B FIELDS FROM A LUDDITE PERSPECTIVE

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ABSTRACT. In this talk, we discuss the geometric realization of $B$ fields and higher $p$-form potentials on a manifold $M$ as connections on affine bundles over $M$. We realize D branes on $M$ as special submanifolds of these affine bundles. As an application of this geometric understanding of the $B$ field, we give a simple geometric explanation for the Chern-Simons modification of the field strength of the heterotic $B$ field.

1. Introduction

The mathematical description of $B$–fields in terms of gerbes (see for example [4]) is too abstract to be useful for many basic computations. In this talk we discuss some of our recent work [10],[9] developing a simple, geometric representation of $B$ fields and higher $p$–form potentials on a manifold $M$ as connections on various affine bundles over $M$.

Our geometric representation has three ingredients:

1. the representation of the $B$ field on a manifold $M$ as a connection on an affine bundle $E$ over $M$,
2. a dictionary between string sigma model fields and differential operators on $E$, and
3. the representation of D branes as submanifolds of $E$ (not $M$).

The choice of $E$ depends on the sector of string theory under consideration. Choosing $E$ to be an affine bundle is probably useful only in a low energy approximation. In fact, heterotic string theory already requires more complicated (principal affine) bundles in order to see the full Green-Schwarz mechanism.

Our geometric model of $B$ fields and D branes closely agrees with our stringy expectations. Some easy consequences of this model include

• A derivation of noncommutative Yang-Mills associated to a $B$ field which suggests generalizations to deformations associated to higher $p$-form potentials.
• A geometric representation of topological B type D branes corresponding to coherent sheaves that need not be locally free and that may be "twisted ".
• A simple geometric explanation for the introduction of the Yang-Mills gauge transformations of $B$ fields in heterotic string theory.

We will not discuss noncommutative deformations here, but give a simple example in section 3 of the geometric realization of a topological B type D
brane corresponding to a coherent sheaf which is not locally free. In section 4, we show how the geometry of affine bundles leads to the introduction of Yang-Mills gauge transformations of B fields in heterotic string theory.

2. THE MODEL

In [9], T duality considerations suggested that B fields should be realized as connections on a space of connections. Here we consider a finite dimensional analog: B fields as connections on affine bundles locally modelled on $T^*M$. (See [5] for related ideas.) First we review some basic facts about connections on $T^*M$.

Local coordinates $(x^i)$ on $M$ define vector fields $\frac{\partial}{\partial x^i}$ on $M$. These coordinates also determine local coordinates $(x^i, p_s)$ on $T^*M$ which define vector fields $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial p_s}$ on $T^*M$. Here we are abusing notation in a standard way, using the same symbol, $\frac{\partial}{\partial x^i}$, to denote two different vector fields, one on $M$ and the other on $T^*M$. The coordinates thus define a lifting of vector fields on $M$ to vector fields on $T^*M$ given in a notationally confusing manner as $\frac{\partial}{\partial x^i} \rightarrow \frac{\partial}{\partial x^i} + p_n \Gamma^n_{is} \frac{\partial}{\partial p_s}$. (1)

This lift is obviously coordinate dependent. The coordinate dependence may be removed by introducing a connection. For example, with the Levi Civita connection, we have the globally well defined lift given by

$$\frac{\partial}{\partial x^i} \rightarrow \frac{\partial}{\partial x^i} + p_n \Gamma^n_{is} \frac{\partial}{\partial p_s}.$$ (1)

We now break the vector space structure on $T^*M$ to an affine structure by introducing new allowed local coordinate transformations

$$(x, p) \rightarrow (x, p + \lambda(x)),$$

where $\lambda$ is a locally defined 1 form on $M$. Then (1) is no longer coordinate independent. To fix this we introduce a 1 form $\mu$ and a 2 form $b$ and define

$$\frac{\partial}{\partial x^i} \rightarrow \frac{\partial}{\partial x^i} + (\mu_{i;s} + b_{is} + p_n \Gamma^n_{is}) \frac{\partial}{\partial p_s}.$$ (2)

This lift is coordinate independent if

$$\mu \rightarrow \mu + \lambda, \text{ and } b \rightarrow b + d\lambda, \text{ when } p \rightarrow p + \lambda.$$

We denote the new bundle equipped with this affine structure, $T_B^*M$. Here the $b$ field is the component of the connection on an affine bundle corresponding to the translation subspace of the affine transformations.

