particles 1 and 2. The corresponding dual complex is drawn in the LHS of figure (9). Following the same path as previously we can gauge fix the amplitude and collapse the complex to the LHS of figure (9) where the gauge fixed edges are crossed. Only four faces of the collapsed dual triangulation contribute non trivially, its evaluation gives

$$\int dk \delta(k \tilde{g}_{1,f} \tilde{g}_{2,f} k^{-1} \tilde{g}_{2,i} \tilde{g}_{1,i}) \delta(k \tilde{g}_{2,f} k^{-1} \tilde{g}_{1,i}) \delta_{\theta_1} (\tilde{g}_{2,f}) \delta_{\theta_2} (\tilde{g}_{1,f}). \quad (148)$$
The effect of the exchange on the scalar product of two spin network is then
\[
\langle \Phi_f | \Phi_i \rangle = \int \left( \prod_p dg_p \delta_{\theta_p}(g_p) \right) \Phi_f(g_1g_2g_1^{-1}, g_1, \ldots g_n)\Phi_i(g_1, g_2, \ldots, g_n)\delta(g_1 \cdots g_n). \tag{149}
\]

The effect of the exchange of particles 1, 2 therefore results in the following operator acting on the physical Hilbert space
\[
\mathcal{R}_{\theta_1, \theta_2} \Phi_P(u_1, u_2, \cdots, u_{n-1}) = \Phi_P(u_1 h_{\theta_1} u_1^{-1} u_2, u_1, \cdots, u_{n-1}). \tag{150}
\]

We recognize the action of the R-matrix of \( D_\kappa(SU(2)) \) acting on the vector space \( \mathcal{H}_{\theta_1,0} \otimes \mathcal{H}_{\theta_2,0} \) [51].

The effect of the exchange of two spinning particles also result in the action of the R matrix of \( D_\kappa(SU(2)) \) on the physical Hilbert space
\[
\mathcal{R}_{(\theta_1,s_1),(\theta_2,s_2)} \Phi_P(u_1, u_2, \cdots, u_{n-1}) = \Phi_P(u_1 h_{\theta_1} u_1^{-1} u_2, u_1, \cdots, u_{n-1}), \tag{151}
\]
where \( \phi_P \in \otimes_i \mathcal{H}_{s_i} \).

In the appendix we present, starting from a triangulation, the computation of the propagator in the case of a genus \( g \) surface. The result is given by the following propagator
\[
G(a_i, b_i; a_i', b_i') = \int dk \delta(g \prod_i [a_i, b_i] ) \prod_i \delta(a_i' k a_i^{-1} k^{-1}) \delta(b_i' k b_i^{-1} k^{-1}), \tag{152}
\]
where \([a, b] = aba^{-1}b^{-1}\) denotes the group commutator. The general transition amplitude for a genus \( g \) with \( n \) punctures can be deduced and we recover the expected transition amplitude.
\[
\langle \Phi | \Phi \rangle_{\Sigma_{g,n}} = \int \prod_{i=1}^{g} da_i db_i \prod_{p=1}^{n} du_p |\Phi(a_i, b_i, u_p h_{\theta_p} u_p^{-1})|^{2} \delta(\prod_{i=1}^{g} [a_i, b_i] \prod_{p=1}^{n} u_p h_{\theta_p} u_p^{-1}). \tag{153}
\]

It is interesting to compute the vacua to vacua scalar product
\[
\langle 0 | 0 \rangle_{\Sigma_{g,n}} = \sum_j \frac{1}{(d_j)^{2g-n}} \prod_{p=1}^{n} \chi_j(h_{\theta_p}) d_j, \tag{154}
\]
where \( d_j = 2j + 1 \) is the dimension of the spin \( j \) representation. This formula gives the volume of the moduli space of flat SU(2) connections [52].
D. Closed manifolds and finiteness of the modified PR model

