Electric/magnetic reciprocity in premetric electrodynamics with and without magnetic charge, and the complex electromagnetic field

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We extend an axiomatic approach to classical electrodynamics, which we developed recently, to the case of non-vanishing magnetic charge. Then two axioms, namely those of the existence of the Lorentz force (Axiom 2) and of magnetic flux conservation (Axiom 3) have to be generalized. Electric/magnetic reciprocity constitutes a guiding principle for this undertaking. The extension of the axioms can be implemented at a premetric stage, i.e., when metric and connection of spacetime don’t play a role. Complex Riemann-Silberstein fields of the form \((E \pm i \mathcal{H}, D \pm i B)\) have a natural place in the theory, independent of the Hodge duality mapping defined by any particular metric.

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I. MAGNETIC CHARGE AND ELECTRIC/MAGNETIC RECIPROCITY

An axiomatic approach to classical electrodynamics without magnetic charges (monopoles) has been developed in [10] (see also [23] and [30]). In this approach, the initial structure of spacetime is that of a 4-dimensional differentiable manifold foliated by means of a monotonically increasing ‘topological time’ parameter $\sigma$ (see also [8]). Maxwell’s equations are expressed in terms of the twisted excitation 2-form $H$, the twisted electric current 3-form $J$, and the untwisted field strength 2-form $F$:

$$dH = J, \quad dF = 0. \quad (1)$$

The different fields decompose into time and space pieces according to

$$H = -\mathcal{H} \wedge d\sigma + \mathcal{D}, \quad F = E \wedge d\sigma + B, \quad (2)$$

$$J = -j \wedge d\sigma + \rho, \quad (3)$$

where, in 3 dimensions, $\mathcal{H}$ and $E$ are 1-forms, $\mathcal{D}$, $B$, and $j$ are 2-forms, and $\rho$ is a 3-form. Eqs. (2) and (3) provide the physical interpretation of the 4-dimensional diffeomorphism and frame invariant Maxwell equations (1) in terms of measurable quantities, namely the electric and the magnetic excitations $\mathcal{D}, \mathcal{H}$, the electric and magnetic field strengths $E, B$, and the electric charge density $\rho$ and its current density $j$, respectively. Up to here, we are still on a premetric level: Neither a metric of spacetime nor a connection are assumed to exist. Additional spacetime structures, including the light cone and other elements of a Lorentzian metric, emerge only after a further condition is imposed via a spacetime relation defining the electromagnetic properties of spacetime (of the ‘vacuum’) [22] (see also [11]).

In Maxwell’s theory as well as in our axiomatics a clear asymmetry is built in between electric and magnetic charges. Magnetic charges have been a matter of numerous theoretical discussions, see, e.g., Schwinger [25, 26], and of experimental investigations. No experimental evidence exists for magnetic charges or monopoles, see He [9], Abbott et al. [1], and Kalbfleisch et al. [16]. Nevertheless, in this note, following a discussion with Kaiser, see [12], we want to address the question how our axiomatics can be adapted to the introduction of magnetic charges.

Let us remind ourselves that for the derivation of Maxwell’s equations (1) we need Axiom 1, namely charge conservation $dJ = 0$, Axiom 2, the existence of the Lorentz force density

$$f_\alpha = (e_\alpha \rfloor F) \wedge J \quad (4)$$
(here \( e_\alpha \) is the frame or vierbein field), and Axiom 3, magnetic flux conservation \( dF = 0 \). Furthermore, for our theoretical analysis Axiom 4 is required, the (still premetric) localization of electromagnetic energy and momentum by means of the twisted kinematic energy-momentum current 3-form

\[
k_\Sigma_\alpha := \frac{1}{2} [F \wedge (e_\alpha] H) - H \wedge (e_\alpha] F)] .
\]

As shown in [10], this current, in contrast to Maxwell’s equations themselves, is electric/magnetic reciprocal, i.e., it is invariant under the substitutions

\[
\begin{align*}
H &\to \zeta F,
\mathcal{H} &\to -\zeta E, \\
\mathcal{D} &\to \zeta B, \\
E &\to \frac{1}{\zeta} \mathcal{H}, \\
B &\to -\frac{1}{\zeta} \mathcal{D},
\end{align*}
\]

with a constant twisted 0-form \( \zeta \) of the dimension of an admittance. In [6], [7], a magnetic quantity is replaced by an electric one and an electric quantity by a magnetic one. Accordingly, we can speak of an electric/magnetic reciprocity of the energy-momentum current \( k_\Sigma_\alpha \). Observe the non-trivial minus sign on the left-hand-side of (7) which is necessary because of the structure of \( k_\Sigma_\alpha \).

