Spin and Statistics in Classical Mechanics

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Abstract

The spin-statistics connection is obtained for classical point particles. The connection holds within pseudomechanics, a theory of particle motion that extends classical physics to include anticommuting Grassmann variables, and which exhibits classical analogs of both spin and statistics. Classical realizations of Lie groups can be constructed in a canonical formalism generalized to include Grassmann variables. The theory of irreducible canonical realizations of the Poincaré group is developed in this framework, with particular emphasis on the rotation subgroup. The behavior of irreducible realizations under time inversion and charge conjugation is obtained. The requirement that the Lagrangian retain its form under the combined operation $CT$ leads directly to the spin-statistics connection, by an adaptation of Schwinger’s 1951 proof to irreducible canonical realizations of the Poincaré group of spin $j$: Generalized spin coordinates and momenta satisfy fundamental Poisson bracket relations for $2j=$ even, and fundamental Poisson antibracket relations for $2j=$ odd.
I. INTRODUCTORY

"In conclusion we wish to state, that according to our opinion, the connection between spin and statistics is one of the most important applications of the special relativity theory.”
W. Pauli, 1940

The spin-statistics connection was originally established as a theorem in relativistic quantum field theory. It has acquired a formidable reputation, not just from the undeniable difficulty of the papers which originally established the connection in field theory, but also from the vicissitudes of later attempts to simplify and streamline proofs of the theorem. Over time, these latter efforts have increasingly come to concentrate on the quantum-mechanical at the expense of the relativistic. In contrast, this paper presents an example of a classical dynamical system that obeys a spin-statistics relation as a consequence of local Poincaré invariance. This project is very much in line with the sentiment, quoted above, expressed more than sixty years ago by Pauli in his landmark paper. It does go-somewhat-against the grain of much of the subsequent research on spin and statistics.

Developments in the intervening decades have fallen, roughly speaking, under three heads: First, the basic result has seen a deepening and strengthening considered as a theorem in relativistic quantum field theory. A collection of the significant papers marking this evolution has been reprinted in the critical retrospective volume by Duck and Sudarshan (vide. also the paper by Greenberg.) Second, while special relativity in the form of local Poincaré symmetry is a sufficient condition for a local quantum field theory to obey the spin-statistics connection, numerous investigations have sought a weaker set of necessary conditions. Topological considerations in a non-relativistic setting predominate in these studies. In the course of this search, the spin-statistics connection has been extended to settings sometimes far removed from relativistic field theory. Topological theorems have been obtained for strings and solitons on two-dimensional surfaces. A notable aspect of the topological theorems is that, while they make no explicit use of relativity or of quantum field theory, they do require the existence of antiparticles.

Third, since the axiomatic proofs of Burgoyne and Lüders and Zumino, there has been continuing interest in finding simple and elementary proofs of the spin-statistics relation. All demonstrations of the spin-statistics theorem in quantum theory amount to proving-
with greater, or lesser, amounts of travail—what can be stated very simply: The operation of exchanging the position of two identical particles is equivalent to the rotation of one of them by $2\pi$. Feynman, Neuenschwander, Duck and Sudarshan and others have addressed this goal explicitly. It is at least implicit in the long search for topological theorems by Balachandran et al. and, from a different perspective, the proof of Berry and Robbins.

A common thread in both the search for necessary conditions and for simplified proofs is retreat both from field theory and from explicit reliance upon relativistic formalism. Despite this last, it appears that any proof of the spin-statistics connection requires assumptions traceable to local Poincaré symmetry. All proofs depend upon rotational symmetry. The topological proofs, in addition, require antiparticles. But, whatever the ultimate status of relativistic assumptions in the topological theorems, they demonstrate that the necessary conditions for the spin-statistics relation can be weak indeed.

This paper addresses a variation on that observation: It is not necessary for a physical system to be quantum-mechanical in order to obey a spin-statistics relation. For this statement to make sense, one needs classical notions of spin, and of statistics, both of which exist, and appear in the following. That the spin-statistics connection is not intrinsically a quantum mechanical relation should not come entirely as a surprise. The early topological theorem for kinks by Rubinstein and Finkelstein invokes few assumptions of an overtly quantum nature, while Mickelsson explicitly proved a topological theorem valid for classical as well as quantum systems.

Apart from Mickelsson’s paper, little attention seems to have been devoted to classical analogs of the spin-statistics connection. There has been great interest, on the other hand, in classical descriptions of spin, and of spinning electrons. In particular, a classical theory of spinning particles constructed from anticommuting Grassmann variables finds practical use in constructing path integral formulations of supersymmetry. This formulation of classical mechanics of anticommuting dynamical variables has odd features, and has come to be labeled ”pseudoclassical” mechanics in consequence. However, the pseudoclassical theories prove suitable for exhibiting a classical spin-statistics connection.

While there is no attempt in this paper to evade explicit reliance upon relativistic symmetry in the form of local Poincaré invariance, the reasoning, in common with most proofs in recent decades, is not, in fact, all that relativistic in detail. The uses made of Poincaré
invariance amount to two: (1) The properties of the rotational subgroup, specifically the properties of irreducible canonical realizations of spin degrees of freedom, and (2) The combination of the discrete symmetries time-reversal $T$ and charge conjugation $C$.

Our starting point is a review of the properties of Grassmann variables, and the extension of the canonical formalism to classical Grassmann variables. The properties of anticommuting Grassmann variables supply a classical equivalent of fermionic exchange symmetry. Classical Lagrangians constructed from these variables are the simplest models of classical half-integral spin.

Next, canonical realizations of continuous (Lie) symmetry groups are described. The infinitesimal generators of transformations produced by a Lie group form a finite algebra, the Lie algebra of the group. The algebra is expressed in terms of bracket relations. The Lie bracket used in unitary representations of symmetries acting on quantum-mechanical Hilbert spaces is the familiar commutator (or anticommutator). In the canonical formalism, the place of quantum-mechanical irreducible unitary representations is taken by that of irreducible canonical realizations. Commutators are replaced by the equivalent Poisson brackets. Canonical equivalents exist for the entire apparatus of unitary representations in a Hilbert space, including ladder and Casimir operators.

The theory of canonical realizations of the Poincaré group for massive particles is developed, with emphasis on the rotation subgroup and the properties of dynamical variables corresponding to a definite spin. The classical analog of an elementary particle then becomes an irreducible canonical realization of the Poincaré group. The (anticanonical) time inversion operation $T$ is defined and its action on irreducible canonical realizations is exhibited. A classical analog of charge-conjugation $C$ is introduced. The composition of $T$ and $C$, called strong time inversion, is also sometimes called Weyl time inversion.

Finally, following Schwinger, it is shown that invariance of the Lagrangian under strong time inversion implies the spin-statistics connection.