In the previous example we have seen that the Ponzano-Regge transition amplitudes are explicitly finite after gauge fixing. In the case of closed manifold this is not true in general. Even after gauge fixing the Ponzano-Regge partition function is not necessarily finite for closed manifold. This is clearly seen if one tries to compute for instance the partition function \( Z(\Sigma_{g,n} \times S^1) \) which can be expressed as the trace of the propagator (153). It should be so since such an expression is computing the dimension of the physical Hilbert space which is of course infinite. One natural question to ask is whether we can find manifolds for which the gauge fixed Ponzano-Regge partition function is finite. One can find necessary conditions, for example if a closed manifold \( M \) contains a non contractible torus then the partition function is infinite even after gauge fixing.

Let’s consider, for instance, a mapping \( \text{Tori } \Sigma_f = \Sigma \times I / \sim_f \) where \( f \) is a mapping class group element (a non trivial diffeomorphism of \( \Sigma_{g,0} \)) and the equivalence relation is \( x \times \{0\} = f(x) \times \{1\} \). It is then known that if \( \Sigma_f \) is hyperbolic then the manifold is atoroidal. One would expect the gauge fixed Ponzano-Regge partition function to be finite in that case, but we haven’t checked. Also the evaluation (63) of the partition function shows that the invariant is defined by counting the number of flat \( SU(2) \) connections. This is very similar to the original definition of the Casson-Walker invariant [53] and we therefore expect the gauge fixed PR partition function to be finite for integral homology spheres.

IX. CONCLUSION

In this paper, we have given a complete treatment of the quantization of 3 dimensional Euclidean gravity in the spin foam language including an analysis of local Poincaré invariance, finiteness, topological invariance and the insertion of massive spinning particles. We have given a general prescription allowing us to compute quantum transition amplitudes with interacting particles. We have introduced the notion of particle graphs functionals which generalize the notion of Feynman graphs for theories coupled to three dimensional gravity. We have sketched a new hamiltonian treatment of 3d gravity coupled to spinning particles and showed that our amplitude prescription computes its physical scalar product. We have presented the link between the spin foam quantization and the combinatorial quantization of Chern-Simons.

We feel that our work opens the way to many new developments. First, the treatment we have presented of the gauge fixing of the symmetries should allow us to tackle the more challenging problem of the spin foam quantization of Lorentzian 3d gravity with particles. Then, our treatment of particle insertions in spin foam shows a clear link between spin foams and Feynman graphs which need to be better understood and eventually generalized to higher dimensional gravity. It would also be very interesting to construct explicitly the field theory reproducing the amplitude we have given. Our work gives many hints towards the answer. The spin foam formalism we have developed here to include particles should be naturally extendable to the case where a cosmological constant is present. One also needs to develop a better understanding concerning the insertion of time in the quantum amplitude, and the corresponding semi-classical interpretation. Eventually, we have to see whether the structures we have introduced here to include matter can be exported for the study of 4d quantum gravity amplitudes.
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APPENDIX A: NOTATIONS ON SU(2)

We use the following notations for SU(2). The elements of the Cartan subgroup $H = U(1)$ are represented using

$$h_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}. \quad (A1)$$

The Weyl group consists of the identity and the reflection $h_\phi \rightarrow h_{-\phi}$. Let us choose a (non necessarily normalized) Haar measure on the group, the integration over the group can be written as an integration over the conjugacy classes using the Weyl integration formula

$$\int_G dg f(g) = \int_H \frac{\Delta(\theta)^2}{|W|} \left( \int_{G/H} f(xh_\theta x^{-1})dx \right) d\theta \quad (A2)$$

where $H = U(1)$ denote the cartan subgroup, $\Delta(\theta) = \sin \theta$ and $|W|$ is the order of the Weyl group.