Our new $b$ field immediately runs into a problem. The cohomology class of $db$ vanishes. To realize B fields with cohomologically nontrivial field strength, we pass to quotients of $T_B^*M$. There are two obvious ways to do this. The first is to consider discrete quotients; then our fibers become products of tori and affine spaces. This cure allows the field strength of $b$ to lie in a discrete subgroup of $H^3(M)$ but restricts the geometry of $M$.

A second solution is given by quotienting by ”gauge equivalence class”. In other words, consider sections $p \mod \text{exact sections } df$. This does not give
a finite dimensional bundle; so, to stay in the geometric regime, we consider
the finite dimensional approximation to this equivalence given by 1 jets of
sections quotiented by 1 jets of exact sections. The resulting quotient space
is an affine space locally modelled on \( \bigwedge^2 T^* M \). If we allow affine changes
of coordinates \((x, p) \rightarrow (x, p + \lambda)\), where now \( p \) and \( \lambda \) are 2 forms, then we
may define lifts of vector fields of the form

\[
\frac{\partial}{\partial x^i} \rightarrow \frac{\partial}{\partial x^i} + (b_{ji; k} + b_{ik; j} + c_{ijk} + p_{nk} \Gamma^p_{ij} + p_{jn} \Gamma^p_{ik}) \frac{\partial}{\partial p_{jk}},
\]

where \( b \) and \( c \) are locally defined 2 and 3 forms respectively. This is well
defined if \( b \rightarrow b - \lambda \), and \( c \rightarrow c + d\lambda \) when \( p \rightarrow p + \lambda \).

If we restrict to closed translations, \( \lambda \), then we may choose \( c = 0 \) and obtain
b fields with arbitrary field strength without restricting the topology of \( M \).
For nonclosed \( \lambda \) it is necessary to include the 3 form potential \( c \). Once
again, its field strength is cohomologically trivial in this formulation, and
this may be remedied by passing to quotients. Discrete quotients constrain
the geometry of \( M \) and lead to field strengths of \( c \) lying in discrete subgroups
of \( H^4(M) \). Gauge quotients lead to higher p-form potentials. We see that
there is an analog of our p-form potential construction for all \( p < \text{dim} M \).

We now see how these bundles are related to string theory. Let \( s, t \) denote
coordinates for a string with \( t \) a timelike parameter and \( s \) the position along
the string. Let \( X^\mu(s, t) \) denote coordinates for the string world sheet. We
would like to study a quantum mechanical system which reflects some low
energy information about the string sigma model. The simplest method to
do this is to consider only the average value \( x^\mu(t) \) of \( X^\mu(s, t) \) and its conjun-
gate momentum. This leads to the quantum mechanics of a point particle
moving on the target space \( M \). This system loses too much string data.
Motivated by the quantum mechanics of the (affine) 1 jet approximation of
the string maps, we include the average value of \( X^\mu_\pi \) in our system. The
average value of \( X^\mu_\pi \) is just \( (X^\mu(\pi, t) - X^\mu(0, t))/\pi \). Jet space constructions
suggest that we represent \( X^\mu(\pi, t) - X^\mu(0, t) \) as a differential operator \( \frac{1}{i} \frac{\partial}{\partial p_\mu} \)
tangent to the fiber of an affine bundle, whose coordinates \( p_\mu \) we think of
as velocities, or conjugate momenta to the \( (X^\mu(\pi, t) - X^\mu(0, t))/\pi \).