Notation: Except as otherwise indicated in the text, lower case Greek letters can be either even or odd Grassmann variables. When it is desirable to distinguish the even variables, these will sometimes be lower case Latin letters. An asterisk denotes complex conjugation. A spacelike convention is assumed for the Minkowski metric $\eta = \text{diag}(-1,1,1,1)$. The summation convention applies to repeated indices.
II. GRASSMANN VARIABLES IN CLASSICAL MECHANICS

A. Grassmann variables

The classical analogs of quantum-mechanical fermionic and bosonic exchange symmetry are found in the properties of Grassmann variables.\textsuperscript{49,50} The creation and annihilation operators of field theory are familiar examples of Grassmann variables in quantum theory. Even Grassmann variables commute, and correspond to the usual bosonic $c$-number variables of classical mechanics. Even and odd variables, in either order, commute. A set of $n$ odd real Grassmann variables obeys anticommutation relations

$$\xi_\mu \xi_\nu + \xi_\nu \xi_\mu = 0$$

for $\mu, \nu < n$. Thus,

$$\xi_\mu^2 = 0.$$  \hspace{1cm} (1)

Anticommutativity of odd classical Grassmann variables is a classical form of the exclusion principle.

Differentiation on Grassmann variables can act from the right or the left. The sign of the derivative of a product, for example, depends upon which derivative is used. Left differentiation, in accord with the convention in Berezin\textsuperscript{49} is assumed in the following.

Determining the behavior of Grassman variables under the time-reversal transformation used in Sections IV and V requires their properties under complex conjugation. Given two real odd variables $\xi^R$ and $\xi^I$ a complex Grassmann variable is defined by

$$\xi = \xi^R + i\xi^I$$

with modulus squared

$$\xi^* \xi = i(\xi^R \xi^I - \xi^I \xi^R) = 2i\xi^R \xi^I.$$  \hspace{1cm} (4)

One desires this quantity to be real. Equating it to its complex conjugate,

$$(\xi^* \xi)^* = -i((\xi^R \xi^I)^* - (\xi^I \xi^R)^*)$$

$$= 2i(\xi^I \xi^R)^* = 2i\xi^R \xi^I.$$  \hspace{1cm} (5)

Since this relation must hold for an arbitrary complex Grassmann variable, it must be that for any two real Grassmann variables $\eta$ and $\xi$,

$$(\eta \xi)^* = \xi \eta.$$  \hspace{1cm} (6)
It follows\textsuperscript{51} that the complex conjugate of a product of complex Grassmann variables is
\[(\xi_1 \xi_1 \cdots \xi_n)^* = \xi_n^* \cdots \xi_2^* \xi_1^* .\] (7)

B. Extension of Canonical Formalism to Grassmann variables

We consider the canonical formalism for massive particles only.\textsuperscript{52} Let \( q_i, p_i, \ i = 1, m \) be coordinates and momenta of even variables, and \( \xi_\alpha, \pi_\alpha, \ \alpha = 1, n \) be coordinates and momenta of odd variables. Given a Lagrangian
\[ L = L(q_i, p_i, \xi_\alpha, \pi_\alpha) \] (8)
the generalized Hamiltonian is given by
\[ H = q_i p^i + \xi_\alpha \pi^\alpha - L \] (9)
and Hamiltons’ equations become
\[ \dot{p}^i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p^i} \] \hspace{1cm} (10)
\[ \dot{\pi}^\alpha = -\frac{\partial H}{\partial \xi_\alpha}, \quad \dot{\xi}_i = -\frac{\partial H}{\partial \pi^\alpha}. \] (11)

The momenta are defined by
\[ p^i = \frac{\partial L}{\partial \dot{q}_i}, \quad \pi^i = \frac{\partial L}{\partial \dot{\xi}_i}. \] (12)

In a theory containing even and odd Grassmann variables, the definition of the Poisson bracket generalizes. The Poisson bracket of two even variables \( f, g \) is given by\textsuperscript{53,54}
\[ [f, g] = \left\{ \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p^i} \right\} + \left\{ \frac{\partial f}{\partial \xi_\alpha} \frac{\partial g}{\partial \pi^\alpha} - \frac{\partial g}{\partial \xi_\alpha} \frac{\partial f}{\partial \pi^\alpha} \right\} \]
\[ = -[g, f]. \] (13)

The bracket of two odd variables \( \theta, \pi \) is given by
\[ [\theta, \psi] = \left\{ \frac{\partial \theta}{\partial q_i} \frac{\partial \psi}{\partial p^i} + \frac{\partial \psi}{\partial q_i} \frac{\partial \theta}{\partial p^i} \right\} - \left\{ \frac{\partial \theta}{\partial \xi_\alpha} \frac{\partial \psi}{\partial \pi^\alpha} + \frac{\partial \psi}{\partial \xi_\alpha} \frac{\partial \theta}{\partial \pi^\alpha} \right\} \]
\[ = [\psi, \theta]. \] (14)
and is called an antibracket. When it is desired to emphasize the difference between brackets of two even variables and antibrackets, these will be written \([f, g]^-\) and \([\theta, \pi]^+\), respectively.

Between an odd and an even variable,

\[
[\theta, f] = \left\{ \frac{\partial \theta}{\partial q_i} \frac{\partial f}{\partial p^i} - \frac{\partial f}{\partial q_i} \frac{\partial \theta}{\partial p^i} \right\} - \left\{ \frac{\partial \theta}{\partial \xi_\alpha} \frac{\partial f}{\partial \pi^\alpha} + \frac{\partial f}{\partial \xi_\alpha} \frac{\partial \theta}{\partial \pi^\alpha} \right\}
\]

\[= -[f, \theta].\]

With these definitions the brackets form a (Grassmann) ring.

Casalbuoni\textsuperscript{53} shows that the set of Poisson brackets and antibrackets in pseudomechanics comprises a graded Lie algebra. A graded Lie algebra is a Lie algebra containing both symmetric and antisymmetric bracket relations.\textsuperscript{55,56} Thus, in a quantum field theory which contains bosons and fermions and respects the symmetries of a Lie group, the set of field commutators and anticommutators make up a graded Lie algebra. The dynamical variables are "graded" by a degree that labels the symmetry of their brackets, e. g.:

\[
\delta_{q_i} = \delta_{p^i} = 0
\]

\[
\delta_{\xi_\alpha} = \delta_{\pi^\alpha} = 1,
\]

where \(\delta_{\sigma}\) is degree(\(\sigma\)). The degree of the product of dynamical variables is the sum of their respective degrees modulo(2), so that

\[
\delta_{\xi_\alpha} + \delta_{\pi^\alpha} = 0.
\]

Thus, in order for the free Lagrangian for an odd dynamical variable to be an even quantity, it must be of the form

\[
L = i\pi \xi.
\]

The equation of motion of an anticommuting dynamical variable must therefore be first order.\textsuperscript{53}

The generalized Jacobi identity\textsuperscript{53,57}

\[
(-1)^{\delta_{\rho}\delta_{\xi}} [\gamma, [\rho, \pi]] + (-1)^{\delta_{\rho}\delta_{\gamma}} [\rho, [\gamma, \pi]] + (-1)^{\delta_{\gamma}\delta_{\pi}} [\pi, [\rho, \gamma]] = 0
\]

finds use in the derivation of canonical angular momentum ladder operators in Section III B.