The representations of SU(2) are labelled by a half-integer $j$ and are realized on the spaces $V^j \sim \mathbb{C}^{2j+1}$. The matrix elements of representations are given by the Wigner functions $D^j_{mn}(g)$ satisfying the orthogonality property

$$\int G D^j_{mn}(g) D^{j'}_{m'n'}(g) dg = \delta_{j,j'} \frac{V_G}{d_j} \delta_{m,m'} \delta_{n,n'}, \quad (A3)$$

$V_G$ being the volume of the group. This relation can be written in terms of convolution product for characters

$$\int_G \chi_{j'}(g) \chi_j(gx) dg = V_G \delta_{j,j'} \frac{\chi_j(x)}{d_j}. \quad (A4)$$

We define the distribution $\delta_\phi(g)$ to be the distribution forcing $g$ to be in the conjugacy class of $h_\phi$. It is invariant under conjugation $\delta_\phi(g) = \delta_\phi(xg^x^{-1})$ and normalized by

$$\int_G \delta_\phi(g) f(g) dg = \int_{G/H} f(xh_\phi x^{-1}) dx. \quad (A5)$$

We can write this distribution in terms of characters

$$\delta_\phi(g) = \frac{1}{V_H} \sum_j \chi_j(h_\phi) \chi_j(g), \quad (A6)$$

where $V_H = 2\pi$ is the volume of the Cartan subgroup. The Weyl integration formula imply that

$$\int_{H/W} d\phi \Delta^2(\phi) \delta_\phi(h_\phi) = 1. \quad (A7)$$

This means that we can relate this distribution to $\delta_\phi(\theta)$ the delta function on $H/W$,

$$\delta_\phi(h_\theta) = \frac{\delta_\phi(\theta)}{\Delta^2(\phi)}. \quad (A8)$$
**APPENDIX B: A DELTA FUNCTION IDENTITY**

In this section we prove the following formula

\[
\int d^3x e^{i\text{tr}(Xg)}(1 \pm \epsilon(g)) = 8\pi\delta(\pm g),
\]

(B1)

where \( X = x^i\sigma_i \) is in the Lie algebra, \( g = \exp(i\theta n^i\sigma_i) \) is a SU(2) group element with \( \theta \in [0, \pi] \) and \( n^i n_i = 1 \); \( \sigma^i \) are the Pauli matrices, \( \text{tr}\sigma_i\sigma_j = \delta_{ij} \), \( \epsilon(g) = \text{sign}(\cos \theta) \), and \( \delta(g) \) is the delta function on the group with respect to the normalized Haar measure. First we evaluate

\[
\int d^3x e^{i\text{tr}(Xg)} = (2\pi)^3\delta(3)(\sin \theta \vec{n}).
\]

(B2)

We can use the familiar identity \( \delta^{(3)}(\vec{X}) = \frac{1}{4\pi |X|} \delta(|X|) \) to write this evaluation as

\[
\frac{2\pi^2}{(\sin \theta)^2} \delta(|\sin \theta|) = \frac{2\pi^2}{(\sin \theta)^2} \sum_{n \in \mathbb{Z}} \delta(\theta - \pi n) \frac{1}{|\cos \theta|},
\]

(B3)

\[
= \frac{2\pi^2}{(\sin \theta)^2} \sum_{n \in \mathbb{Z}} (\delta(\theta - \pi 2n) + \delta(\theta - \pi (2n + 1))).
\]

(B4)

The normalized Haar measure on the group is given by

\[
dg = \frac{2}{\pi} d\theta (\sin \theta)^2 d^2 n,
\]

(B5)

where \( d^2 n \) is the normalized measure on \( S^2 \), therefore we can write

\[
\delta(g) = \frac{\pi}{2(\sin \theta)^2} \sum_{n \in \mathbb{Z}} \delta(\theta - \pi 2n),
\]

(B6)

\[
\delta(-g) = \frac{\pi}{2(\sin \theta)^2} \sum_{n \in \mathbb{Z}} \delta(\theta - \pi (2n + 1)).
\]

(B7)

Together with (B3) this proves that

\[
\int d^3x e^{i\text{tr}(Xg)} = 4\pi (\delta(g) + \delta(-g)).
\]

(B8)

A similar computation shows that

\[
\int d^3x e^{i\text{tr}(Xg)}\epsilon(g) = 4\pi (\delta(g) - \delta(-g))
\]

(B9)

which proves eq(B1).

