The double torus as a 2D cosmos: 
Groups, geometry, and closed geodesics.

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Abstract.

The double torus provides a relativistic model for a closed 2D cosmos with topology of genus 2 and constant negative curvature. Its unfolding into an octagon extends to an octagonal tesselation of its universal covering, the hyperbolic space $H^2$. The tesselation is analysed with tools from hyperbolic crystallography. Actions on $H^2$ of groups/subgroups are identified for $SU(1,1)$, for a hyperbolic Coxeter group acting also on $SU(1,1)$, and for the homotopy group $\Phi_2$ whose extension is normal in the Coxeter group. Closed geodesics arise from links on $H^2$ between octagon centers. The direction and length of the shortest closed geodesics is computed.

1 Introduction.

In a publication \cite{7} entitled Cosmic Topology, M. Lachieze-Rey and J.-P. Luminet review topological alternatives for pseudoriemannian manifolds of constant negative curvature as cosmological models and discuss their implications for observations. The general classification of spaces of constant curvature was given by Wolf \cite{11}. Some of these manifolds admit closed (null-) geodesics. Test particles (photons) travelling along such geodesics produce images of their own source and therefore observable effects under conditions described in \cite{7} pp. 189-202. All these simple geometric models of a cosmos neglect the influence of varying mass distributions. These distort locally the curvature and the geodesics assumed in the models for test particles, compare \cite{7} p. 173. Of particular interest for observing the topology, compare \cite{7} for details, are the shortest closed geodesics in at least three ways: (i) They produce the
brightest images of a source and are considered less sensitive to the local distortions. (ii) They determine the first peaks in the autocorrelation function of the matter distribution. (iii) By their length they determine a global radius of the model. If no correspondence between sources and images can be observed, this puts a lower limit on this global radius.

For comparison we sketch an analysis for the 2D torus manifold with Riemannian metric. Its fundamental polygon can be taken as a unit square. Its universal covering manifold is a plane which admits a tesselation by copies of one fundamental initial square. The Euclidean metric on the plane \( E^2 \) induces on the torus a Riemannian metric with zero curvature. The homotopy group of the torus is \( \mathbb{Z} \times \mathbb{Z} \). This homotopy group is isomorphic to a 2D translation group acting on the plane whose elements \((m, n)\) yield the square lattice of center positions of all squares, seen from the initial square. The straight lines on the plane, when pulled back to the reference square, determine its geodesics. Consider now a straight line passing through the center of the initial square. If and only if it hits for the first time the center \((m, n)\) of another square, then \(m, n\) are relatively prime, and the geodesic when pulled back to the initial square will close on the torus. The slope of this closed geodesic is \(n/m\), its length is \(s = \sqrt{m^2 + n^2}\). The problem of closed geodesics on the torus is thus solved by simple crystallographic computations on the plane. These relations are illustrated in Fig. 1. The shortest closed geodesics of length \(s = 1\) clearly run between the centers of the initial square and its 4 edge neighbours. By giving the squares a general fixed edge length one could introduce a global radius of this torus model.

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Fig. 1. The universal covering manifold of the torus is a plane \( E^2 \) tesselated by unit squares. The homotopy group \( \mathbb{Z} \times \mathbb{Z} \) of the torus acts on \( E^2 \) by translations \((m, n)\). The Euclidean metric on \( E^2 \) induces on the torus a Riemannian metric. Geodesic sections between centers of squares on \( E^2 \) become closed geodesics on the torus. On
the left, a geodesic line on $E^2$ of slope $2/3$ and of length $s = \sqrt{3^2 + 2^2}$ connects the centers $(0, 0)$ and $(3, 2)$ on $E^2$. On the right, this geodesic is pulled back to the initial square.

In what follows we examine the continuous and discrete groups and the geometry for the 2D double torus as a paradigm for a cosmos with closed geodesics. The analysis will in concept follow similar steps as sketched above for the torus. In contrast and in detail it will involve the action of crystallographic groups on hyperbolic space with pseudoriemannian metric, a field in which the present authors have been interested [5, 6]. We believe that similar techniques apply to some other topologies of [7].

We start from the information on the double torus given in [7]. The fundamental domain of the double torus is a hyperbolic octagon. It was described by Hilbert and Cohn-Vossen in 1932 [2]. The universal covering manifold of the double torus must admit a tesselation by these octagons. This topological condition enforces the hyperbolic space $H^2$ as the universal covering of the double torus. The homotopy group of the double torus as discussed by Seifert and Threlfall [10] pp. 6, 174 and by Coxeter [11] p. 59 can be converted into a group $\Phi_2$ acting on the universal covering manifold $H^2$. The generators and the single relation for this group were given by Coxeter [11] and Magnus [8]. On the universal covering manifold we shall see that the search for pre-images of closed geodesics becomes a problem of hyperbolic crystallography.

In section 2 we describe the group $SU(1, 1)$, its action and relevant cosets. In section 3 we review the universal covering of the double torus, the hyperbolic space $H^2 \sim SU(1, 1)/U(1)$. Equipped with a pseudoriemannian structure it has constant negative curvature and is invariant under the action of $SU(1, 1)$. In section 4 we give the general group description and group action for $H^2$. The octagonal fundamental domain of the double torus is identified as a double coset of $SU(1, 1)$ with respect to the left action of the homotopy group $\Phi_2$ and the right action of $U(1)$. In section 5 we analyze the homotopy group as a subgroup of a hyperbolic Coxeter group. An extension of the homotopy group is shown to be a normal subgroup of this hyperbolic Coxeter group which by conjugation yields the octagonal symmetry. In section 6 we turn to the geometric action of the homotopy group. A geometric origin of the relation between the generators is given. We characterize on $SU(1, 1)/U(1)$ the condition for closed geodesics. We determine the next neighbour octagons in the tesselation and from them find the shortest closed geodesics on the double torus.

### 2 Groups and actions on Minkowski space $M(1, 2)$.

Here we describe the group $SU(1, 1)$, its universal covering relation to $SO_{1}(1, 2)$, and their action on Minkowski space $M(1, 2)$. 
2.1 The group $SU(1,1)$.

We first discuss group elements $g$ of $SU(1,1)$ and their parameters.

\[
g \in SU(1,1) : \quad gMg^\dagger M = e, \quad M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M^2 = e,
\]

\[
g = \begin{bmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{bmatrix}, |\lambda|^2 - |\mu|^2 = 1,
\]

\[
\lambda = \xi_0 + i\xi_1, \quad \mu = \xi_3 - i\xi_2.
\]

Here $\xi_j$ are real parameters.

Exponential parameters arise as follows: Define in terms of the Pauli matrices the $2 \times 2$ matrix representation of the Lie algebra $su(1,1)$,

\[
\tau_0 := i\sigma_3, \tau_1 := \sigma_2, \tau_2 := \sigma_1, \quad \tau_0^2 = \tau_1^2 = \tau_2^2 = 1,
\]

\[
g = \exp(\tilde{\alpha}h), \quad h + Mh^\dagger M = 0, \quad \tilde{\alpha} \text{ real},
\]

\[
h = \sum_0^2 \omega_j \tau_j = \begin{bmatrix} i\omega_0 & \omega_2 - i\omega_1 \\ \omega_2 + i\omega_1 & -i\omega_0 \end{bmatrix}, \quad (\tilde{\alpha}h)^2 = -(\tilde{\alpha})^2(\omega_0^2 - \omega_1^2 - \omega_2^2)e.
\]

With the exponential parameters, elements of $SU(1,1)$ are described by an angle $\tilde{\alpha}$ and by the vector $\omega = (\omega_0, \omega_1, \omega_2)$ in $M(1,2)$, compare eq. 8. Depending on the value of $|tr(g)|/2 = |(\lambda + \bar{\lambda})|/2 : \langle < 1, > 1, = 1 \rangle$ we have three cases, the elliptic case

\[
1 = (\omega_0^2 - \omega_1^2 - \omega_2^2), \quad \omega_0 = \cos(\tilde{\alpha})e + \sin(\tilde{\alpha})(\omega_0\tau_0 + \omega_1\tau_1 + \omega_2\tau_2),
\]

\[
\xi_0 = \cos(\tilde{\alpha}), \quad \xi_j = \sin(\tilde{\alpha})\omega_{j+1},
\]

the hyperbolic case

\[
-1 = (\omega_0^2 - \omega_1^2 - \omega_2^2), \quad \omega_0 = \cosh(\tilde{\alpha}) + \sinh(\tilde{\alpha})(\omega_0\tau_0 + \omega_1\tau_1 + \omega_2\tau_2),
\]

\[
\xi_0 = \cosh(\tilde{\alpha}), \quad \xi_j = \sinh(\tilde{\alpha})\omega_{j+1},
\]

and the Jordan case

\[
0 = (\omega_0^2 - \omega_1^2 - \omega_2^2), \quad \omega_0 = e + (\omega_0\tau_0 + \omega_1\tau_1 + \omega_2\tau_2),
\]

\[
\xi_0 = 1, \quad \xi_j = \omega_{j+1},
\]
where the real parameters $\xi_j$ of eq. [1] are identified in each case.