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C. Pseudoclassical Lagrangians

Classical Lagrangian theories of anticommuting Grassmann variables have been studied by Berezin and Marinov,\textsuperscript{44} Galvao and Teitelboim,\textsuperscript{45} Casalbuoni,\textsuperscript{53,54} Gomis \textit{et al.},\textsuperscript{58–60} and others. In these investigations, the goal was to devise a supersymmetric model of classical point particles suitable for path integral quantization. These models possess three kinds of symmetry: Poincaré invariance, which preserves the distinction between even and odd Grassman variables; supersymmetry under transformations which do not respect that distinction, instead relating even and odd Grassmann variables; and invariance under arbitrary monotone reparameterizations of the proper time. This last is a form of gauge invariance. Constructing particle solutions with a well-defined world line that satisfies all these symmetry requirements turns out to be quite involved, especially the constraint analysis. Since we are unconcerned with quantization, we limit our attention to Poisson brackets. The construction of Dirac brackets described in Gomis \textit{et al.} and others receives no further discussion, other than to note that Dirac brackets necessarily have the same symmetry under exchange of arguments as do the Poisson brackets from which they are computed.

Of the three symmetries listed, only Poincaré invariance is of concern for the present discussion. The simpler Lagrangian given by Di Vecchia and Ravndal\textsuperscript{61} and Ravndal\textsuperscript{62} will serve as an illustrative example for the subsequent discussion, although the method of proof given in Section \textsuperscript{V} applies to the more involved models just mentioned as well. Like those models, the Di Vecchia and Ravndal Lagrangian is invariant under supersymmetric transformations, but it differs from them in having a simple parameterization of proper time along a particle trajectory. It is

\[
L = \frac{1}{4} [\dot{q}^\mu \dot{q}_\mu - i \xi^\mu \dot{\xi}_\mu]. \tag{20}
\]

Here \(q_\mu\) is an even position variable, and the intrinsic spin tensor for spin 1/2 is constructed from the spatial components of the Grassmann variable \(\xi_\mu\). The higher-spin generalization of eqn (20) is described below. An overdot denotes the total differentiation with respect to a timelike affine parameter along a particle trajectory; in this model, the affine parameter is just proper time.

The canonical momenta for free motion are

\[
p_\mu = \frac{1}{2} \dot{q}_\mu \tag{21}
\]
\[ \pi_\mu = \frac{1}{4} i \xi_\mu. \]  
(22)

The free particle Hamiltonian is constant:

\[ H = p^2 = -m^2. \]  
(23)

In the presence of an electromagnetic field, the canonical momentum conjugate to \( \xi_\mu \) is given by minimal coupling to a vector potential in the usual manner. In order to construct a classical theory for spin 1/2 which, upon quantization, yields the Dirac equation, the pseudomechanical models usually introduce a further odd Grassmann variable \( \xi_5 \), generalized suitably as needed for higher spin.\(^{44}\) The Di Vecchia and Ravndal Lagrangian does not include this extra dynamical variable.

While it is necessary for Poincaré invariance that the \( \xi_\mu \) be elements of a four-vector, the models must also impose \( \xi_0 = 0 \) in some manner if the components of \( \xi_\mu \) are to form an irreducible realization of the angular momentum algebra. Berezin and Marinov,\(^{44}\) for example, posit an additional symmetry relation that results in the exclusion of \( \xi_0 \) from the equations of motion.

III. IRREDUCIBLE CANONICAL REALIZATIONS OF THE POINCARÉ GROUP

In quantum field theory, the notion of a particle is frequently identified with an irreducible unitary representation of the Poincaré group.\(^{63,64}\) The classical model of a massive particle used in this paper is the counterpart in the canonical formalism of such an irreducible unitary representation.\(^{48}\) One speaks, instead, of irreducible canonical realizations, and replaces commutation relations amongst the matrix generators of infinitesimal Lorentz transformations and translations with Poisson brackets relating infinitesimal generators of canonical transformations. Similarly, a function on phase space which is a function solely of the generators of the Lie algebra and which is an invariant in all realizations of the Lie group is called a Casimir invariant. Casimir invariants serve as the canonical equivalents of quantum-mechanical Casimir operators.
A. The Poincaré Group

The Lie algebra of the Poincaré group has a ten-parameter set of infinitesimal generators $\mathcal{M}$ and $\mathcal{P}$ satisfying commutation relations:

\[
\mathcal{M}^{\mu\nu} \mathcal{M}^{\alpha\beta} \mathcal{M}^{\nu\mu} - \mathcal{M}^{\alpha\beta} \mathcal{M}^{\mu\nu} = \eta^{\nu\beta} \mathcal{M}^{\mu\alpha} - \eta^{\nu\alpha} \mathcal{M}^{\mu\beta} - \eta^{\mu\beta} \mathcal{M}^{\nu\alpha} + \eta^{\mu\alpha} \mathcal{M}^{\nu\beta}
\]

\[
\mathcal{M}^{\mu\nu} \mathcal{P}_\alpha - \mathcal{P}_\alpha \mathcal{M}^{\mu\nu} = -\delta_\alpha^{\nu} \mathcal{P}^\mu + \delta_\alpha^\mu \mathcal{P}^\nu \quad (24)
\]

\[
\mathcal{P}_\mu \mathcal{P}_\nu - \mathcal{P}_\nu \mathcal{P}_\mu = 0.
\]

Here $\eta$ is the Minkowski metric introduced earlier and $\mu, \nu$ range from 0-3. The derivation of these relations is sketched in Appendix A. They give the Lie algebra of the generators of infinitesimal inhomogeneous Lorentz transformations near the origin. Discrete transformations not deformable to the identity are described below. $\mathcal{M}_{\mu\nu}$ is the generator of rotations in the $\mu - \nu$ plane, while $\mathcal{P}_\mu$ similarly generates spacetime translations in the $\mu$-direction.