Assuming the \( p_\mu \) are coordinates for an affine fiber is clearly at best a low
energy approximation. For example, if \( X^1 \) wraps a small circle then \( X(s, t) \)
will not lie in a single coordinate chart for all \( s \) and the average value of \( X_s \)
will also encode winding number (and only winding number for the closed
strings). This dictates that in the \( p_1 \) direction, the noncompact affine fiber
be replaced by a circle fiber dual to the wrapped circle, thus leading to dis-
crete quotients of the affine fibers as required for cohomologically nontrivial
B field field strengths. Then the commutator

\[
\left[ \frac{1}{i} \frac{\partial}{\partial p_1}, e^{2\pi i wp_1} \right] = 2\pi we^{2\pi i wp_1}
\]
with \( p_1 \) corresponding to \( X^1(\pi, t) - X^1(0, t) \) suggests we interpret \( p_1 \) as the infinitesimal generator of motion along the circle. (Further deformations of the affine geometry of the fibers which are suggested by supersymmetry are considered in [10, Section 4.5].)

These considerations lead to the rough dictionary

\[
\text{average } X^\mu_s \rightarrow \frac{\partial}{\partial p_\mu}, \quad \text{and}
\]

\[
\text{average } X^j_t \rightarrow g^{ij} \left( \frac{\partial}{\partial x^i} + (\mu_i + b_{i\nu} + p_n \Gamma^n_{i\nu}) \frac{\partial}{\partial p_\mu} \right).
\]

Here \( \pi^j \) denotes the total momentum in the \( j \)th direction. The interpretation of \( p_1 \) as a generator of motion in the \( X^1 \) direction further suggests we associate

\[
X^\mu_s(\pi, t) \rightarrow p_\mu.
\]

3. D branes

We now use the dictionary of the preceding section to see what form D branes must take in our model. An \( n \)-brane (in \( M^n \)) is given by fully Neumann boundary conditions: \( X^\nu_t(0) = X^\nu_t(\pi) = 0 \). From our dictionary, we see that this corresponds to a zero section (in a choice of local affine coordinates) of \( T^*_B M \). A change in time parameter for the string world sheet \((s,t) \rightarrow (s, \tau(s,t))\) induces an affine change

\[
\frac{\partial}{\partial s} \rightarrow \frac{\partial}{\partial s} + \frac{\partial \tau}{\partial s} \frac{\partial}{\partial \tau},
\]

and correspondingly an affine change of coordinates in \( T^*_B M \). Thus the B field reflects the nonuniqueness of the time coordinate.

Consider next a \( p \) brane corresponding to a \( p \) dimensional submanifold \( S \) of \( M \). Choose local coordinates so that \( S \) is given by \( x^\nu = 0 \) for \( \nu > p \). Then the \( p \)-brane boundary conditions \( X^\nu_t = 0 \), \( X^\nu_s \) free, for \( \nu > p \) and \( X^\nu_t = 0 \), \( X^\nu_s \) free, for \( \nu > p \) for some choice of local affine coordinates. This is an \( n \) dimensional submanifold \( Z \) of \( T^*_B M \) which, in a choice of local affine coordinates, is the conormal bundle of \( S \). In particular, a zero brane is just an affine fiber over a point.

Fixing coordinates so that \( Z \) is identified with a conormal bundle removes the gauge freedom to vary \( \mu_\nu \) arbitrarily for \( \nu \leq p \). Hence the D brane comes equipped with a locally defined 1 form \( \mu \) on \( S \). When the B field is trivial, this defines the gauge field for a line bundle on \( S \) (equipped with a local frame). On the overlap of two coordinate neighborhoods, \( U_\alpha \) and \( U_\beta \), we have \( b^\alpha - d\mu^\alpha = b^\beta - d\mu^\beta \). Thus, \( \mu \) determines a connection on a line bundle only if the field strength of \( b \) vanishes in \( H^3(S) \). This is, of course, to be expected. In the presence of a B field with nontrivial field strength, the gauge field of a D brane is not that of a vector bundle, but of a "twisted bundle" (see for example [11],[6],[2]) or more generally perhaps
of an infinite dimensional $C^*$ algebra [1]. Interpreting $\mu$ as a coordinate of the brane in a bundle is, perhaps, geometrically simpler than working with infinite dimensional $C^*$ algebra bundles.