An element of elliptic type may be written, compare eqs. [10] and [12] as

\[ g = r_3(\alpha) b_2(\theta) r_3(\gamma) r_3(-\gamma) b_2(-\theta) r_3(-\delta) \]
\[ = \begin{bmatrix}
  \cos(\tilde{\alpha}) + i \cosh(2\theta) \sin(\tilde{\alpha}) & -i \sinh(2\theta) \sin(\tilde{\alpha}) \exp(2i\alpha) \\
  i \sinh(2\theta) \sin(\tilde{\alpha}) \exp(-2i\alpha) & \cos(\tilde{\alpha}) - i \cosh(2\theta) \sin(\tilde{\alpha})
\end{bmatrix}, \]
\[ \omega = (\cosh(2\theta), \sinh(2\theta) \cos(2\alpha), \sinh(2\theta) \sin(2\alpha)). \]

The adjoint action of $SU(1, 1)$ on its Lie algebra can be written by use of eq. [1] as

\[ (g, h) \to ghg^{-1}, \]
\[ (g, -ihM) \to g(-ih)g^{-1}M = g(-ihM)g^\dagger, \]
\[ (-ihM)^\dagger = -ihM. \]

By the adjoint action, the vector components $\omega(g) = (\omega_0, \omega_1, \omega_2)$ associated with $g$ are linearly transformed according to the homomorphism $SU(1, 1) \to SO_\uparrow(1, 2)$, see subsection 2.2. Any vector $\omega(g)$ is fixed under the action of $g$. The vectors $\omega$ for elliptic, hyperbolic and Jordan case correspond to and transform as time-like, space-like and light-like vectors on $M(1, 2)$.

### 2.2 Group action and coset spaces.

We adopt a vector and matrix notation in Minkowski space $M(1, 2)$:

\[ x = (x_0, x_1, x_2), \]
\[ \langle x, y \rangle := x_0y_0 - x_1y_1 - x_2y_2, \]
\[ \tilde{x} := \begin{bmatrix} x_0 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 \end{bmatrix}. \]

The group action on $M(1, 2)$ for $g \in SU(1, 1)$ we define by

\[ (g, \tilde{x}) \to g\tilde{x}g^\dagger, \]
\[ (g, x_i) \to (gx)_i = \sum_j L_{ij}(g)x_j. \]

This action preserves hermiticity, the determinant, and hence the metric on $M(1, 2)$. $L(g)$ with $L(-g) = L(g)$ is the two-to-one homomorphism of $SU(1, 1)$ to $SO_\uparrow(1, 2)$, the orthochronous Lorentz group, compare [4]. We have chosen the action of $SU(1, 1)$
on $M(1, 2)$ in eq. 9 to agree with the adjoint action eq. 7 on the Lie algebra with the exponential parameters eq. 2. We define the elements

$$r_3(\alpha) = \begin{bmatrix} \exp(i\alpha) & 0 \\ 0 & \exp(-i\alpha) \end{bmatrix}, \quad \omega = (1, 0, 0),$$

$$b_2(\theta) = \begin{bmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{bmatrix}, \quad \omega = (0, 0, 1),$$

of $SU(1, 1)$ whose homomorphic images

$$L_3(2\alpha) = L(r_3(\alpha)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(2\alpha) & -\sin(2\alpha) \\ 0 & \sin(2\alpha) & \cos(2\alpha) \end{bmatrix},$$

$$L_2(2\theta) = L(b_2(\theta)) = \begin{bmatrix} \cosh(2\theta) & \sinh(2\theta) & 0 \\ \sinh(2\theta) & \cosh(2\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are a rotation from $U(1)$ or a boost respectively. The factor 2 in the angular parameters reflects our emphasis on the group $SU(1, 1)$. A general group element of $SU(1, 1)$ admits the Euler-type parametrization

$$g = g(\Omega) = g(\alpha, \theta, \gamma) := r_3(\alpha)b_2(\theta)r_3(\gamma).$$

We shall need the multiplication rule for two elements $g_1, g_2$ written in these parameters. A special case of this multiplication is

$$g_1(0, \theta_1, \gamma_1)g_2(0, \theta_2, 0) = g(\alpha, \theta, \gamma),$$

$$\cosh(2\theta) = \cosh(2\theta_1) \cosh(2\theta_2) + \sinh(2\theta_1) \sinh(2\theta_2) \cos(2\gamma_1),$$

$$\sinh(2\theta) \sin(2\alpha) = \sinh(2\theta_2) \sin(2\gamma_1),$$

$$\sinh(2\theta) \sin(2\gamma) = \sinh(2\theta_1) \sin(2\gamma_1).$$

For general products $g_1(\alpha_1, \theta_1, \gamma_1)g_2(\alpha_2, \theta_2, \gamma_2)$ we conclude from eq. 13 that

$$\cosh(2\theta(g_1g_2)) = \cosh(2\theta_1) \cosh(2\theta_2) + \sinh(2\theta_1) \sinh(2\theta_2) \cos(2(\gamma_1 + \alpha_2)).$$

The general multiplication law is obtained from eq. 14 by right- and left- multiplications with rotation matrices of type $r_3(\alpha)$.

The hyperbolic space $H^2$, compare section 3, is the coset space $SU(1, 1)/U(1)$. It is a homogeneous space under the action of $SU(1, 1)$. The points of $SU(1, 1)/U(1)$ we choose as the elements $c \in SU(1, 1)$ whose Euler angle parameters from eq. 12 obey $\gamma(c) = 0$. Any element $g \in SU(1, 1)$ can now be written as

$$g = ch, \ h \in U(1).$$

The vectors $x \in M(1, 2), \langle x, x \rangle = 1$ of $H^2$, and the coset elements $c$ are in one-to-one correspondence, $x \leftrightarrow c$. 6
2.3 Involutive automorphisms of $SU(1, 1)$.

There are two involutive automorphisms $\psi_1, \psi_2$, $\psi_1^2 = e, \psi_2^2 = e$ of $SU(1, 1)$:

\[
\begin{align*}
(\psi_1, g) & \to \overline{g}, \\
(\psi_2, g) & \to (g^\dagger)^{-1}.
\end{align*}
\]

The action of these automorphisms is given explicitly by

\[
\begin{align*}
\psi_1\left(\begin{array}{cc}
\lambda & \mu \\
\overline{\mu} & \overline{\lambda}
\end{array}\right) & = \left(\begin{array}{cc}
\overline{\lambda} & \overline{\mu} \\
\mu & \lambda
\end{array}\right), \\
\psi_2\left(\begin{array}{cc}
\lambda & \mu \\
\overline{\mu} & \overline{\lambda}
\end{array}\right) & = \left(\begin{array}{cc}
\lambda & -\mu \\
-\overline{\mu} & \overline{\lambda}
\end{array}\right).
\end{align*}
\]

The stability subgroups $H_{\psi_i}$ under these automorphisms are easily seen to be $H_{\psi_1} = SO(1, 1) = \langle b_2(\theta) \rangle$, $H_{\psi_2} = U(1) = \langle r_3(\alpha) \rangle$. It follows that the automorphisms must map the coset spaces with respect to these subgroups into themselves.

We shall combine these automorphisms together with the left action $(g_1, g) \to g_1 g$ into a larger group. This larger group admits the involutions $\psi_1(g_1) := g_1 \circ \psi_1 \circ g_1^{-1}, \psi_2(g_2) := g_2 \circ \psi_2 \circ g_2^{-1}$ which act on $SU(1, 1)$ as

\[
\begin{align*}
(\psi_1(g_1), g) & \to (g_1 \circ \psi_1 \circ g_1^{-1})g = g_1(\overline{g_1})^{-1}\overline{g}, \\
(\psi_2(g_2), g) & \to (g_2 \circ \psi_2 \circ g_2^{-1})g = g_2 g_2^{-1}(g^\dagger)^{-1}.
\end{align*}
\]

Here and in what follows we use the symbol $\circ$ to emphasize operator multiplication. From the stability groups for $\psi_1, \psi_2$ it follows that all $\psi_1(g_1), \psi_2(g_2)$ are in one-to-one correspondence to the points of the cosets $g_1 \in SU(1, 1)/SO(1, 1), g_2 \in SU(1, 1)/U(1)$.

2.4 Weyl reflections and involutive automorphisms.

A Weyl reflection acting on $M(1, 2)$ with a Weyl vector $k$ is defined as the involutive map

\[
(W_k, x) \to y = x - 2(\langle x, k \rangle/\langle k, k \rangle) k.
\]

It preserves the scalar product eq. 8, and leaves any point $x$ of the plane $\langle k, x \rangle = 0$ fixed. Weyl reflections act on the coset spaces in $M(1, 2)$. We can distinguish two types of Weyl reflections for $k$ time-like or space-like. All Weyl operators $W_k$ in $M(1, 2)$ with $k$ space-like may be expressed with

\[
\begin{align*}
k & = L(r_3(\alpha))L(b_2(\theta))(0, 1, 0) \\
& = L(r_3(\alpha))L(b_2(\theta))L(r_3(\pi/4))(0, 0, -1) \\
& = (\sinh(2\theta), \cosh(2\theta) \cos(2\alpha), \cosh(2\theta) \sin(2\alpha)).
\end{align*}
\]
Here we let $L(g)$ represent actions $g \in SU(1, 1)$ on $M(1, 2)$. As representative it proves convenient to pass to $k^0 = (0, 0, -1)$ with

$$W_{(0,0,-1)}(x_0, x_1, x_2) \rightarrow (x_0, x_1, -x_2). \quad (21)$$

For Weyl reflections we have the general conjugation property

$$L(g) \circ W_k \circ L(g^{-1}) = W_{L(g)k} \quad (22)$$

which applies in particular to $k^0 = (0, 0, -1)$.