**APPENDIX C: EXPLICIT COMPUTATIONS**

In this appendix, we present explicit computations using the gauge fixing procedure we introduced. In particular, we compute partition functions for manifolds \( \Sigma_g \times S^1 \) where \( \Sigma_g \) is the genus \( g \) surface, and transition amplitudes and propagator for \( \Sigma_g \times I \). We present the explicit computations for the genus 1 case and the results for the general genus, which can be obtained in the same way.
1. Partition function

First, the results are the following: the gauge fixed partition function for a manifold \( \Sigma_g \times S^1 \) is given by

\[
Z[\Sigma_g \times S^1] = \int_{G^{2g+1}} d\mu(a_1, b_1) \prod_{i=2}^g da_i db_i \delta(\prod_i a_i b_i a_i^{-1} b_i^{-1}) \prod_i \delta(a_ik a_i^{-1} k^{-1}) \delta(b_ik b_i^{-1} k^{-1}).
\]

(C1)

We give the explicit proof in the genus 1 case i.e \( \Sigma_1 \times S^1 \) which is the 3 dimensional torus. A triangulation is obtained by considering the triangulation of \( \Sigma_1 \) with two triangles. It induces a decomposition of \( \Sigma_1 \times I \) into two prisms, each prism can be triangulated with three tetrahedra (see figure 10). By identifying the past and future faces, we obtained a triangulation of \( \Sigma_1 \times S^1 \) with 6 tetrahedra, 12 faces, 7 edges and 1 vertex (see figure 11). The 6 tetrahedra (dual vertices) are denoted \( P_1, I_1, F_1, P_2, I_2, F_2 \).

FIG. 10: Triangulation of a prism with 3 tetrahedra. The tetrahedron owning past face is called \( P \), the one owning future face is called \( F \), the intermediate one is called \( I \).

The non-gauge fixed partition function for \( \Sigma_1 \times S^1 \) takes the form (44) with 12 group elements and 7 \( \delta \)-functions. The 7 dual faces corresponding to the 7 edges are

\[
\begin{align*}
F_1 P_1 I_2 P_2 F_2 I_1, & \quad F_2 P_2 I_1 P_1 F_1 I_2, & \quad I_1 P_2 P_1 I_2 F_1 F_2, & \quad P_1 P_2 I_2 F_2 F_1 I_1, \\
P_1 P_2 I_2 F_2 F_1 I_1, & \quad F_2 I_1 P_1 I_2, & \quad P_2 I_1 F_1 I_2, & \quad F_1 F_2 P_2 P_1.
\end{align*}
\]

(C2) (C3)

FIG. 11: Triangulation of \( \Sigma_1 \times S^1 \) and its dual 1-skeleton

The translational symmetry does not need to be gauge fixed since the triangulation possess only one vertex. We only have to perform the gauge fixing of the Lorentz symmetry. There are 6 dual vertices, we choose to gauge fix to the identity the following 5 group elements (see figure 12).

\[
gI_1 F_1, \ gI_1 P_1, \ gI_2 F_2, \ gI_2 P_2, \ gP_1 P_2.
\]