In a straightforward way, we can try to introduce an untwisted magnetic current 3-form \( K \) that is conserved according to \( dK = 0 \), in analogy to Axiom 1, and that again could be verified just by counting the monopoles. If this new Axiom 3’ is suitably formulated, we would find, by de Rham’s theorem, the magnetic current \( K \), analogously to the electric current \( J \), as exact 3-form: \( dF = K \). If we uphold Axioms 1, 2, and 4, then we expect an inconsistency since, according to Axiom 2, the electromagnetic field would only act on electric not, however, on magnetic charges.

Before we continue this line of arguments, let us take a look at the ‘new’ Maxwell equations:

\[
dH = J, \quad dF = K.
\]

If we now introduce, in addition to [6], [7], a new substitution for the electric and magnetic currents

\[
J \to \zeta K, \quad K \to -\frac{1}{\zeta} J,
\]
then not only the energy-momentum current is electric/magnetic reciprocal but also the Maxwell equations themselves.

Let us come back to the last but one paragraph. In order to remove the inconsistency mentioned, we have, in addition to Axiom 3, also to change at least one other axiom. As argued, Axiom 2 would now appear to be shaky. This axiom of standard (monopole-free) electrodynamics defines the electromagnetic field strength \( F \) by means of the Lorentz force density acting on a moving electric charge density. The assumption of the existence of magnetic charge requires to modify the Lorentz force axiom correspondingly by also including a piece for the force acting on the magnetic charge. However, magnetic monopoles were never observed, and thus we cannot appeal to experiment in formulating the modification of Axiom 2. As a matter of fact, there exists quite an ambiguity in choosing the form of the new “Lorentz” force density, and we can only rely on theoretical arguments. Among other possibilities, the use of electric/magnetic reciprocity as a guiding principle provides a self-consistent and mathematically nice framework.

If we supplement, in accordance with (6), (7), and (9), the right-hand-side of (4) such that it becomes electric/magnetic reciprocal, we find

\[
    f_\alpha = (e_\alpha [F] \wedge J) - (e_\alpha [H] \wedge K).
\]

The second term describes the force acting on a magnetic charge. Then, following along the same line of thought as in our standard premetric axiomatics, we naturally arrive at the unchanged Axiom 4 for the electromagnetic energy-momentum current, as we specified it in (8). Accordingly, by assuming the existence of hypothetical magnetic charges, one can develop a straightforward generalization of the axiomatic premetric approach by stressing specifically the notion of electric/magnetic reciprocity. The latter can be considered as a premetric counterpart of electric-magnetic duality.

Incidentally, the density of the magnetic monopole force \(- (e_\alpha [H] \wedge K)\) is not unprecedented: For magnetic (north and/or south) poles, a Coulomb like law can be formulated and experimentally verified to some approximation, as was done first by Coulomb himself, see (2). Then the force acting on the unit pole defines the magnetic excitation \( \mathcal{H} \) in much the same way as the electric (monopole) force defines the electric field strength \( E \). Accordingly, we assume that the hypothetical magnetic monopoles behave in an analogous way as the effective magnetic poles of conventional electrodynamics. Needless to say that a magnetic
north pole cannot be separated from its south pole. Therefore the magnetic pole concept
doesn’t survive on a fundamental level.

In order to verify (10), we substitute (8) in (10):

\[ f_\alpha = (e_\alpha]F) \wedge dH - (e_\alpha]H) \wedge dF. \] (11)

This is eq.(B.5.1) of ref.[10] on page 163 in the case of \( a = 1 \). After some simple algebra,
which is spelled out in [10], we arrive at eq.(B.5.4), loc.cit. (with \( a = 1 \)):

\[ f_\alpha = d[F \wedge (e_\alpha]H) - H \wedge (e_\alpha]F)] - F \wedge (\mathcal{L}_{e_\alpha}H) + H \wedge (\mathcal{L}_{e_\alpha}F) - (e_\alpha]F) \wedge dH + (e_\alpha]H) \wedge dF. \] (12)

We denote by \( \mathcal{L}_{e_\alpha} \) the Lie derivative with respect to the frame \( e_\alpha \). We can substitute (5) in (12). Then, with the definition of the force density that is left over,

\[ X_\alpha := -\frac{1}{2} (F \wedge \mathcal{L}_{e_\alpha}H - H \wedge \mathcal{L}_{e_\alpha}F), \] (13)

we find

\[ f_\alpha = 2 d\Sigma_\alpha + 2 X_\alpha - (e_\alpha]F) \wedge dH + (e_\alpha]H) \wedge dF. \] (14)

The last two terms in (14) add up to \(-f_\alpha \), see (10). Thus,

\[ f_\alpha = (e_\alpha]F) \wedge dJ - (e_\alpha]H) \wedge dK = d\Sigma_\alpha + X_\alpha. \] (15)

This conclusion coincides with an earlier result of Kaiser [12].