We now wish to define operators acting on $SU(1, 1)$ by left multiplication and by involutive automorphisms which correspond to Weyl reflections. In the disc model of section 3.2, the Weyl reflection of eq. 21 corresponds to complex conjugation, and we therefore shall associate with $k^0 = (0, 0, -1)$ the automorphism $\psi_1$ of eq. 18. For operator products we have the identity

$$\psi_1 \circ g \circ \psi_1 = \overline{g}. \quad (23)$$

It allows to convert operator products containing an even number of automorphisms into pure left multiplications. Guided by eqs. 20 22 we define an involutive automorphism corresponding to a general Weyl vector $k$ by forming in correspondence to eq. 22 the operator product

$$s_k := g \circ \psi_1 \circ g^{-1} = g \circ \psi_1 \circ g^{-1} \circ \psi_1 \circ \psi_1 = g(\overline{g})^{-1} \circ \psi_1, \quad (24)$$

$$g = r_3(\alpha)b_2(\theta)r_3(\pi/4),$$

$$g(\overline{g})^{-1} = r_3(\alpha)b_2(\theta)r_3(\pi/2)b_2(-\theta)r_3(\alpha)$$

$$= \begin{bmatrix} i \cosh(2\theta) \exp(2i\alpha) & i \sinh(2\theta) \\ i \sinh(2\theta) & -i \cosh(2\theta) \exp(-2i\alpha) \end{bmatrix}.$$ 

The matrix $g$ is taken from eq. 20 but expressed on the level of $SU(1, 1)$.

3 The two-dimensional hyperbolic manifold $H^2$.

In this section we introduce the hyperbolic manifold $H^2$ as a hyperboloid in $M(1, 2)$, the hyperbolic disc model, and review the pseudoriemannian structure on $H^2$.

3.1 $H^2$ as a hyperboloid.

In $M(1, 2)$ we consider the hyperboloid $H^2 : \langle x, x \rangle = x_0^2 - x_1^2 - x_2^2 = 1, \ x_0 > 0$, compare Ratcliffe \[9\] pp. 56-104. Its representative point $x^0 = (1, 0, 0)$ has the stability group $U(1) \sim SO(2, R)$ generated by rotations. Therefore the hyperbolic
manifold $H^2$ is the coset space $SU(1,1)/U(1)$. A parametrization of $H^2$ is obtained from eqs. 8-9: 

$$x = L(r_3(\alpha)b_2(\theta))x^0,$$

$$\tilde{x} \rightarrow \tilde{x}(\theta,\alpha) = r_3(\alpha)b_2(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} b_2^\dagger(\theta)r_3^\dagger(\alpha)$$

$$= \begin{bmatrix} \cosh(2\theta) & \sinh(2\theta)\exp(2i\alpha) \\ \sinh(2\theta)\exp(-2i\alpha) & \cosh(2\theta) \end{bmatrix}.$$ 

The action of $SU(1,1)$ on $H^2$ is given in eqs. 8-9. We introduce in $M(1,2)$ intersections of the coset space $SU(1,1)/U(1)$ with planes through $(0,0,0)$ characterized by their normal vector $k$, $\langle k,x \rangle = 0$. These intersections exist only if $k$ is space-like. They always define geodesics, compare [9] pp. 68-70. For $k = (0,1,0)$ we can write the intersection points as $(x_0,x_1,x_2) = (\cosh(2\rho),0,\sinh(2\rho))$. Then $2\rho$ is a geodesic length parameter, compare section 3.2. The application of $r_3(\alpha)b_2(\theta)$ to $k$ and $x$ from eqs. 10-13 yields

$$k \rightarrow (\sinh(2\theta), \cosh(2\theta)\cos(2\alpha), \cosh(2\theta)\sin(2\alpha)), \quad (26)$$

$$x(\rho) = (\cosh(2\theta)\cosh(2\rho), \cos(2\alpha)\sinh(2\theta)\cosh(2\rho) - \sin(2\alpha)\sinh(2\rho), \sin(2\alpha)\sinh(2\theta)\cosh(2\rho) + \cos(2\alpha)\sinh(2\rho)).$$

All the vectors $k$ normal to planes which intersect $H^2$ are space-like and may be written in terms of $(\alpha,\beta)$. Geodesics are mapped into geodesics under $SU(1,1)$. There follows

**Prop 1:** The expression eq. 26 yields the most general geodesics on $H^2$ in a form parametrized by the geodesic length $2\rho$. If we wish to determine the geodesic between two given time-like points $x^1 \neq x^2$, we can determine the normal and the parametrized geodesic from the vector product [9] p. 64-66 $k \sim (x^1 \times x^2)$, which is perpendicular to $x^1, x^2$, by an appropriate normalization and parametrization.

### 3.2 The hyperbolic disc.

The hyperbolic disc parametrization of $H^2$, compare [9] pp. 127-135, is given by the map of the hyperboloid to the interior of the unit circle,

$$x, \langle x,x \rangle = 1 \rightarrow z = (x_1 + ix_2)/(1 + x_0), \quad x_0 \geq 1, |z| \leq 1. \quad (27)$$

with inverse

$$x_0 = (1 + |z|^2)/(1 - |z|^2), \quad x_1 + ix_2 = z 2/(1 - |z|^2). \quad (28)$$
The map from the hyperboloid to the hyperbolic disc is conformal [9] p.8, i.e. preserves the angle between geodesics. The hyperbolic disc represents \( SU(1,1)/U(1) \) within \( M(1,2) \). The action of Lorentz transformations on the disc is nevertheless described by the group \( SU(1,1) \). It is given by the linear fractional transform

\[
g = \begin{bmatrix}
\lambda & \mu \\
\bar{\mu} & \bar{\lambda}
\end{bmatrix},
\]

\[
(g, z) \rightarrow w = \frac{\lambda z + \mu}{\bar{\mu}z + \bar{\lambda}}.
\] (29)

Extend the complex variable \( z \) to the full complex plane \( \mathbb{C} \) and consider a circle of radius \( R \) centered at \( q \in \mathbb{C} \). Its equation is

\[
(z - q)(\bar{z} - q) - R^2 = zz - q\bar{z} - qz + \bar{q}q - R^2 = 0.
\] (30)

Define the vector \( k = k_0(1, \text{Re}(q), \text{Im}(q)) \), \( k_0 > 1 \). The vector \( k \) is space-like if \( |q|^2 \geq 1 \).

For the intersection of the plane perpendicular to \( k \) with \( H^2 \) we find with eqs. 27, 28

\[
0 = \langle k, x \rangle = \frac{1}{2}k_0(1 + x_0)(zz - \bar{q}z - q\bar{z} + 1).
\] (31)

Comparing eq. 30 we find: The geodesic intersection of the plane perpendicular to space-like \( k \) in the hyperbolic disc model is a circle with center \( q = q(k) \), \( |q|^2 \geq 1 \) and of radius \( R = \sqrt{|q|^2 - 1} \). Comparison with the unit circle \( |z|^2 = 1 \) shows that the two circles have perpendicular intersections. An example of a geodesic circle is given in Fig. 2 below. This circle contains an edge line of an octagon.

Since planar intersections in the hyperboloid model map into circles in the hyperbolic disc model, we look for the action of Weyl reflections on the hyperbolic disc. They must be reflections in the circles.

**Prop 2**: The Weyl reflection with space-like unit Weyl vector \( k, \langle k, k \rangle = -1 \) in the hyperbolic disc model determines a non-analytic fractional map

\[
q = (k_1 + ik_2)/(k_0) = (\cosh(2\theta)/(\sinh(2\theta)) \exp(2i\alpha),
\]

\[
k = k_0(1, \text{Re}(q), \text{Im}(q)), \quad k_0 = R^{-1} = 1/(\sqrt{\bar{q}q} - 1),
\]

\[
(W_k, z) \rightarrow w = s_kz = g\bar{g}^{-1}\psi_1z = g\bar{g}^{-1}z
\]

\[
= \frac{a\bar{z} + b}{c\bar{z} + d'} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = g\bar{g}^{-1},
\]

where \( g\bar{g}^{-1} \) is given in eq. 24.