If we have two distinct D branes $Z_1$ and $Z_2$ which correspond after distinct affine transformations to the normal bundle of a single submanifold $S$ of $M$, then we see that we have a gauge enhancement. In addition to the previously identified gauge fields, we also have the 1 form measuring the relative displacement in the fiber between the 2 branes.

In general, we would like to define a D brane to be an n dimensional submanifold, $Z$ of $T^*_B M$ which corresponds to a choice of boundary condition, Dirichlet or Neumann, for each coordinate. If the Neumann boundary condition were well defined, then for $Z$ to represent a choice of boundary condition would imply that the symplectic form $\omega := dx^\mu \wedge dp_\mu$ vanishes when pulled back to $Z$. I.e., $Z$ is Lagrangian. Neither the form $\omega$ nor the Neumann condition is well defined. One can define a connection dependent analog of each, but we will instead use the provisional definition:

**Provisional Definition 3.1.** A D brane is an n dimensional submanifold $Z$ of $T^*_B M$ such that for every point $p \in Z$ there exists a neighborhood of $p$ in $Z$ which, after an affine choice of coordinates can be identified with an open set in the conormal bundle of some submanifold $S$ of $M$.

BPS conditions will restrict the possible $Z$ which occur. This definition may be too broad. In particular, the distinction between $p$ brane and $p'$ brane for $p \neq p'$ becomes somewhat fuzzy. Nonetheless, it does provide a geometrical framework which includes sheaves. For example, assume now that $M$ is a complex manifold, and consider the ideal sheaf $\mathcal{I}_D$ of a divisor $D$. Let $z$ be a local defining function for $D$. The connection form $dz/z$ of $\mathcal{I}_D$ is singular along $D$. Hence the n brane $Z_D$ given by the graph of $dz/z$ becomes vertical as it approaches $D$. This singularity cannot be removed by a (finite) affine change of coordinates and reflects the fact that $\mathcal{I}_D$ is not the sheaf of sections of a vector bundle. Assuming in this example that $T^*_B M = T^* M$, we may try to deform $Z_D$ to the zero section $Z_O$, which is the n brane corresponding to the trivial sheaf $O$. Allowing only bounded affine shifts, we are left in the limit with

$$Z_D \to Z_O \cup Z_{O_D},$$

where $Z_{O_D}$ is the brane corresponding to the sheaf $O_D$. This gives a geometrical analog of the exact sequence of sheaves

$$0 \to \mathcal{I}_D \to O \to O_D \to 0.$$

4. **Chern-Simons augmentation of the field strength**

In this section we show for heterotic strings how our treatment of B fields leads to the modification of the B field field strength by the addition of Chern-Simons terms. We will only treat the Yang-Mills term in detail. Some
Let $g$ be the Lie algebra of $SO(32)$ or $E_8 \times E_8$. Let $\mathfrak{h}$ be a Cartan subalgebra. Let $P \subset \mathfrak{h}^*$ denote the set of roots of $g$ with respect to $\mathfrak{h}$, and let $\{\alpha_i\}_{1 \leq i \leq 16}$ be a basis of simple roots. Let $\Gamma$ denote the lattice in $\mathfrak{h}^*$ generated by $P$. With respect to an inner product $(\cdot, \cdot)$ on $\mathfrak{h}$ determined by the Killing form, $\Gamma$ is self dual and every element of $P$ has square length 2.