In Appendix B it is shown that to each commutator of generators $A_s$ of a representation of a Lie group,

\[
A_r A_s - A_s A_r = C_{rs}^t A_t \quad (25)
\]

corresponds the Poisson bracket of generators of the equivalent canonical transformation

\[
[A_r, A_s]^- = C_{rs}^t A_t + d_{rs} \quad (26)
\]

where the $d$-matrices are constants. In the case of the Poincaré group (but not the Galilei group), it is possible to define the canonical generators $M_{\mu\nu}$ and $P_\mu$ equivalent to $\mathcal{M}_{\mu\nu}$ and $\mathcal{P}_\mu$ in such a way that the $d$-matrices vanish identically, thus

\[
[A_r, A_s]^- = C_{rs}^t A_t. \quad (27)
\]

The Poisson bracket relations for the canonical realization of the Poincaré group are therefore

\[
[M^{\mu\nu}, M^{\alpha\beta}]^- = \eta^{\nu\beta} M^{\mu\alpha} - \eta^{\nu\alpha} M^{\mu\beta} - \eta^{\mu\beta} M^{\nu\alpha} + \eta^{\mu\alpha} M^{\nu\beta}
\]

\[
[M^{\mu\nu}, P_\alpha]^- = -P_\nu \delta_\alpha^\mu + P^\nu \delta_\alpha^\mu \quad (28)
\]

\[
[P_\mu, P_\nu]^- = 0.
\]

The action of the ten generators of infinitesimal canonical transformations can be grouped as three spatial translations, one temporal translation, three boosts, and three rotations.
Of these, only the bracket relations for the generators of rotational canonical transformations

\[ J_i = \epsilon_{ijk} M^{jk}; \]  
(29)

\[ [J_i, J_j]^- = \epsilon_{ijk} J_k. \]  
(30)

\((i, j, k=1-3)\) find use in what follows.

\section*{B. Angular momentum, spin, and irreducible canonical realizations in pseudomechanics}

A (unitary) representation of a group is a set of linear transformations induced by a set of (unitary) matrices that gives a realization of the group; \textit{i.e.}, the matrix commutators are isomorphic to the group bracket relations. In a canonical realization, the commutators are replaced by the equivalent Poisson brackets. The discussion can be limited to irreducible realizations without loss of generality. Within an irreducible realization of a Lie group, any two points of phase space can be connected by a canonical transformation representing the action of some element of the group. An irreducible realization possesses no nontrivial invariants. Thus, Casimir invariants reduce to numbers in an irreducible canonical realization.

The properties of irreducible canonical realizations with definite angular momentum is obtained in a manner closely analogous to the corresponding quantum mechanical theory. Consider the transformation properties of dynamical variables in the canonical formalism under rotations. The bracket relations obeyed by canonical angular momentum variables form a subgroup of the Poincaré group decoupled from the boost degrees of freedom. In particular, the spin degrees of freedom of a massive particle in the rest frame are treated exactly as in the nonrelativistic case. Fix the direction of the \(z\)-axis along the spatial part of \(\xi_\mu\) and write \(\xi\) for its magnitude. The infinitesimal canonical transformation induced by the generator of rotations about the \(z\)-axis in a dynamical variable is

\[ \xi \Rightarrow \xi + \delta \phi [\xi, J_z]. \]  
(31)

If the variable \(\xi\) is rotationally symmetric about the axis defining \(\phi\), the effect of this transformation must be equivalent to multiplication by a phase:

\[ \xi \Rightarrow \xi + im\delta \phi \xi, \]  
(32)
or

\[ [J_z, \xi] = -im\xi. \]  \hfill (33)

This relation amounts to a kind of eigenvector condition. Define the ladder operators

\[ J_\pm = J_z \pm iJ_y. \]  \hfill (34)

These have the following properties, closely analogous to the familiar quantum-mechanical identities

\[ [J_+, J_-]^- = -2iJ_z \]  \hfill (35)

\[ [J_z, J_\pm]^- = \mp J_\pm. \]  \hfill (36)

Note that these relations are obtained from the quantum definitions by setting \( i\hbar \equiv 1 \).

It easily verified that the quantity

\[ J^2 = J_x^2 + J_y^2 + J_z^2 \]  \hfill (37)

\[ = J_z^2 + \frac{1}{2}[J_+J_- + J_-J_+] \]  \hfill (38)

has vanishing brackets with all the generators of rotations in an irreducible realization. It is thus a Casimir invariant which, in any irreducible canonical realization, is a constant number so that

\[ [J^2, \xi] = const.\xi \equiv j(j+1)\xi. \]  \hfill (39)

The irreducible realizations are labeled by the value of \( j \). In the remainder of this section, and the next, it is convenient to label \( \xi \) by both eigenvalues \( j \) and \( m \) as \( \xi_{jm} \), just as for irreducible tensor operators in spherical coordinates.

Now consider the quantity

\[ [J_\pm, \xi_{jm}]. \]  \hfill (40)

We can determine the z-projection of its angular momentum by computing

\[ [J_z, [J_\pm, \xi_{jm}]] \]  \hfill (41)

with the aid of the Jacobi identity eqn (19) and the bracket relations of the ladder operators from eqn (36):

\[ [J_z, [J_\pm, \xi_{jm}]] = -im[J_\pm, \xi_{jm}] \mp [J_\pm, \xi_{jm}] \]  \hfill (42)
or

\[ [J_z, [J_{\pm}, \xi_{jm}]] = -i(m \pm 1)[J_{\pm}, \xi_{jm}]. \]  

(43)

Comparing this expression with eqn (33) shows the result of taking the Poisson bracket of one of the ladder operators with a dynamical variable of spin \( j \) is a dynamical variable with the \( z \)-projection \( m \) of its dimensionless angular momentum changed by unity:

\[ [J_{\pm}, \xi_{jm}] = \lambda \xi_{jm \pm 1}. \]  

(44)

where \( \lambda \) will depend on \( j \) and \( m \).

Proceeding in this vein, the action of the ladder operators in the canonical formalism is obtained in close analogy to the quantum case. In particular \( ^{69} \)

\[ [J_{\pm}, \xi_{jm}] = -i \sqrt{(j \mp m)(j \pm m + 1)} \xi_{jm \pm 1}. \]  

(45)

In a manner entirely analogous to the quantum case \( ^{68-69,72} \) it can be shown that the eigenvectors of \( J_z \) given by eqn (33) have integer eigenvalues

\[ -j \leq m \leq j \]  

(46)

and that they span a \( 2j+1 \) dimensional subspace of the Hilbert space of canonical realizations of the rotation subgroup. It is the behavior one expects, say, from a spherical harmonic of angular momentum \( j \), with \( z \)-projection \( m \). Poisson brackets being dimensionless, the quantities \( j \) and \( m \) that label distinct elements of an irreducible canonical realization with definite angular momentum do not set a scale for the physical angular momentum associated with the dynamical variable \( \xi \). The machinery developed in this section may, therefore, have the appearance of a mathematical analogy devoid of physical content, but it is required in order to obtain the properties of classical Grassman dynamical variables under time inversion in Section IV.