**Proof**: The expression eq. 32 arises by application of the automorphism eq. 24. The automorphism \( \psi_1 \) acting on the hyperbolic disc yields \( z \rightarrow \bar{z} \) and is followed by the fractional transform with the matrix given in eq. 24.
In the hyperbolic disc model, any geodesic circle divides the unit circle into two parts. These two parts are mapped into one another under a Weyl reflection with space-like Weyl vector. Given two points $z^1, z^2$, $z^1 \neq 0$ in the hyperbolic disc model, the unique geodesic circle which passes through both of them can be obtained by inserting $z^1, z^2$ into eq. 30 and solving for the center $q$. One finds

$$q = q(z^1, z^2) = \left( z^1 z^2 - z^1 z^2 \right)^{-1} (z^2 - z^1 - z^1 z^2(z^2 - z^1)).$$ \hspace{1cm} (33)

In case $z^1 z^2 - z^1 z^2 = 0$ when eq. 33 does not apply the geodesic is a straight line $z = \tanh(\rho) z^1 / |z^1|$ through the origin of the unit disc.

3.3 Pseuordiernannian structure and curvature on $H^2$.

For the notation we refer to [7] pp. 146-7. On 2D surfaces in $M(1, 2)$ with coordinates $u^\mu$ we define a pseudodiernannian metric with line element

$$ds^2 = \sum_{\mu\nu} g_{\mu
u} du^\mu du^\nu$$ \hspace{1cm} (34)

$$= \sum_{\mu,\nu} \left( \frac{\partial x_0}{\partial u^\mu} \frac{\partial x_0}{\partial u^\nu} - \frac{\partial x_1}{\partial u^\mu} \frac{\partial x_1}{\partial u^\nu} - \frac{\partial x_2}{\partial u^\mu} \frac{\partial x_2}{\partial u^\nu} \right) du^\mu du^\nu.$$

On $H^2$ with coordinates eq. 25 we obtain with $u^1 = 2\theta, u^2 = 2\phi$

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & -(\sinh(2\theta))^2 \end{bmatrix},$$ \hspace{1cm} (35)

$$g^{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & -(\sinh(2\theta))^{-2} \end{bmatrix},$$

$$ds^2 = -(2d\theta)^2 - (\sinh(2\theta))^2 d(2\phi)^2 \leq 0.$$

For any curve $\theta = \rho, \phi = c_0$ on $H^2$ with parameter $\rho$ we find

$$\left( \frac{ds}{d(2\rho)} \right)^2 = -1,$$ \hspace{1cm} (36)

so that $2\rho$ is a geodesic length parameter. General geodesics with the same length parameter can then be constructed as given in eq. 26.

In general relativity, the geodesics are the world lines followed by free test particles. Null geodesics $ds^2 = 0$ are followed by photons, and geodesics $ds^2 > 0$ are world lines for massive test particles. Note that all points on $H^2$ have space-like separation and so can be connected only by geodesics with $ds^2 < 0$.

From the only non-vanishing derivative $\partial g_{22} / \partial u^1 = -2 \sinh(2\theta) \cosh(2\theta)$ we get as non-vanishing Christoffel symbols

$$\Gamma^1_{22} = -2 \sinh(2\theta) \cosh(2\theta), \hspace{0.5cm} \Gamma^2_{12} = \Gamma^2_{21} = \cosh(2\theta) / \sinh(2\theta).$$ \hspace{1cm} (37)
The element of interest of the Riemannian curvature tensor becomes
\[ R_{1212} = -\left(\sinh(2\theta)\right)^2, \]
and from it the scalar curvature \( R \) and the Ricci tensor \( R_{ij} \) become
\[
R = -2, \quad (R_{ij}) = -\begin{bmatrix} 1 & 0 \\ 0 & (\sinh(2\theta))^2 \end{bmatrix}.
\]
So \( H^2 \) with the metric eq. 35 has constant negative curvature, compare [7] and [9] p. 5.

4 Topology and metric of the double torus.

We wish to consider a 2D cosmological model with the topology of a double torus. The double torus may be unfolded into an octagon as described by Klein [3] pp. 264-268 and by Hilbert and Cohn-Vossen [2] p. 265. This octagon in topology is denoted as the fundamental domain [7] of the double torus. The pairwise gluing of the octagon edges then reflects the topology. All the eight vertices of this octagon represent the same point of the double torus. The universal covering manifold of the double torus from this unfolding must admit a tesselation by octagons. Such a tesselation requires that eight octagons be arranged without overlap around a single vertex. This topological condition enforces the hyperbolic space \( H^2 \) and its geometry as the universal covering [7] of the octagon.

4.1 Group description of the topology.

The topology of the double torus can be characterized by its homotopy group \( \Phi_2 \) whose elements are closed paths starting at the same point. This homotopy group is described in [1, 2]. It has four generators with a single relation between them. Two pairs of generators are associated each to a single torus. The group relation, see eq. 51 below, arises from the gluing of the two tori into the double torus along a glue line, compare Fig. 3 below.

Upon embedding the octagon into its universal covering \( H^2 \), there must exist a fixpoint-free action of the homotopy group \( \Phi_2 \) on \( H^2 \). This action was explicitly given by Magnus [8] and will be described in subsection 5.1. To any element of \( \Phi_2 \) there must correspond one and only one position of an octagon on \( H^2 \). The topological fundamental domain property of the octagon now implies that any orbit on \( H^2 \) under \( \Phi_2 \) has one and only one representative on a single octagon. This property assures the fundamental domain property from the point of view of group action. Points on the intersection of different octagons require an extra treatment.

We now describe the various manifolds associated to the double torus in terms of actions, subgroups and cosets of the group \( SU(1,1) \).

Select as the fundamental domain of \( \Phi_2 \) the octagon \( X_0 \) centred at \( z^0 = (1, 0, 0) \) on \( H^2 \), corresponding to \( z_0 = 0 \) on the disc, see Fig. 2. For any point \( z \) of this
octagon there exists an element \( p \in SU(1,1)/U(1) : z_0 \rightarrow z \). For an arbitrary point \( c \sim z \in H^2 \) interior to an octagon, the tesselation of \( H^2 \) by octagons as fundamental domains implies that there exist unique elements \( \langle \phi \in \Phi_2, p \in SU(1,1)/U(1) \rangle \) such that

\[
c = \phi p, \ \phi \in \Phi_2.
\]  

Combined with eq. 15 we find for arbitrary elements of \( SU(1,1) \) a unique factorization

\[
g = \phi ph, \ \phi \in \Phi_2, \ h \in U(1).
\]  

This is a double coset factorization of \( SU(1,1) \) with respect to the left subgroup \( \Phi_2 \) and the right subgroup \( U(1) \). The group elements \( p \) which generate the points of the initial octagon \( X_0 \) are the representatives of the corresponding double cosets \( \Phi_2 \backslash SU(1,1)/U(1) \).

For the group actions we obtain from eq. 10. Under the left action of \( \Phi_2 \) we find \( (\tilde{\phi}, \phi ph) \rightarrow (\tilde{\phi} \phi) ph \) governed by group multiplication within \( \Phi_2 \). The general left action of \( SU(1,1) \), \( (\tilde{g}, \phi ph) \rightarrow \phi \phi'h' \) implies on \( H^2 \) that, starting from any fixed point of an arbitrary fixed octagon, we can find its unique image on a unique image octagon. The map \( (\tilde{g}, p) \rightarrow p' \) depends on \( \phi \) but is independent of \( h' \). It determines a transitive action of \( SU(1,1) \) on the octagon \( X_0 \) modulo the group \( \Phi_2 \). The group \( SU(1,1) \) with this action on \( X_0 \) is the most general one compatible with the pseudoriemannian metric.

### 4.2 Geodesics on the double torus.

Consider geodesics on \( H^2 \), equipped with the pseudoriemannian metric of subsection 3.3. Viewing them as sections of the hyperboloid in \( M(1,2) \) with planes through \((0,0,0)\) perpendicular to space-like vectors \([9]\) one concludes that they are always infinite. Any such geodesic starting at a point of the initial octagon \( X_0 \) will cross a sequence of octagons. To get the geodesic on the double torus, we must pull its points back to the initial octagon. Both the geodesic property and the geodesic distance are unchanged under this pull-back. The full geodesic on the octagon will consist of all these pull-backs. Take the center \( x^0 = (1,0,0) \) of the initial octagon \( X_0 \) and a geodesic \( x(\tau), x(0) = x^0 \) on \( H^2 \) of fixed direction. Suppose that that geodesic on \( H^2 \) hits at \( \tau = \tau' \) for the first time a center \( x^1 = x(\tau') = L(\phi)x(0) \) of another octagon in the tesselation. Then the pull-back of the geodesic to \( X_0 \) must hit the center of \( X_0 \) after a finite geodesic distance and therefore must close on \( X_0 \). Thus the search for closed and finite geodesics through the center of the initial fundamental domain is converted on the hyperbolic covering manifold \( H^2 \) into the crystallographic search for sections of geodesics, characterized by their length and direction, which connect the centers of octagons in the tesselation. In what follows we shall focus on the representative closed geodesics passing through the center of the octagon. In
subsections 6.2 and 6.4 we shall determine the directions and the shortest geodesic distances between centers of octagons. In the double torus model, these geodesics become the shortest closed geodesics. As explained in the introduction and in detail in [7], the shortest closed geodesics are of interest for observing the topology of the model.