We are now coming to the central point: Note that all the expressions in Axiom 2’, see (10), and in Axiom 4, see (5), are invariant under the electric/magnetic reciprocity mapping \( \otimes \) defined by

\[ \otimes H = \zeta F, \quad \otimes F = -\frac{1}{\zeta} H, \] (16)

\[ \otimes J = \zeta K, \quad \otimes K = -\frac{1}{\zeta} J. \] (17)

Since \( F \) has the dimension of an action and \( H \) that of a charge, \( \zeta \) has the dimension of an admittance or reciprocal impedance. Furthermore, to keep \( \otimes H \) twisted and \( \otimes F \) twist-free,
ζ must transform as a *pseudoscalar*, changing sign under orientation-reversing diffeomorphisms. Of course, by reciprocity, Axiom 1 is mapped to Axiom 3, and vice versa. Accordingly, as we saw already, the ‘new’ Maxwell equations are also electric/magnetic reciprocal. In the absence of a metric, reciprocity precedes *electric-magnetic duality*, as we will see below. In the generalized Lorentz force density (10), the excitation $H$ (generated by the electric current $J$ via $dH = J$) yields a force on the magnetic current $K$ via $- (e_\alpha H) \wedge K$. The magnetic current $K$ generates the field strength $F$ (via $dF = K$) and $F$, in turn, yields the force $(e_\alpha F) \wedge J$ on $J$. This closes the circle.

It is interesting to look into the physical meaning of the modified Axiom 2 and Axiom 3, namely (10) and (8). In simple terms, they tell us that both electromagnetic fields, $H$ as well as $F$, are simultaneously of the type of excitations (i.e., quantities) *and* of field strengths (i.e., intensities). Thus, if magnetic charges exist, the electric/magnetic reciprocity washes out the difference between these notions and we naturally come to the conclusion that these fields might be combined into a single electric/magnetically symmetric object.

**II. COMPLEX STRUCTURES**

We know that the expressions in (8), (10), and (5) are invariant under the electric/magnetic reciprocity mapping (16). This suggests that we introduce the complex electromagnetic field and its conjugate complex $U$:

$$U := H + i\zeta F, \quad \overline{U} = H - i\zeta F.$$  \hfill (18)

Then the reciprocity mapping yields

$$\circledast U = -iU, \quad \circledast \overline{U} = +i\overline{U},$$  \hfill (19)

that is, $U$ and $\overline{U}$ are *eigenvectors* of the reciprocity mapping and acquire a nontrivial meaning thereby. In physical terms this means that $U$ is, up to a factor, electric/magnetic reciprocal. Furthermore, by implication, $\circledast \circledast U = -U$ and $\circledast \circledast \overline{U} = -\overline{U}$. Therefore the reciprocity mapping induces locally an *almost complex structure* on the underlying spacetime manifold.

If we introduce the complex electric/magnetic current

$$\mathfrak{J} := J + i\zeta K,$$  \hfill (20)
then we can display the complex Maxwell equation as

\[ dU = \mathfrak{J}. \]  

(21)

If and only if we have a nontrivial magnetic current \( K \neq 0 \), electric/magnetic reciprocity extends to the Maxwell equation, since

\[ \otimes \mathfrak{J} = -i \mathfrak{J}. \]  

(22)

However, conventionally, with vanishing magnetic current \( K = 0 \), reciprocity is confined to the energy-momentum 3-form of the electromagnetic field (see (5)), where we observed it in the first place:

\[ \Sigma_\alpha(U) = \frac{i}{4\zeta} \left( \overline{U} \wedge e_\alpha \right) U - \overline{U} \wedge e_\alpha \overline{U}, \]

\[ \otimes \Sigma_\alpha(U) = \Sigma_\alpha(U). \]  

(23)

Our earlier results are already incorporated in this generally covariant and premetric framework developed so far in this section. To make them explicitly visible, we can use a foliation and the “topological” time \( \sigma \). Quite generally, a 2-form decomposes according to

\[ U = -U_\perp \wedge d\sigma + \overline{U}, \]  