We use the bosonized description of the heterotic string. So, let $X$ be a 26 dimensional manifold which is a torus bundle with 10 dimensional base $M$ and fiber isometric to $\mathfrak{h}/\Gamma$. Consider an affine cotangent bundle over $X$.

Quotient the affine fibers by the action generated by 1 forms dual to Killing vectors generating the torus lattice to obtain an affine bundle $T^*_{hetB}M$.

For $\alpha \in P$, let $\tau_\alpha$ denote the vector field on $\mathfrak{h}/\Gamma$ which satisfies $\tau_\alpha \beta = (\alpha, \beta)$, for all $\beta \in P$.

Up to the familiar cocycle difficulty [3, Section 6.4 Volume 1] (most easily corrected by adjoining gamma matrix prefactors), the vector fields $e^{2\pi i \alpha} \tau_\alpha$, $\alpha \in P$ and $\tau_i, 1 \leq i \leq 16$ generate an algebra isomorphic to $g$, when endowed with the Lie bracket given by their commutator composed with projection onto their span.

Let $B$ denote a $B$-field on $X$. We modify our prior assumption that $B$ is locally a 2 form pulled back from $X$ with the (chiral) assumption that it is allowed to vary in the $\mathfrak{h}^*/\Gamma$ fiber but only (in an appropriate frame) as $B(x, t, s) = B(x, t+s)$, where $x$ is a local coordinate on $M$, and $t$ and $s$ denote coordinates in $\mathfrak{h}/\Gamma$ and $\mathfrak{h}^*/\Gamma \simeq \mathfrak{h}/\Gamma$ respectively. More precisely, we assume that $B$ has the form

$$B = b_\nu dx^\nu \wedge dx^0 + A_\mu \epsilon^{2\pi i \alpha(t+s)} dx^\nu \wedge d\alpha + A_i dx^\nu \wedge d\alpha_i + \text{massive},$$

where other terms may occur but will be thought of as massive and ignored at this level. Only those terms in $B$ which are invariant in the fiber and which are annihilated by interior multiplication by vectors tangent to the torus fiber descend to a 2 form on $M$. Thus, these lead to the ”$B$ field” $b$ on $M$.

We also allow affine transformations $p \rightarrow p + \lambda$ to vary in the torus fiber; we assume that $\lambda$ has the form

$$\lambda = \lambda^{-\alpha}(x) e^{2\pi i \alpha(t+s)} d\alpha + \lambda^i(x) d\alpha_i + \lambda^0(x) + \text{massive}.$$  

Here $\lambda^0(x)$ is the pull back of a 1 form on $M$ and $\lambda^\alpha$ and $\lambda^i$ are the pull back of functions on $M$. Under this transformation

$$b \rightarrow b + d\lambda^0 - A_\mu dx^\mu \wedge d\lambda^{-\alpha} - A_i dx^\mu \wedge d\alpha_i(\lambda^0)/2,$$

which under a natural identification can be written as

$$b \rightarrow b + d\lambda^0 + Tr A \wedge d\lambda.$$

Similarly we obtain a variation in $A$ which can be written as

$$A \rightarrow A + d(\lambda - \lambda^0) + [A, \lambda].$$
The new term in the variation of the $b$ (not $B$) field differs by an exact term from the usual Yang-Mills modification of the heterotic $B$ field. This leads as usual to the Yang-Mills Chern-Simons modification of the field strength of $b$:

$$H = db - \omega_{YM},$$

but now $H$ arises simply as the component of $dB$ which descends to a form on $M$. Here $\omega_{YM}$ denotes the Yang-Mills Chern-Simons form.

To obtain the gravitational Chern-Simons form, $\omega_L$, requires a bit more work, but seems likely to cast more light on the relation of heterotic string theory to M theory. The affine bundle must be replaced by a principal affine bundle. Then the modified field strength

$$H = db + \omega_L - \omega_{YM}$$

again arises by taking the component of $dB$ which descends to $M$.

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References