The canonical formalism just sketched accommodates intrinsic spin, in much the same manner as its quantum-mechanical counterpart. The infinitesimal canonical generator of rotations \( M^{\mu \nu} \) for point particle motion can be written

\[ M^{\mu \nu} = x^{\mu} p^{\nu} - x^{\nu} p^{\mu} + S^{\mu \nu}. \]  

(47)

\( S^{\mu \nu} \), of course, is the classical intrinsic spin tensor of the particle. This result may be obtained with aid of the classical Pauli-Lubanski vector \( ^{74} \) or directly from even \( ^{67} \) and odd \( ^{58} \)
irreducible canonical realizations. From eqn (47), the intrinsic spin vector of a particle in its rest frame is obtained as

$$S_i = \epsilon_{ijk}S^{jk}$$  \hspace{1cm} (48)

with

$$[S_i, S_j] = \epsilon_{ijk}S_k.$$

(49)

In any frame in which the momentum $p$ vanishes, $J = S$. Pauri and Prosperi\textsuperscript{75} construct the irreducible canonical realizations of spin $j$, and note that they supply the classical equivalent of a particle with spin. Gomis et al.\textsuperscript{60} extend the construction to arbitrary spin Grassmann variables with the canonical version of the Bargmann-Wigner formalism.\textsuperscript{76,77}

The corresponding generalization of the Di Vecchia and Ravndal model spin tensor in terms of a set of spin 1/2 Grassman variables $\xi_\lambda$ is given by

$$S^{\mu\nu} = -\frac{i}{2} \sum_{\lambda=1}^{N} \xi_\lambda^\mu \xi_\lambda^\nu,$$

(50)

where the spin is $N/2$. The spin portion of the Lagrangian becomes

$$L_{\text{spin}} = -\frac{i}{4} \eta_{\mu\nu} \sum_{\lambda=1}^{N} \xi_\lambda^\mu \dot{\xi}_\lambda^\nu,$$

(51)

$$\equiv -\frac{i}{4} \Theta^\mu \cdot \dot{\Theta}_\mu,$$

(52)

by way of defining both the dynamical variable $\Theta$ for spin $N/2$ and the scalar product on spin indices. The form of the spin tensor in eqn (50) creates problems when either spacetime index $\mu$ or $\nu$ is 0.\textsuperscript{44} In the Di Vecchia and Ravndal Lagrangian, the spin angular momentum tensor satisfies

$$p_\mu S^{\mu\nu} = 0$$

(53)

identically, so that the spin has only spatial components in the particle rest frame.\textsuperscript{62}

IV. TIME REVERSAL, WEAK AND STRONG, IN PSEUDOMECHANICS

A. Time reversal invariance and anticanonical transformations

The preceding Section developed the properties of continuous coordinate transformations upon classical Grassmann variables, specifically rotations, necessary for proving the spin-statistics connection. A complete realization of the Poincaré group for massive spinning
particles must also include the discrete transformations of parity, time reversal, and charge conjugation. The effects of parity and time reversal are given by

\[ \phi(\mathbf{x}, t) = \phi(-\mathbf{x}, t) \]  

\[ T(\phi(\mathbf{x}, t)) = \phi(\mathbf{x}, -t) \]  

The full Poincaré group thus has four components related by the various combinations of the parity and time-reversal transformations. The classical analog of charge conjugation is discussed in Section IV D. Schwinger’s proof of the spin-statistics connection depends upon the effects of time-reversal and charge-conjugation on states of definite spin. The parity operation is of no further concern here.

The operation \( T \) commutes with the generators of spatial translations and rotations, but anticommutes with the generators of boosts and, in particular, of time translations:

\[ T \mathcal{P}^0 = -\mathcal{P}^0 T \]  

Let \( \hat{T} \) be the operator which realizes \( T \) on functions in phase space \((q, p)\)

\[ \hat{T}(\phi(q, p)) = \phi(q', p') \]  

where the primed variables are related to the unprimed ones by a canonical transformation

\[ q' = q'(q, p) \]

\[ p' = p'(q, p). \]  

We have, from eqn (56),

\[ \hat{T}([E, \phi]) = -[E, \phi] \hat{T}, \forall \phi, \]  

while from the definition of a canonical transformation

\[ [\phi, \gamma]_{qp} = [\phi, \gamma]_{q'p'} \]

we have

\[ \hat{T}[\phi, \gamma] = \{\hat{T}(\phi), \hat{T}(\gamma)\}, \]
so that

\[(\hat{T}(E), \hat{T}(\phi)) = -[E, \hat{T}(\phi)]\]  

(63)

for arbitrary \(\phi\). \(\hat{T}(E)\) and \(-E\) can therefore differ by only a constant. As \(\hat{T}^2\) must equal unity,

\[\hat{T}(E) = -E.\]  

(64)

But this is awkward, because the generator of time translations is interpreted as the energy, and should be positive definite. The solution\(^{67}\) is to realize time reversal \(T\) as an anticanonical operation \(\hat{T}\),

\[T([\phi, \gamma]) = -[\hat{T}(\phi), \hat{T}(\gamma)]\]  

(65)

with

\[T(E) = E.\]  

(66)

The anticanonical time reversal operation commutes with the generator of boosts and anticommutes with generators of rotation and translations. In particular,

\[T(J) = -J.\]  

(67)

The quantum mechanical realization of time reversal is an antilinear and antiunitary transformation. Wigner\(^{79}\) shows it is always possible to write such a transformation as the composition of a unitary transformation with complex conjugation. To maintain consistency with the quantum case-by way of inverting the correspondence principle-\(T\) is defined as an antilinear, as well as anticanonical, operation

\[T(a\phi + b\gamma) = a^*T(\phi) + b^*T(\gamma).\]  

(68)

An antilinear operation is likewise the composition of complex conjugation with a linear transformation. Note that the action of \(T\), as defined, on a scalar quantity is that of complex conjugation. This observation finds use in Section [IV C] for finding the effect of time inversion on products of dynamical variables.