(24)

see [10]. Here \( U_\perp \), a 1-form in 3 dimensions, is the timelike (or longitudinal) piece of \( U \) and \( \overline{U} \), a 2-form, the spacelike (or transversal) piece of \( U \). By means of (2) and (24), we find

\[ U_\perp = \mathcal{H} - i\zeta E, \quad \overline{U} = \mathcal{D} + i\zeta B. \]  

(25)

Thus, the timelike component of \( U \), that is, \( U_\perp \), is, apart from the factor \(-i\), the Riemann-Silberstein field \( \zeta E + i\mathcal{H} \) and the spacelike component \( \overline{U} \) is \( \mathcal{D} + i\zeta B \). Accordingly, we can decompose the Maxwell equation (21) by the method quoted. We find,

\[ \dot{\overline{U}} - dU_\perp = \mathfrak{J}_\perp, \quad d\overline{U} = \mathfrak{J}. \]  

(26)

Here the dot is defined by \( \dot{()} := \mathcal{L}_n \), where \( n \) denotes the normal vector of the spacetime folio with respect to which the Lie derivative is taken. In this way, we have separated the evolution equation (26)1 from the constraint equation (26)2; and the extensive quantity the time evolution of which we are studying is the Riemann-Silberstein field \( \mathcal{D} + i\zeta B \).
Complex combination of the type $E \pm iH$ and $D \pm iB$ have appeared in the literature for many years. Bialynicki-Birula [5] has traced the occurrence of such combinations back to Riemann and Silberstein, thus he calls them *Riemann-Silberstein fields*. They have been rediscovered many times and applied to a wide variety of electromagnetic phenomena. Some examples which we are familiar with include the work of Robinson on null fields [24], Trautman on analytic solutions of Lorentz-invariant equations [28, 29], Newman’s interpretation of the analytically continued Coulomb field as the electromagnetic part of the Kerr-Newman solution [13, 20, 21], Mashhoon’s treatment of wave propagation in a gravitational background [17, 18, 19], Kaiser’s construction of electromagnetic wavelets [14], and Bialynicki-Birula’s work on the photon wave function [5] and electromagnetic vortices [6, 7, 15].

The “exterior square” of the complex electromagnetic field

$$U \wedge U = H \wedge H - \zeta^2 F \wedge F + 2i\zeta H \wedge F ,$$ (27)

if decomposed according to (2),

$$U \wedge U = 2d\sigma \wedge (D + i\zeta B) \wedge (H - i\zeta E),$$ (28)

contains both Riemann-Silberstein fields in a symmetric form. The premetric modulus of $U$ turns out to be

$$\overline{U} \wedge U = H \wedge H + \zeta^2 F \wedge F$$
$$= 2d\sigma \wedge (H \wedge D - \zeta^2 B \wedge E).$$ (29)

Let us stress that all the equations (11) to (29) are generally covariant under coordinate and frame transformations. There is no metric involved and no Poincaré group, but rather the diffeomorphism group and the local linear group alone. In other words, this is really a *premetric* approach to Maxwell’s theory.

Let us now turn to *Maxwell-Lorentz electrodynamics* (“linear vacuum electrodynamics”) by postulating Axiom 5, that is, the spacetime relation

$$H = \zeta^* F .$$ (30)

The spacetime metric comes in with the Hodge star operator $\ast$. In SI (the International System of Units), the vacuum admittance is $\zeta = \sqrt{\varepsilon_0/\mu_0}$ and the speed of light $c = 1/\sqrt{\varepsilon_0\mu_0}$. Furthermore $\sigma = t$, with the metric time $t$. Then we find

$$U = \zeta (\ast F + iF), \quad \overline{U} = \zeta (\ast F - iF),$$ (31)
which reduce to the Hodge eigenvectors

\[ *U = iU, \quad \overline{*U} = -i\overline{U}, \]

compare (19). Clearly, \[ **U = -U \] and \[ **\overline{U} = -\overline{U} \]. In this way, the Hodge star operator inherits the almost complex structure from our electric/magnetic reciprocity mapping. Reciprocity is primary, Hodge duality secondary. If \( \star \) is the 3-dimensional Hodge star, then we can decompose (30) as

\[ D = \frac{\zeta}{c} \star E = \varepsilon_0 \star E, \quad B = \frac{1}{c\zeta} \star H = \mu_0 \star H. \]  

(33)

The exterior square \( U \wedge U \) of (27), in the Maxwell-Lorentz case, reduces to

\[ U \wedge U^{ML} = -2\zeta^2 (F \wedge F + i^* F \wedge F). \]  