(C4)
After this gauge fixing, the partition function can be written as

\[ Z[\Sigma_1 \times S^1] = \int dF_1 dF_2 dp_1 dp_2 dp_3 dp_4 dp_5 dp_6 dp_7 dp_8 dp_9 dp_{10} dp_{11} dp_{12} \]

\[ \delta(g_{F_1}p_1 g_{F_2}p_2 g_{F_3}p_3 g_{F_4}p_4 g_{F_5}p_5 g_{F_6}p_6 g_{F_7}p_7 g_{F_8}p_8 g_{F_9}p_9 g_{F_{10}}p_{10} g_{F_{11}}p_{11} g_{F_{12}}p_{12}) \delta(g_{F_1}p_1 g_{F_2}p_2 g_{F_3}p_3 g_{F_4}p_4 g_{F_5}p_5 g_{F_6}p_6 g_{F_7}p_7 g_{F_8}p_8 g_{F_9}p_9 g_{F_{10}}p_{10} g_{F_{11}}p_{11} g_{F_{12}}p_{12}). \]  

(C5)

Solving first the last \( \delta \) function gives \( g_{F_1}F_2 = 1 \), then the three ones remaining on last line give

\[ g_{I_1,P_2} = a = g^{-1}_{I_2,F_1}, \quad g_{P_1,P_2} = b = g^{-1}_{F_2,F_1}, \quad g_{P_1,F_1} = k = g^{-1}_{F_2,F_2}. \]  

(C6)

This leads for the gauged fixed partition function to

\[ Z[\Sigma_1 \times S^1] = \int_{G^3} dadbdk \, \delta(aba^{-1}b^{-1}) \delta(aka^{-1}k^{-1}) \delta(kb^{-1}k^{-1}). \]  

(C7)

Finally we use the measure \( d\mu(a,b) \) to take into account the remaining gauge invariance at the last dual vertex.

The previous computation can be systematically generalized to the genus \( g \) case. We can triangulate \( \Sigma_g \) with \( 4g - 2 \) triangles. This induces a triangulation of \( \Sigma_g \times S^1 \) with \( 12g - 6 \) tetrahedra and still one vertex. The Lorentz gauge fixing procedure can be systematically conducted along the same lines and we obtained the general result announced.

2. Propagator and transition amplitudes

The propagator for genus \( g \) case is given by

\[ G(a_i, b_i; a'_i, b'_i) = \int dk \delta \left( \prod_i a_i b_i a_i^{-1} b_i^{-1} \right) \prod_i \delta(a'_i k a_i^{-1} k^{-1}) \delta(b'_i k b_i^{-1} k^{-1}). \]  

(C8)

Again we consider only the genus 1 case, i.e the case of the manifold \( \Sigma_1 \times I \), where the boundaries are a past and a future surface \( \Sigma_1 \). We consider the same way to triangulate this manifold than before, with two triangles on each boundary and 6 tetrahedra (except that there is no more past/future identification). We first consider the case of the propagator between boundary connections. Each boundary carries three fixed group elements representing the fixed boundary data, and we have 14 group elements living on the internal dual edges. There are 4 flatness conditions associated to internal edges of \( \Delta \), and 6 flatness conditions around edges of the boundary (see figure 13)

The dual graph is given by figure 13. We now perform a gauge fixing according to figure 13. The gauge fixing on the boundary spin-network leave only two group elements. Writing
explicitly the gauge fixed partition function and solving the delta functions, the remaining propagator is

$$G(a, b; a', b') = \int dk \delta(a ba^{-1} b^{-1}) \delta(a'ka^{-1} k^{-1}) \delta(b'kb^{-1} k^{-1}).$$  \hspace{1cm} (C9)

Recall that we use the measure $d\mu(a, b)$ (see (62)) to fix the remaining gauge invariance at the last vertex. The gauge fixing procedure we proposed can be generalized in a systematic way for the $\Sigma_g \times I$ case. We obtain for the propagator

$$G(a_i, b_i; a'_i, b'_i) = \int dk \delta\left(\prod_i a_i b_i a_i^{-1} b_i^{-1}\right) \delta\left(\prod_i a'_i k a_i^{-1} k^{-1}\right) \delta\left(\prod_i b'_i k b_i^{-1} k^{-1}\right).$$  \hspace{1cm} (C10)