**B. Weak Time-reversal symmetry in pseudomechanics**

We next address the effect of time inversion on the angular momentum relations given earlier, in particular the raising and lowering operations eqn (45). First, consider the effect
of $T$ on eqn (33)

$$T(\xi_{jm}) \Rightarrow T(\xi_{jm}) - im\delta\phi T(\xi_{jm}),$$

or

$$[J_z, T(\xi_{jm})] = imT(\xi_{jm}),$$

from which

$$T(\xi_{jm}) = \varsigma_m \xi_{j-m}$$

where $\varsigma_m$ may depend on $j$ as well. Second, the anticanonical action of $T$ gives

$$T([J_\pm, \xi_{jm}]) = -[T(J_\pm), T(\xi_{jm})].$$

Now, recalling eqns (45), (65) and (67)

$$T([J_\pm, \xi_{jm}]) = i\sqrt{(j \mp m)(j \pm m + 1)}\varsigma_{m \pm 1} \xi_{j-(m \pm 1)}$$

$$= -[T(J_\pm), T(\xi_{jm})] = [J_x \mp iJ_y, \varsigma_m \xi_{j-m}]$$

$$= -i\sqrt{(j \mp m)(j \pm m + 1)}\varsigma_m \xi_{j-(m \pm 1)}. $$

Dividing out common terms in eqns (73) and (74) gives

$$-\varsigma_m = \varsigma_{m+1}$$

or

$$\varsigma_m = \varsigma(-1)^{-m}. $$

It remains to fix $\varsigma$. The choice $\varsigma_j = 1$ ensures that $T$ applied to a real dynamical variable will give a real result, giving

$$\varsigma_m = (-1)^{j-m}. $$

Thus,

$$T(\xi_{jm}) = (-1)^{j-m} \xi_{j-m}. $$

C. Weak $T$ on products of dynamical variables

The Lagrangian is constructed from invariant scalar combinations of dynamical variables and their derivatives. In pseudomechanical models, the invariant used is the scalar product
of two dynamical variables with the same spin. The spin portion of the Lagrangian from eqn (52) may be written as a sum of scalar products of the form

$$\sigma \cdot \pi = (-1)^k \sigma_k \pi_{-k}. \quad (79)$$

Consider the action of $T$ on the scalar product of two dynamical variables belonging to the same irreducible canonical realization of spin $j$. Recall that the antilinear action of $T$ on a scalar quantity is that of complex conjugation and, from eqn (7), that complex conjugation inverts the order of factors in a product:

$$(\sigma \pi)^* = \pi^* \sigma^* \quad (80)$$

Thus,

$$T(\sigma \cdot \pi) = (-1)^k \pi^*_{-k} \sigma^*_k = (-1)^{-j} \pi^*_k (-1)^{-j} \sigma^*_k$$

$$= (-1)^{-j} \pi^*_k T(\sigma_k) = (-1)^{-2j} (-1)^k T(\pi_k) T(\sigma_{-k}) \quad (81)$$

$$= (-1)^{2j} T(\pi) \cdot T(\sigma).$$

The inversion of the order of factors under $T$ clearly generalizes by induction to an arbitrary number of them.

D. Charge conjugation in pseudomechanics and strong time-reversal invariance

Schwinger’s proof of the spin-statistics connection relies upon "strong", or "Weyl", time reversal, as opposed to the "weak" or "Wigner" time reversal $T$ as defined above. The condition of strong time reversal invariance is that the form of the classical action be preserved if evolution from an initial to a final state is replaced by the evolution of a time-reversed state from the final state to the initial one. That is, in addition to reversing the sign of the locally timelike variable in all dynamical quantities, initial and final states are exchanged in the action, and the affine parameter labeling proper time changes sign as well:

$$\tau \Rightarrow -\tau \quad (82)$$

$$\frac{d}{d\tau} \Rightarrow -\frac{d}{d\tau} \quad (83)$$
Costella et al.\textsuperscript{83} show that this operation is the classical analog of charge conjugation. The connection with Stückelberg’s identification of antiparticle motion with time-reversed particle motion is clear; in fact, Feynman\textsuperscript{84} examined the classical formulation of this concept prior to the debut of his theory of positrons.\textsuperscript{29} Strong time reversal is therefore the composition of the weak time reversal operation $T$ with charge conjugation $C$.\textsuperscript{5,10,18,33}

V. CONNECTION BETWEEN SPIN AND STATISTICS

With the results from preceding sections on properties of classical Grassmann variables of definite spin under time-reversal and charge-conjugation in hand, we are now in position to impose the condition of invariance under strong time-reversal transformation upon a pseudomechanical system and show that the spin-statistics connection necessarily follows. Invariance of the pseudoclassical description of particle motion under strong $T$ inversion requires that the form of the action functional

$$S = \int_{\tau_1}^{\tau_2} d\tau L(\theta, \dot{\theta})$$

be unaltered by inverting both the sign of $t$ and the definition of proper time, to include the order of the initial and final proper time of a segment of a particle orbit. The former operation is $T$; the latter, $C$. Thus, if

$$S = \int_{\tau_1}^{\tau_2} d\tau L(\theta, \dot{\theta}) \Rightarrow \int_{\tau_1}^{\tau_2} d\tau' L(T(\theta), \frac{d}{d\tau'} T(\theta))$$

under the combined operation of $T$ and $C$, the form of the action will be preserved. The total Lagrangian in pseudoclassical models is made of quadratic forms in the dynamical variables. It suffices to consider the contribution $L_j$ for an irreducible realization of spin $j$.\textsuperscript{85}

Apply the operations of $T$ and $C$ to $L_j$. First $T$:

$$T(L_j) = (-1)^{2j} L_j'(T(\theta), \frac{d}{d\tau'} T(\theta))$$

$$= (-1)^{2j} L_j'(T(\theta), -\frac{d}{d\tau'} T(\theta))$$

where the superscript $t$ on $L_j$ indicates transposition of the order of all factors in the Lagrangian.

Next, applying $C$ changes the sign of the proper time derivative:

$$CT(L_j) = (-1)^{2j} L_j'(T(\theta), \frac{d}{d\tau'} T(\theta))$$
Invariance of the form of the action under $CT$ is guaranteed if

$$L_j(T(\theta), \frac{d}{d\tau} T(\theta)) = (-1)^{2j} L_j(T(\theta), \frac{d}{d\tau} T(\theta)).$$ (88)

This last will hold if the sign change attendant upon inverting the order of odd Grassmann variables is compensated by a factor of minus one in front, while no compensating sign change accompanies inverting the order of even Grassmann variables. We conclude classical spin variables which are irreducible canonical realizations of spin $j$ must be commuting, even Grassmann variables if $j$ is an integer, and anticommuting, odd Grassmann variables if $j$ is half-integral. From the symmetry properties of brackets in pseudomechanics given earlier follows immediately the conclusion that irreducible canonical realizations for integral $j$ obey Poisson bracket relations, while realizations for half-integral $j$ obey Poisson antibracket relations. The Poisson brackets for the spin degrees of freedom are

$$[\theta_\mu, \theta_\nu]^- = [\pi_\mu, \pi_\nu]^- = 0$$ (89)

$$[\theta_\mu, \pi_\nu]^- = \eta_{\mu\nu}$$

for $2j=$even, and

$$[\theta_\mu, \theta_\nu]^+ = [\pi_\mu, \pi_\nu]^+ = 0$$ (90)

$$[\theta_\mu, \pi_\nu]^+ = -\eta_{\mu\nu}$$

for $2j=$odd. The vectorial position variables $q_\mu$ and $p_\mu$ satisfy Poisson bracket relations for any value of $j$. This is the spin-statistics theorem stated in the language of the canonical formalism for pseudomechanics.