(34)

In the Lagrangian, \( F \wedge F \) would be a surface term and \( ^* F \wedge F \) be proportional to the Maxwell Lagrangian. Thus \( U \wedge U \) qualifies as a Lagrangian. It can also be expressed as \( U \wedge U^{ML} = -i^* U \wedge U \). For better insight, we can still decompose (34) into 1+3. Then (28) translates into

\[ U \wedge U^{ML} = 2i dt \wedge \star (H - i\zeta E) \wedge (H - i\zeta E). \]  

(35)

Hence the square of \( U \) is closely linked to the modulus of the Riemann-Silberstein covector \( H - i\zeta E \). The Minkowski space analog of (35) is the well-known quantity \( \psi := (\zeta E + iH) \cdot (\zeta E + iH) \). It is, of course, also Poincaré-invariant. Among other things, it plays the key role of polarization scalar in the definition of electromagnetic vortices [6, 7, 15].

We substitute (30) into (29) and find for the modulus of \( U \):

\[ \overline{U} \wedge U^{ML} = \zeta^2 (\star F \wedge \star F + F \wedge F) \]

\[ = \zeta^2 (** F \wedge F + F \wedge F) = 0. \]  

(36)

Thus, in Minkowski space — and, more generally, in all pseudo-Riemannian spacetimes — the 4-form \( U \wedge U \) vanishes.

\section{III. DISCUSSION AND CONCLUSION}

In this paper, we formulated an extension of the axiomatic premetric approach to the case of magnetic charges. Electric/magnetic reciprocity plays a central role in such an extension.
It is instructive to check the absolute dimensions of the physical quantities involved. Let us recall [10] that the absolute dimension of the electric current is that of electric charge, $[J] = q$.

As a result of Axiom 1 and Axiom 2, the absolute dimensions of the electromagnetic field 2-forms are $[H] = q$ and $[F] = h/q$, i.e., electric charge and action per charge, respectively.

Now, looking at the modified Lorentz force density (10), we conclude that the absolute dimension of the magnetic current 3-form is that of action per charge: $[K] = h/q$. Thus the dimension of magnetic charge turns out to be fixed. As a result classical electrodynamics may only have 4 or less independent dimensional units (contrary to Sommerfeld [27], e.g., who operated with 5 units). From a dimensional point of view, the reciprocity operator maps electric charge into action per charge, and vice versa. This fact obviously underlies the well-known charge quantization which arises naturally in the framework of an electrodynamics that includes magnetic monopoles.

Denoting the elementary electric and magnetic charges by $e$ and $g$, respectively, we conclude that the 2-forms $eF$ and $gH$, as well as the 3-forms $eK$ and $gJ$, all have the absolute dimension of an action $h$. In particular,

$$[g] = \frac{h}{q}, \quad [e] = q \implies [g] \times [e] = h. \quad (37)$$

The ultimate quantized nature of the action then clearly indicates that the product of the electric and magnetic charges should be an integer multiple of $h$. Such a qualitative conclusion, based on sheer dimensional analysis within the premetric approach, appears to be in complete agreement with the old observation of Bialynicki-Birula [3, 4] which shows, in essence, that the 2-form $eF - gH = \text{exact \, 2-form} + \text{a possible } \delta \text{-like 2-form}$. Taking the exterior derivative of this relation and using the generalized Maxwell equations (8), one then arrives at the quantization condition of the electric and magnetic charge in the form of Schwinger and Zwanziger, see [3].

The complex electromagnetic field $U$ represents a 4-dimensional version of the Riemann-Silberstein fields. It provides a compact formulation of the electrodynamic field equations with and without magnetic charges and demonstrates the primary character of the metric-free reciprocity operator as compared to the metric-dependent Hodge duality operator.

From a mathematical point of view, the modified axiomatics proposed here is consistent and it naturally generalizes the approach developed earlier in [10]. However, we should honestly mention that some potentially serious physical problems may arise in this frame-
work. In the standard formulation, electromagnetism manifests itself in two fundamentally different physical quantities: excitation and field strength. In simple terms, they represent the answers to the questions “How many?” and “How strong?”, respectively. When we allow for magnetic charge, this important and clear-cut physical distinction of quantities and intensities is lost, which may lead to a possible confusion in the interpretation of the electromagnetic fields $H$ and $F$. When combined with the lack of any experimental evidence for monopoles [1, 9, 16], such a theoretical problem makes us skeptical about the existence of magnetic charges in nature.

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