We can compute the geodesic distance from \( x^0 \) to the center \( x^1 = L(\phi)x^0 \) of the image octagon from the group element \( \phi \in \Phi_2 \). When the group element is written in terms of the Euler angles eq. 12 as \( \phi = g(\alpha, \theta, \gamma) \), the scalar product of the vectors \( x_0, x_1 \) pointing to the two centers is determined by the second hyperbolic Euler angle as

\[
\langle x^0, x^1 \rangle = \langle x^0, L(\phi(\alpha, \theta, \gamma))x^0 \rangle = \cosh(2\theta), \theta = \theta(\phi).
\]

Any other point \( y^0 \) inside the octagon \( X_0 \) from eqs. 39 40 can be reached by application of a fixed group element \( p : y^0 = L(p)x^0 \). The image \( y^1 \) of \( y^0 \) under \( \phi \in \Phi_2 \) is in the octagon whose center is \( L(\phi)x^0 \) and located at the point \( L(\phi p)x^0 \). By construction, the geodesic from \( y^0 \) to its image \( \phi y^0, \phi \in \Phi_2 \) is closed on the double torus. The geodesic distance from \( y^0 \) to \( y^1 \) is determined by the hyperbolic cosine

\[
\langle y^0, y^1 \rangle = \langle x^0, L((p^{-1}\phi p)(\alpha', \theta', \gamma'))x^0 \rangle = \cosh(2\theta'), \theta' = \theta'(p^{-1}\phi p).
\]

Here we used the invariance of the scalar product under the Lorentz group \( SO(1, 2) \). This expression generalizes eq. 11. It means that the search for general closed geodesics will depend on the initial point of the octagon. In subsection 6.2 we shall extend the search to vertices of the octagon. An analysis of eq. 12 for general points requires crystallographic elaboration and will not be given here.

The pull-back of any geodesic on \( H^2 \) which does not connect corresponding points for any pair of octagons does not close on \( H^0 \).

In the next sections we describe the details of the octagon and of the homotopy group.

5 The Coxeter and the homotopy group.

We describe the homotopy group of the double torus as a subgroup of a hyperbolic Coxeter group.

5.1 The hyperbolic Coxeter group.

Consider the hyperbolic Coxeter group generated by three Weyl reflections eq. 19 with Weyl vectors and relations

\[
R_1 : k = (0, 0, -1), \quad R_2 : k = (0, \cos(3\pi/8), \sin(3\pi/8)),
\]

(43)
\[ R_3 : k = (\sinh(2\theta), -\cosh(2\theta), 0), \cosh(2\theta) = \cot(\pi/8), \]
\[ R_1^2 = R_2^2 = R_3^2 = e, \]
\[ (R_1R_2)^2 = (R_2R_3)^8 = (R_3R_1)^8 = e. \]

We follow Magnus \cite{magnus} pp. 81-95 who denotes the Coxeter group by \( T(2, 8, 8) \). Products of Weyl reflections in \( M(1, 2) \) generate Lorentz transformations
\[ (R_1R_2) = L(B), (R_2R_3) = L((AB)^{-1}), (R_3R_1) = L(A). \] (44)

which Magnus \cite{magnus} pp. 87-88 expresses as representations of elements \( A, B \in SU(1, 1) \):
\[
A : = \frac{i}{\sin(\alpha_0)} \begin{bmatrix} \cos(\alpha_0) & \rho \\ -\rho & -\cos(\alpha_0) \end{bmatrix},
B : = \begin{bmatrix} \exp(i\alpha_0) & 0 \\ 0 & \exp(-i\alpha_0) \end{bmatrix},
\]
\[
AB : = \frac{i}{\sin(\alpha_0)} \begin{bmatrix} \cos(\alpha_0) \exp(i\alpha_0) & \rho \exp(-i\alpha_0) \\ -\rho \exp(i\alpha_0) & -\cos(\alpha_0) \exp(-i\alpha_0) \end{bmatrix},
\]
\[
\alpha_0 := \pi/8, \cot(\alpha_0) = ((1 + \sqrt{1/2})/(1 - \sqrt{1/2}))^{1/2}, \]
\[
\rho = \sqrt{\cos(\pi/4)} = \sqrt{1/2}. \]

Here the fixed angle \( \alpha_0 = \pi/8 \) is taken from Magnus and should be carefully distinguished from the general Euler angle \( \alpha \) used in eq. 10. All three matrices are of elliptic type. By use of eq. 6 and eq. 12 we find for the exponential and Euler parameters the values
\[
A : \tilde{\alpha} = \pi/2, \omega(A) = (\cot(\alpha_0), -\rho/\sin(\alpha_0), 0), \]
\[
z(A) = \exp(i\pi)/\sqrt{\cos(2\alpha_0)/(1 + \sin(2\alpha_0))}, \]
\[
\Omega(A) = (\pi/2, \arccosh(\cot(\alpha_0)), 0), \]
\[
B : \tilde{\alpha} = \pi/8, \omega(B) = (1, 0, 0), z(B) = 0, \]
\[
\Omega(B) = (\alpha_0, 0, 0), \]
\[
AB : \tilde{\alpha} = \pi - \alpha_0, \omega(AB) = ((\cot(\alpha_0))^2, -\rho \cos(\alpha_0)/(\sin(\alpha_0))^2, \rho/\sin(\alpha_0)), \]
\[
z(AB) = \exp(i7\pi/8)/\sqrt{\cos(2\alpha_0)}, \]
\[
\Omega(AB) = (\pi/2, \arccosh(\cot(\alpha_0)), \alpha_0). \]

The points \( z(g) = (\omega_1 + i\omega_2)/(1 + \omega_0), g = A, B, AB \) represent the fixpoints \( \omega(g) \) in the hyperbolic disc model.
The matrices $A, B, AB$ have the following properties which are relevant under the homomorphic map to the Lorentz group:

$$A^2 = -e, \quad B^8 = -e, \quad (AB)^8 = -e. \quad (47)$$

The vectors $\omega(B), \omega(AB), \omega(A)$ are the vertices of the triangular fundamental domain on $H^2$ for the Coxeter group $T(2,8,8)$, compare Fig. 2. The hyperbolic edge lengths of this triangle are given from eqs. 41 and 46 and from hyperbolic trigonometry, [9] p. 86, by

$$\langle \omega(B), \omega(AB) \rangle =: \cosh(2\theta(B, AB)) = (\cot(\alpha_0))^2, \quad (48)$$
$$\langle \omega(AB), \omega(A) \rangle =: \cosh(2\theta(AB, A)) = \cot(\alpha_0),$$
$$\langle \omega(A), \omega(B) \rangle =: \cosh(2\theta(A, B)) = \cot(\alpha_0),$$

The hyperbolic angle $2\theta(B, AB)$ is defined between the unit vectors $\omega(B), \omega(AB)$.

The hyperbolic triangle has two edges of equal length which form an angle $\pi/2$, and two more angles $\pi/8$.

### 5.2 The homotopy group.

With Magnus [8] we pass from hyperbolic triangles to octagons and from the hyperbolic Coxeter group to the homotopy group $\Phi_2$. The octagon, the fundamental domain of the group $\Phi_2$, is obtained by applying all reflections in the planes containing $\omega(B) = (1,0,0)$ and one of the vectors $(\omega(A), \omega(AB))$. It consists of 16 triangles. The vertices of the octagon are obtained from $\omega(AB)$, the midpoints of edges from $\omega(A)$ by applying all powers of $L(B)$ to these two vectors respectively. The images of the octagon under $\Phi_2$ yield a tesselation of $H^2$.

![Octagon](image.png)

Fig. 2. The octagon that forms the fundamental domain of the double torus in the unit circle $|z| = 1$ of the hyperbolic disc model. The eight edges are parts of
geodesic circles as indicated on the left-hand side. The vertices of the fundamental hyperbolic Coxeter triangle are marked by the complex numbers $z(B), z(AB), z(A)$ which represent the rotation axes of these elliptic elements in the hyperbolic disc model.

In the octagon tessellation of $H^2$, any vertex is surrounded by eight octagons sharing that vertex. The Coxeter group eq. has an involutive automorphism by the interchange of the generators $R_1, R_2$. As a consequence, any Coxeter triangle has two equal angles, and the triangular tessellation has two classes of vertices with congruent surroundings. These classes form the centers and the vertices respectively of the octagon tessellation, compare Fig. 2. Seen from any center of an octagon, the hyperbolic distance of its interior points to its center is smaller than their distance to any other octagon center. So the octagons form hyperbolic Voronoi cells. The tessellation seen not from the octagon centers but rather from their vertex set is a copy of the octagon tessellation. The new octagons around the vertices may be called the dual Delone cells and in shape coincide with the octagons.