VI. COMMENTS

The result just obtained is neither the strongest, nor the most general, that could be desired. It is as close to a literal transcription of Schwinger’s 1951 reasoning into the language of the canonical formalism for particle mechanics as could be contrived. The choice to proceed in this manner was not made as a simple matter of filial piety. Rather, it appears the simplest, quickest route to a classical spin statistics relation is to recapitulate Schwinger’s
proof in close to its original form. It is notable that the proof proper in Section V is shorter, and arguably more appealing, than the formal developments necessary to erect the canonical formalism underpinnings which preceded it.

Like that earlier proof, this one applies to free particles, or to particles minimally coupled at most weakly by an interaction that conserves $CT$. This should not be considered a serious shortcoming in the case of the electromagnetic interaction, in which $C$, $P$, and $T$ are each conserved separately. A classical spin-statistics connection valid for electromagnetically interacting particles would seem capable of meeting most needs for that class of relation.

There is another, and more serious, sense in which the result just shown should be regarded as a comparatively weak one. Schwinger’s argument, strictly speaking, applies only to brackets constructed from dynamical variables evaluated at a common point of phase space. The point might seem of limited relevance for eqns (89) and (90), since that is how one normally evaluates Poisson brackets, but the commutation properties of dynamical variables at distinct phase space locations is left undetermined by the present argument. Their extension even to separate points at null interval, which would be the minimum required for consistency with the relation just shown, does not follow without additional assumptions. Schwinger cites “the general compatibility requirement for physical quantities attached to points with a spacelike interval” to justify extending commutation relations from coincident to spacelike intervals. The assumptions of a particular graded Lie algebra structure for the bracket relations in the present discussion could similarly be strengthened by fiat, but only at the price of underscoring the weakness of the result obtained.

The particular form of classical mechanics used in the foregoing may look odd as an exemplar of classical physics-amongst other peculiarities, dynamical variables are allowed to take on complex values, in general. While it is often said that classical dynamical variables should be real-valued functions on spacetime, the classical physics of waves or oscillatory phenomena is too riddled with complex exponentials for this stricture to be altogether convincing. In any event, complex conjugation is required in the present demonstration for, strictly speaking, its effect on real Grassman variables.

What should not be obscured, however, is that pseudomechanics offers an elementary example of a physical theory which respects the spin-statistics connection without being quantum mechanical.
VII. CONCLUSION

"We conclude that the connection between spin and statistics of particles is implicit in the requirement of invariance under coordinate transformations." Schwinger, 1951

Schwinger used this comment as a period for his proof of the spin-statistics connection. In spirit, it is very close to that of Pauli’s, cited at the start of this paper, but in wording it is notably less emphatic. Oddly so, given that the requirement of invariance upon which Schwinger erected his construction of quantum electrodynamics from the action principle was Poincaré invariance. Note that in either statement the tone struck can be interpreted as a classical one.

It has been remarked more than once that proofs of the spin-statistics connection necessarily depend upon some assumption traceable to Poincaré invariance. The dependence may be explicit, as in Pauli’s original proof, or the axiomatic proofs of Burgoyne, Lüders, and Zumino, or it may be implicit, as in Feynman, the topological proof of Balachandran et al., or the proof of Berry and Robbins using topological phases. The topological theorems invoke the existence of antiparticles. Proofs by Weinberg, which use the language of representations of the Poincaré group, or of Berry and Robbins, which do not, invoke no symmetry higher than that of rotational invariance. But the Poincaré group contains rotational symmetry as a subgroup.

The physical world is not more relativistic than it is quantum-mechanical. However, the existence of classical systems obeying the spin-statistics connection allows one to think of that phenomenon as a relativistic one at bottom. Under terrestrial laboratory conditions, Poincaré invariance is an exquisitely accurate symmetry of nature. The lesser symmetries upon which the spin-statistics connection depends, be they the existence of antiparticles, or of rotational invariance, or (as here) invariance under time inversion, are all necessary consequences of Poincaré invariance in nature.

It is usually supposed that relativistic phenomena are significant only for high energies, or for velocities approaching that of light. The effect of the spin-statistics connection on the nature of the everyday world is profound, perhaps most significantly under intrinsically low energy conditions of Fermi degeneracy. It cannot be doubted that, were the spin-statistics connection different, or nonexistent, the resulting world would almost certainly be unrecognizable to us. To suppose such a violent rearrangement of microscopic physics
would leave the macroscopic world sensibly unaltered amounts to invoking a conspiracy of nature for the sake of avoiding that conclusion. Not one of our senses is independent of the accidents consequent upon the connection between spin and statistics, either from the role played by Pauli exclusion in atomic structure and chemical binding, or from the effects of incompressible flow; not sight, nor hearing, nor smell, nor taste, nor touch. One need not invoke exotic conditions to find evidence of relativistic symmetries in the world.

Acknowledgments

I wish to thank Drs. S. Gasster, J. Johnson, and G. Smit for their careful reading of this paper, and for suggestions which substantially improved its clarity.

APPENDIX A: THE LIE ALGEBRA OF THE POINCARÉ GROUP

The general inhomogeneous Lorentz transformation on a four-vector \( x_\mu \) is

\[
x'_\mu = \Lambda_\mu^\alpha x_\alpha + a_\mu
\]  

(A1)

If the four-interval between two points in spacetime is to remain invariant under eqn (A1), then the \( \Lambda \) matrices must satisfy

\[
\Lambda_\mu^\alpha \Lambda_\nu^\beta \eta_{\mu \nu} = \eta_{\alpha \beta}
\]  

(A2)

Let a set of (unitary) matrices \( D \) comprise a representation of inhomogeneous Lorentz transformations satisfying

\[
D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1 \Lambda_2)
\]

(A3)

and consider transformations infinitesimally close to the origin,

\[
\Lambda_\mu^\alpha = \delta_\mu^\alpha + \omega_\mu^\alpha + O(\omega^2)
\]

(A4)

and

\[
a_\mu = \epsilon_\mu
\]