The homotopy group $\Phi_2$ as a subgroup of the Coxeter group $T(2, 8, 8)$ can be generated by (an even number of) Weyl reflections in the hyperbolic edges of this triangle. Center and vertices of the octagon are marked by full and open circles respectively. The homotopy group $\Phi_2$ according to Magnus [8] pp. 92-93 is a subgroup of the hyperbolic Coxeter group and has the generators

$$C_0 = AB^{-2}, \quad C_1 = BC_0B^{-1}, \quad C_4 = B^4C_0B^{-4}, \quad C_5 = B^5C_0B^{-5}. \quad (49)$$

which have the Euler angle parameters

$$C_0 : \quad \Omega = (\pi/2, \arccosh(\cot(\alpha_0)), -\pi/4), \quad (50)$$
$$C_1 : \quad \Omega = (5\pi/8, \arccosh(\cot(\alpha_0)), -3\pi/8),$$
$$C_4 : \quad \Omega = (\pi, \arccosh(\cot(\alpha_0)), -3\pi/4),$$
$$C_5 : \quad \Omega = (9\pi/8, \arccosh(\cot(\alpha_0)), -7\pi/8).$$

We call this set-up the Magnus representation of the homotopy group. Note that the group $\Phi_2$ acts without fixpoints and so, in contrast to the Coxeter group, does not contain reflections or rotations.

The generators are subject to the relation

$$(C_0C_1^{-1}C_0^{-1}C_1)(C_4C_5^{-1}C_4^{-1}C_5) = e. \quad (51)$$

As mentioned in subsection 4.1, this relation in topology expresses the gluing of the double torus from two single tori. A geometric interpretation of this group relation will be given in subsection 6.1.
5.3 The Coxeter group acting on $SU(1,1)$.

We wish to implement the Coxeter group in terms of actions on $SU(1,1)$. Our goal is to obtain the Magnus representation eq. 45 of the homotopy group and its embedding into the Coxeter group on the level of $SU(1,1)$ without use of the Lorentz group. Another reason for this analysis can be seen by comparison with free spin-(1/2) fields in special relativity and their discrete versions [6]. These spinors under the unimodular time-preserving Lorentz group transform according to its covering group $SL(2,C)$. Involutions outside this group like parity or Coxeter reflections for them must be introduced by automorphisms. In the present case, spinors on $H^2$ are two-component fields $(\chi_1(x), \chi_2(x))$. A natural action of $g \in SU(1,1)$ on spinors is defined by

$$\begin{bmatrix} (T_g \chi_1)(x) \\ (T_g \chi_2)(x) \end{bmatrix} := g \begin{bmatrix} \chi_1(L^{-1}(g)x) \\ \chi_2(L^{-1}(g)x) \end{bmatrix}.$$  \hspace{1cm} (52)$$

The operators $T_g$ in eq. 52 obey $T_g \circ T_g = T_g^2$. We turn now to the action of involutions on the spinors and employ the involutive automorphisms of section 2.4. To get preimages of the Coxeter group acting on $SU(1,1)$ we use the Weyl vectors from eq. 43 and the parameters of eq. 20 and define by use of eq. 24 three involutive automorphisms of $SU(1,1)$,

$$s_1 : k = (0,0,-1), \ 2\alpha = -\pi/2, \ \theta = 0,$$

$$s_1 = g_1 \bar{g}_1^{-1} \circ \psi_1,$$

$$g_1(\bar{g}_1)^{-1} = e,$$

$$s_2 : k = (0,\cos(3\pi/8),\sin(3\pi/8)), \ 2\alpha = 3\pi/8, \ \theta = 0,$$

$$s_2 = g_2 \bar{g}_2^{-1} \circ \psi_1,$$

$$g_2(\bar{g}_2)^{-1} = r_3(7\pi/8),$$

$$s_3 : k = (\sinh(2\theta), -\cosh(2\theta), 0), \ 2\alpha = \pi, \ \cosh(2\theta) = \cot(\pi/8),$$

$$s_3 = g_3 \bar{g}_3^{-1} \circ \psi_1,$$

$$g_3(\bar{g}_3)^{-1} = r_3(\pi/2)b_2(2\theta)r_3(\pi/2)r_3(\pi/2) = -r_3(\pi/2)b_2(2\theta)$$

$$= \begin{bmatrix} -i \cosh(2\theta) & -i \sinh(2\theta) \\ i \sinh(2\theta) & i \cosh(2\theta) \end{bmatrix} \hspace{1cm} (53)$$

We compute the pairwise products of these automorphisms. The automorphism $\psi_1$ in the products cancel in pairs due to eq. 23 and we obtain operators acting only by left multiplication. We find by comparison with the matrices eq. 45 in the Magnus representation the left actions

$$(s_1 \circ s_2) = \psi_1 \circ r_3(7\pi/8) \circ \psi_1 = -r_3(\pi/8) \circ \psi_1^2 = -r_3(\pi/8)$$

$$= -B, \ (s_2 \circ s_3) = (AB)^{-1}, \ (s_3 \circ s_1) = -A.$$  \hspace{1cm} (54)$$

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The operator relations of eq. 54 lift into the relations eq. 44 of the Coxeter group due to \( L(g) = L(-g) \).

**Prop 3:** The involutive automorphisms \( s_1, s_2, s_3 \) of eq. 53 with \( L(s_i) := R_i, i = 1, 2, 3 \) generate for the Coxeter group eq. 13 a preimage which acts on \( SU(1, 1) \). We shall speak about the Coxeter group generated by \( \langle s_1, s_2, s_3 \rangle \).

### 5.4 Conjugation of \( \Phi_2 \) by the Coxeter group.

The matrices \( A, B \) under conjugation with the three automorphisms \( s_1, s_2, s_3 \) of eq. 53 transform as

\[
\begin{align*}
X & \quad s_1Xs_1 & s_2Xs_2 & s_3Xs_3 \\
A & \quad -A & -B^{-1}AB & -A \\
B & \quad B^{-1} & B^{-1} & -AB^{-1}A
\end{align*}
\]

(55)

We shall show that the homotopy group \( \Phi_2 \), augmented by the element \( B^4 \), is a normal subgroup of the Coxeter group. The element \( B^4, B^{-4} = -B^4 \) has a simple geometric interpretation: In \( M(1, 2) \) it is a rotation by \( 2\tilde{\alpha} = \pi \) and transforms the two tori of the double torus into one another. Writing the generators eq. 49 of \( \Phi_2 \) in terms of the matrices \( A, B \) one finds the conjugation relations

\[
B^4C_0B^{-4} = C_4, B^4C_1B^{-4} = C_5, B^4C_4B^{-4} = C_0, B^4C_5B^{-4} = C_1
\]

(56)

The subgroup generated by \( B^4 \) consists of the elements \( \langle e, -e, B^4, -B^4 \rangle \). Only the identity element is shared with \( \Phi_2 \). We denote the group generated by \( \langle C_0, C_1, C_4, C_5, B^4 \rangle \) as \( \tilde{\Phi}_2 \). It is a semidirect product with \( \Phi_2 \) as the normal subgroup. Now we study the conjugation of the generators of this group under the three automorphisms eq. 53 and obtain by use of eqs. 49, 55, 56

\[
\begin{align*}
i & \quad s_1C_is_1 & s_2C_is_2 & s_3C_is_3 \\
0 & \quad -C_0B^4 & C_5^{-1} & -C_0^{-1} \\
1 & \quad C_5^{-1} & C_4^{-1} & -C_0C_4C_4^{-1}B^4 \\
4 & \quad -C_4B^4 & C_1^{-1} & C_0C_4^{-1}C_0^{-1} \\
5 & \quad C_1^{-1} & C_0^{-1} & C_0C_4C_4^{-1}B^4
\end{align*}
\]

(57)

We also find from eq. 55

\[
s_1B^4s_1 = -B^4, \quad s_2B^4s_2 = -B^4, \quad s_3B^4s_3 = -C_0C_4^{-1}B^4.
\]

(58)

All these conjugations yield again elements of the group \( \tilde{\Phi}_2 \). The inclusion of \( B^4 \) is necessary since it appears in the conjugations of eq. 57. Therefore we find

**Prop 4:** The extension \( \tilde{\Phi}_2 \) of the homotopy group \( \Phi_2 \) is a normal subgroup of the Coxeter group generated by \( \langle s_1, s_2, s_3 \rangle \).

Corresponding relations hold true on the level of the Lorentz group. In view of this property under conjugation we can call the hyperbolic Coxeter group the discrete symmetry group of the octagonal tessellation associated with the double torus.
6 The homotopy group and closed geodesics.

With the information on the homotopy group, its action and symmetry obtained in section 5 we return to the octagonal tesselation and analyse the closed geodesics as outlined in section 4.

6.1 The action of the generators of $\Phi_2$.

By application of the four generators of $\Phi_2$ and their inverses to an initial octagon $X_0$ whose center is located at $z = 0$, we obtain eight images. It turns out that all of them are different edge neighbours of $X_0$. For $C_0 = AB^{-2}$ we have $\theta(C_0) = \theta(A)$ and from eq. 46 $\cosh(\theta(A)) = \cot(\alpha_0)$. From these expressions we obtain for the geodesic distance eq. 41

$$\langle x^0, L(C_0)x^0 \rangle = \cosh(2\theta(C_0)) = 2(\cot(\alpha_0))^2 - 1$$  \hspace{1cm} (59)

Since $L(A)$ is a rotation by $\pi$ in the midpoint of an edge of the central octagon, the image under $C_0$ is an edge-neighbour. The other generators eq. 49 of $\Phi_2$ are conjugates of $C_0$ by powers of $B$. Therefore all the corresponding images of the central octagon under the generators are edge-neighbours. All pairs $X_0, L(\phi)X_0$ of octagons are shown in Figure 3. In Figs. 3 and 4 we omit the hyperbolic geometry of the octagons as displayed in Fig. 2.

Fig. 3. Schematic view of the octagon, the fundamental domain of the double torus. The images of the octagon under the four generators $C_i$ of the homotopy group $\Phi_2$. 

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and their inverses $C_i^{-1}$ are eight edge neighbours. Outward pointing arrows mark the action of these generators. The double torus can be glued from two single tori whose homotopy groups have the generators $C_0$, $C_1$ and $C_4$, $C_5$ respectively. The gluing of these two tori inside the octagon is marked by an arrowed line.

The action of the generators of $\Phi_2$ described so far refers to the full hyperbolic disc as the covering manifold of the double torus. The relation to the homotopy group as described by closed paths on the double torus arises as follows: Take a homotopic path $P$ which passes from $X_0$ into $L(\phi)X_0$ through a shared edge. Mark first this shared entrance edge as seen from the center of $L(\phi)X_0$. Then identify the preimage of this entrance edge seen from the center of $X_0$. The reentry path $P$ into $X_0$ passes through this edge. This information is presented in Figure 3. The path in the homotopy group for any generator may in fact be represented on the covering manifold by a geodesic section between two octagons that share an edge, compare Fig. 4.

We would like to use the information on the action of single generators to find products of generators which, when applied to the central octagon, give a sequence of pairwise edge-sharing images of $X_0$ around a vertex.

The action of a generator $g_i = C_j$ is always obtained from products of Weyl reflections in the edges of the fixed Coxeter triangle in the central octagon shown in Fig. 2. We call it a passive transformation. Consider a word $g = g_1g_2g_3...$ and rewrite it by use of conjugations as

$$g = g_1g_2g_3... = \cdots ((g_1g_2)(g_1g_2)^{-1})(g_1g_2g_1^{-1})g_1 = \cdots g_3^*g_2^*g_1^*.$$  

The word on the left-hand side is expressed on the right-hand side as a product of conjugates $g_j^*$ whose terms appear in reverse order. When acting on the initial central octagon say $X_0$, the term $g_j^*$ conjugate to $g_j$ on the right-hand side acts on the image $L(g_1g_2...g_{i-1})X_0$ of $X_0$. Therefore the right-hand side of eq. 60 expresses a sequence of active transformations, and we learn that these active transformations appear in the reverse order of the sequence of passive transformations.

We apply the active transformations to the passage around the selected vertex $z = \coth(\theta_0) \exp(i7\pi/8)$ of $X_0$. We use the geometric information shown in Figs. 3,4 to pass counterclockwise around this vertex through a sequence of edges. We find the sequences

$$C_5^{-1}, C_4C_5^{-1}, C_5C_4C_5^{-1}, C_4^{-1}C_5C_4C_5^{-1}, C_4^{-1}C_5^{-1}C_5C_4C_5^{-1}, C_4^{-1}C_5^{-1}C_5C_4C_5^{-1},$$

$$C_0C_4^{-1}C_5C_4C_5^{-1}, C_1C_0C_4^{-1}C_5C_4C_5^{-1}, C_0^{-1}C_1C_0C_4^{-1}C_5C_4C_5^{-1}.  \tag{61}$$

which should finally restore the octagon $X_0$. The full passive sequence from eq. 61 is

$$g = C_5^{-1}C_4C_5C_4^{-1}C_1^{-1}C_0C_1C_0^{-1} = e \tag{62}$$
It becomes the identity by using the single relation eq. 51 of the group $\Phi_2$. We have found a geometric interpretation of this relation on $H^2$: the subwords of this relation in the Magnus representation describe active transformations of the octagon around one of its vertices. Similar results apply to all eight vertices, and we list them in Table 1. Moreover we compute the geodesic distance from the central octagon to these neighbours in Table 2.

Fig. 4. Schematic view of the action of the generators of the homotopy group $\Phi_2$ on the central octagon. The arrowed glue line, compare Fig. 3, is marked inside
each octagon. The image of the central octagon under any generator or its inverse is an edge neighbour. The direction of the map is indicated by a white arrow passing through the edge. A homotopic path on the covering manifold could be taken as a geodesic connecting the centers of octagons sharing an edge. The sequence of generators as shown, multiplied from left to right, produces a sequence of images which share a vertex.

Table 1: The words \(w_\nu\) which transform the octagon into images sharing a vertex 

\[
z_\nu = \sqrt{\cos(2\alpha)} \exp(i(2\nu - 1)\pi/8).\]

The full words yield the identity element due to the group relation eq. [51]. The seven images of the octagon around the chosen vertex appear in counterclockwise order if each word is cut after 1, 2, \ldots, 7 entries, counted from left to right.

<table>
<thead>
<tr>
<th>(\nu)</th>
<th>(w_\nu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(C_4^{-1}C_0C_2C_0^{-1}C_5^{-1}C_4C_3C_4^{-1})</td>
</tr>
<tr>
<td>1</td>
<td>(C_4C_5C_4^{-1}C_1^{-1}C_0C_1C_0^{-1}C_5^{-1})</td>
</tr>
<tr>
<td>2</td>
<td>(C_3C_4^{-1}C_1^{-1}C_0C_1C_0^{-1}C_5^{-1}C_4)</td>
</tr>
<tr>
<td>3</td>
<td>(C_4^{-1}C_1^{-1}C_0C_1C_0^{-1}C_5^{-1}C_4C_5)</td>
</tr>
<tr>
<td>4</td>
<td>(C_5^{-1}C_4C_3C_4^{-1}C_1^{-1}C_0C_1C_0^{-1})</td>
</tr>
<tr>
<td>5</td>
<td>(C_3C_4^{-1}C_1^{-1}C_0^{-1}C_5^{-1}C_4C_5C_4^{-1}C_1^{-1})</td>
</tr>
<tr>
<td>6</td>
<td>(C_1C_0^{-1}C_5^{-1}C_4C_3C_4^{-1}C_1^{-1}C_0)</td>
</tr>
<tr>
<td>7</td>
<td>(C_0^{-1}C_5^{-1}C_4C_5C_4^{-1}C_1^{-1}C_0C_1)</td>
</tr>
</tbody>
</table>

6.2 Vertex neighbours.

We shall use the self-dual property of the octagon tesselation explained in subsection 5.2 as follows: Instead of analyzing eight octagon centers around a single vertex we can equally well consider eight vertices around a single center. Instead of geodesics passing through octagon centers we now choose those passing through octagon vertices.

Choosing a geodesics starting at a fixed vertex, we may require it to run inside the octagon and hit another vertex. In this way we can avoid any pull-back for these shortest geodesics. These geodesics are dual to the geodesics between the centers of octagons sharing a vertex.

The positions of the eight vertices seen from the center \(z = 0\), see Fig. 2, are given from eqs. 45, 46 by

\[
z_\nu = \sqrt{\cos(2\alpha)} \exp(i(2\nu - 1)\pi/8), \quad \nu = 1, \ldots, 7.\]

From eq. 33 we obtain by the insertions \(z_1 \rightarrow z_0, z_2 \rightarrow z_\nu\) for the center of the geodesic circle that connects \(z_0\) with \(z_\nu, \nu \neq 0:\)

\[
q(z_0, z_\nu) = \exp(i\pi(\nu - 1)/8) \cotanh(2\theta(AB))/\cos(\pi\nu/8) \quad (64)
\]
We compute the hyperbolic cosine for all pairs of vertices and find from hyperbolic trigonometry, compare [9] pp. 83-91,
\[
\cosh(2\theta(P_0, P_\nu)) = \cosh(2\theta(P_0)) \cosh(2\theta(P_\nu)) - \sinh(2\theta(P_0)) \sinh(2\theta(P_\nu)) \cos(\nu\pi/4) = (\cot(\alpha))^4(1 - \cos(\nu\pi/4)) + \cos(\nu\pi/4)
\]

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$\nu\pi/4$</th>
<th>$\cos(\nu\pi/4)$</th>
<th>$\cosh(2\theta(P_0, P_\nu))$</th>
<th>multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1, 7</td>
<td>$\pm\pi/4$</td>
<td>$\sqrt{1/2}$</td>
<td>$(1 - \sqrt{1/2})^{-2}(-1/2 + 2\sqrt{1/2})$</td>
<td>8</td>
</tr>
<tr>
<td>2, 6</td>
<td>$\pm\pi/2$</td>
<td>0</td>
<td>$(1 - \sqrt{1/2})^{-2}/(3/2 + 2\sqrt{1/2})$</td>
<td>16</td>
</tr>
<tr>
<td>3, 5</td>
<td>$\pm3\pi/4$</td>
<td>$-\sqrt{1/2}$</td>
<td>$(1 - \sqrt{1/2})^{-2}(7/2 + 2\sqrt{1/2})$</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>$\pm\pi$</td>
<td>$-1$</td>
<td>$(1 - \sqrt{1/2})^{-2}(3/2 + 6\sqrt{1/2})$</td>
<td>8</td>
</tr>
</tbody>
</table>

The multiplicity counts geodesic sections of equal length and starting point but of different directions which result from the octagonal symmetry. The cases $\nu = 1, 7$ yield the same distance as between centers of edge neighbours. The point $P_4$ results from a rotation by $\pi$ and yields the largest hyperbolic distance corresponding to a vertex neighbour.