(A5)

with \( |\omega|, |\epsilon| \ll 1 \). If Eqn (A2) is to be satisfied, \( \omega \) must be antisymmetric in its indices. The corresponding matrix representation of eqns (A4) and (A5) is

\[
D(1 + \omega, \epsilon) = 1 + \frac{1}{2} \omega_{\mu \nu} M^{\mu \nu} - \epsilon_\rho \mathcal{P}^\rho
\]

(A6)
to first order, where $M$ is a constant antisymmetric matrix and $P$ is a vector. In order for eqn (A3) to be satisfied, $M$ and $P$ must obey

$$M^{\mu\nu}M_{\alpha\beta} - M_{\alpha\beta}M^{\mu\nu} = \eta^{\nu\beta}M^{\mu\alpha} - \eta^{\nu\alpha}M^{\mu\beta} - \delta_{\mu}^{\nu}M^{\alpha\beta} + \delta_{\alpha}^{\nu}M^{\mu\beta}$$

$$M^{\mu\nu}P_{\alpha} - P_{\alpha}M^{\mu\nu} = -\delta_{\alpha}^{\nu}P^{\mu} + \delta_{\alpha}^{\mu}P^{\nu}$$

$$P_{\mu}P_{\nu} - P_{\nu}P_{\mu} = 0.$$  \hspace{1cm} (A7)

**APPENDIX B: CANONICAL REALIZATIONS OF A LIE GROUP**

The group action of a symmetry in the canonical formalism leads to a corresponding Lie algebra of Poisson brackets. Let the commutation relations of the infinitesimal generators of a Lie group be written

$$A_r A_s - A_s A_r = C_{rs}^t A_t,$$  \hspace{1cm} (B1)

and let the canonical coordinates and momenta, $(q_\mu, p_\nu)$ even and $(\xi_\mu, \pi_\nu)$ odd, obey fundamental Poisson bracket relations

$$[q_\mu, p_\nu] = \eta_{\mu\nu}$$

$$[\xi_\mu, \pi_\nu] = -\eta_{\mu\nu}$$  \hspace{1cm} (B2)

$$[q_\mu, q_\nu] = [p_\mu, p_\nu] = [\xi_\mu, \xi_\nu] = [\pi_\mu, \pi_\nu] = 0.$$

A canonical realization of a symmetry group is a set of transformations of the canonical coordinates, homomorphic to the symmetry group, that leaves the fundamental bracket relations eqns (B2) unaltered. The infinitesimal canonical transformations are defined so as not to mix even and odd Grassmann variables.\(^{54}\) That is,

$$q'_i = q'_i(q, \{p\}, \{a\})$$  \hspace{1cm} (B3)

$$p'_i = p'_i(q, \{p\}, \{a\})$$  \hspace{1cm} (B4)

where each of the set of parameters $\{a\}$ which characterizes the transformation is even, and

$$\theta'_\alpha = \theta'_\alpha(\theta, \{\pi\}, \{\rho\})$$  \hspace{1cm} (B5)

$$\pi'_a = \pi'_a(\theta, \{\pi\}, \{\rho\})$$  \hspace{1cm} (B6)
where the set of \( \{\rho\}'s \) is odd. With canonical transformations so restricted, the generators of infinitesimal canonical transformations \( A_r \) are even functions of the canonical variables. Then

\[
q'_i = q_i + \delta a^r [A_r, q_i]^- \quad \text{(B7)}
\]

\[
p'_i = p_i + \delta a^r [A_r, p_i]^- \quad \text{(B7)}
\]

and

\[
\theta'_\alpha = \theta_\alpha + \delta a^s [A_s, \theta_\alpha]^- \quad \text{(B8)}
\]

\[
\pi'_\alpha = \pi_\alpha + \delta a^s [A_s, \pi_\alpha]^- \quad \text{(B8)}
\]

correspond to the infinitesimal operations

\[
1 + \delta a^r A_r \quad \text{(B9)}
\]

and

\[
1 + \delta \rho^s A_s \quad \text{(B10)}
\]

respectively. These are the canonical generators of infinitesimal transformations, and their bracket relations must form a realization of the Lie algebra. In particular, to the infinitesimal transformation

\[
1 + \delta a^r \delta b^s (A_r A_s - A_s A_r) \quad \text{(B11)}
\]

must correspond the canonical transformation

\[
q'_i = q_i + \delta a^r \delta b^s \left[ [A_r, A_s]^- , q_i \right] \quad \text{(B12)}
\]

\[
p'_i = p_i + \delta a^r \delta b^s \left[ [A_r, A_s]^- , p_i \right] \quad \text{(B13)}
\]

and to

\[
1 + \delta \rho^t \delta u^u (A_t A_u - A_u A_t), \quad \text{(B14)}
\]

\[
\theta'_\alpha = \theta_\alpha + \delta \rho^t \delta u^u \left[ [A_t, A_u]^- , \theta_\alpha \right] \quad \text{(B15)}
\]

\[
\pi'_\alpha = \pi_\alpha + \delta \rho^t \delta u^u \left[ [A_t, A_u]^- , \pi_\alpha \right] . \quad \text{(B16)}
\]
Inserting eqn (B1) into eqns (B11) and (B14), and comparing eqn (B9) with eqn (B7), and eqn (B10) with eqn (B8), leads to the conclusion that

\[ [A_r, A_s]^- = C_{rs} A_t + d_{rs} \]  

(B17)

where the d’s are constants.

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5 Julian Schwinger, ”Theory of Quantized Fields I”, Phys Rev. 82, 914-927 (1951).
11 Julian Schwinger, Particles and Sources (Gordon and Breach, New York, NY, 1969) pp. 25-27


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46 Robert Geroch, Mathematical Physics (University of Chicago Press, Chicago, 1958) ch. 19
47 W.-K.Tung, Group Theory in Physics (World Scientific, Singapore, 1985) ch. 7-10
50 M. S. Swanson, Path Integrals and Quantum Processes (Academic Press, New York, NY, 1992) ch. 5
51 M. S. Swanson, Path Integrals and Quantum Processes (Academic Press, New York, NY, 1992) p. 120
52 The result to be proved concerns symmetries of Poisson brackets for dynamical variables of a single massive particle; there is no need to examine what is meant by "identical" particles in the present discussion.


In the corresponding classical unitary representation, this result follows immediately from Schur’s lemma.


Strictly speaking, the discrete transformation $CT$ used later is a surrogate for $P$ by virtue of the $CPT$ theorem. However, the discussion here is restricted in such a way that $CPT$ need not be explicitly invoked. The components connected by $T$ make up the orthochorous Poincaré group.


85 There is no loss of generality in this restriction, although the total Lagrangian in pseudoclassical models contains terms corresponding to more than one irreducible realization, *vide* eqn (20), and in general, contains a term which mixes vector and spinor variables.

86 Schwinger's original proof can be generalized by substituting flavor symmetry for CT invariance.