We claim that the four octagons closest to the central octagon are four vertex neighbours. Therefore they determine the shortest closed geodesics of the double torus.
6.3 Products of generators.

We consider products of the generators $C_i$ eq. 19. All possible products of two of them up to some inversions can be written in terms of the matrices $A, B$ eq. 15 according to the following Table 3.

Table 3: Products of pairs of generators of $\Phi_2$ up to inversion.

<table>
<thead>
<tr>
<th>$i,j$</th>
<th>$C_iC_j$</th>
<th>$C_i(C_j)^{-1}$</th>
<th>$(C_i)^{-1}C_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,1</td>
<td>$AB^{-1}AB^{-3}$</td>
<td>$-ABAB^{-1}$</td>
<td>$-B^2ABAB^{-3}$</td>
</tr>
<tr>
<td>1,0</td>
<td>$BAB^{-3}AB^{-2}$</td>
<td>$-BAB^{-1}A$</td>
<td>$-B^3AB^{-1}AB^{-2}$</td>
</tr>
<tr>
<td>0,4</td>
<td>$AB^2AB^{-6}$</td>
<td>$-AB^4AB^{-4}$</td>
<td>$-B^2AB^4AB^{-6}$</td>
</tr>
<tr>
<td>4,0</td>
<td>$B^4AB^{-6}AB^{-2}$</td>
<td>$-B^4AB^{-4}A$</td>
<td>$-B^6AB^{-4}AB^{-2}$</td>
</tr>
<tr>
<td>0,5</td>
<td>$AB^3AB^{-7}$</td>
<td>$-AB^5AB^{-5}$</td>
<td>$-B^2AB^5AB^{-7}$</td>
</tr>
<tr>
<td>5,0</td>
<td>$B^5AB^{-7}AB^{-2}$</td>
<td>$-B^5AB^{-5}A$</td>
<td>$-B^7AB^{-5}AB^{-2}$</td>
</tr>
<tr>
<td>1,4</td>
<td>$BABAB^{-6}$</td>
<td>$-BAB^3AB^{-4}$</td>
<td>$-B^3AB^3AB^{-6}$</td>
</tr>
<tr>
<td>4,1</td>
<td>$B^4AB^{-5}AB^{-3}$</td>
<td>$-B^4AB^{-3}AB^{-1}$</td>
<td>$-B^6AB^{-3}AB^{-3}$</td>
</tr>
<tr>
<td>1,5</td>
<td>$BAB^2AB^{-7}$</td>
<td>$-BAB^4AB^{-5}$</td>
<td>$-B^3AB^4AB^{-7}$</td>
</tr>
<tr>
<td>5,1</td>
<td>$B^5AB^{-6}AB^{-3}$</td>
<td>$-B^5AB^{-4}AB^{-1}$</td>
<td>$-B^7AB^{-4}AB^{-3}$</td>
</tr>
<tr>
<td>4,5</td>
<td>$B^4AB^{-1}AB^{-7}$</td>
<td>$-B^4ABAB^{-5}$</td>
<td>$-B^6ABAB^{-7}$</td>
</tr>
<tr>
<td>5,4</td>
<td>$B^5AB^{-3}AB^{-6}$</td>
<td>$-B^5AB^{-1}AB^{-4}$</td>
<td>$-B^7AB^{-1}AB^{-6}$</td>
</tr>
<tr>
<td>0,0</td>
<td>$AB^{-2}AB^{-2}$</td>
<td>$e$</td>
<td>$e$</td>
</tr>
<tr>
<td>1,1</td>
<td>$BAB^{-2}AB^{-3}$</td>
<td>$e$</td>
<td>$e$</td>
</tr>
<tr>
<td>4,4</td>
<td>$B^4AB^{-2}AB^{-6}$</td>
<td>$e$</td>
<td>$e$</td>
</tr>
<tr>
<td>5,5</td>
<td>$B^5AB^{-2}AB^{-7}$</td>
<td>$e$</td>
<td>$e$</td>
</tr>
</tbody>
</table>

6.4 Geodesic distances for products of generators.

We wish to have a geometric interpretation for the action of products of generators on the octagon. Consider the product $C_i C_j$ of two generators. If we define $C_j X_0 := X_j$ as a reference octagon, we have $X_0 = C_j^{-1} X_j$, $(C_i C_j) X_0 = C_i X_j$. Therefore we find that $X_0$ and $(C_i C_j) X_0$ are edge neighbours to the single octagon $X_j$. When running through all products of generators, we run through all pairs of different edge neighbours to a single octagon. We expect to find only four different geodesic distances between such pairs.

We now compute for the products of generators the geodesic distances from the matrix products of Table 3. It suffices to compute the second hyperbolic Euler angles for the matrices $AB^\mu A$, $\mu = 1, \ldots, 7$ since all the group elements in Table 2 arise from $AB^\mu A$ by right- and left-multiplication with powers of $B$. To these triple products we apply eqs. 13, 14 and use the Euler angle parameters from eq. 16. First we find from $AB^\mu A$ for the angle corresponding to $2\gamma_1$ in eq. 13: $\cos(2\gamma_1) =$
\[
\cos(\mu \pi/4 + \pi) \quad \text{with values given in column 3 in Table 4. From eq. 14 and eq. 46 we then get}
\]
\[
\cosh(2\theta(AB^\mu A)) = (\cosh(2\theta(A)))^2 + (\sinh(2\theta(A)))^2 \cos(\mu \pi/4 + \pi) \\
= (2(\cot(\alpha))^2 - 1)^2(1 + \cos(\mu \pi/4 + \pi)) - \cos(\mu \pi/4 + \pi)
\]

which gives Table 4.

**Table 4:** Hyperbolic cosine for geodesic distances of octagon images under the products of Table 3.

<table>
<thead>
<tr>
<th>(\mu)</th>
<th>(\mu \pi/4 + \pi)</th>
<th>(\cos(\mu \pi/4 + \pi))</th>
<th>(\cosh(2\theta(AB^\mu A)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\pi)</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1, 7</td>
<td>5(\pi/4, 3\pi/4)</td>
<td>(-\sqrt{1/2})</td>
<td>((1 - \sqrt{1/2})^2(3/2 + 2\sqrt{1/2}))</td>
</tr>
<tr>
<td>2, 6</td>
<td>3(\pi/2, 5\pi/2)</td>
<td>0</td>
<td>((1 - \sqrt{1/2})^2(11/2 + 6\sqrt{1/2}))</td>
</tr>
<tr>
<td>3, 5</td>
<td>7(\pi/4, 9\pi/4)</td>
<td>(\sqrt{1/2})</td>
<td>((1 - \sqrt{1/2})^2(19/2 + 10\sqrt{1/2}))</td>
</tr>
<tr>
<td>4</td>
<td>2(\pi)</td>
<td>1</td>
<td>(\cosh(4\theta(A)) = (1 - \sqrt{1/2})^2(19/2 + 14\sqrt{1/2}))</td>
</tr>
</tbody>
</table>

From the expressions in Table 1 we can infer in particular all the products of two generators which move an octagon counterclockwise around a vertex. They are: 

\(C_1^{-1}C_0, C_0C_1, C_1C_0^{-1}, C_0^{-1}C_5^{-1}, C_5^{-1}C_4, C_4C_5, C_5C_4^{-1}, C_4^{-1}C_1^{-1}\). 

All of them from Table 3 correspond to \(\mu = 7\). We therefore expect that these cases in Table 4 yield the same geodesic distance as \(\nu = 2\) in Table 2 which is the case. The case \(\mu = 4\) in Table 4 gives twice the geodesic distance of an edge neighbour, the centers of the three octagons are on a single geodesic. The pull-back of this geodesic section runs twice over the shortest closed geodesic.

### 7 Conclusion and Outlook.

The double torus provides a model for a 2D octagonal cosmos with a topology of genus 2. We study the geometry of this model by passing to its universal covering, the hyperbolic space \(H^2\). Insight into the model is gained from the action of groups and subgroups. The continuous group \(SU(1,1)\) acts on \(H^2\) and, modulo the homotopy group \(\Phi_2\), on the octagon. This action determines the homogeneity of both spaces. A discrete hyperbolic Coxeter group acts as a subgroup on \(SU(1,1)\). An extension of the homotopy group \(\Phi_2\) of the double torus is a normal subgroup of this hyperbolic Coxeter group. So the hyperbolic Coxeter group expresses by conjugation the discrete symmetry of the octagonal tesselation of \(H^2\). Representative closed geodesics of the double torus model have on \(H^2\) the crystallographic interpretation of geodesic links between octagon centers or vertices. Four shortest geodesic sections connect pairs of vertices of the octagon. The directions and geodesic distances for these and the next closed geodesics are explicitly computed.
8 Acknowledgments.

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