MAGNETIC BACKGROUNDS
AND NONCOMMUTATIVE FIELD THEORY*

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This paper is a rudimentary introduction, geared at non-specialists, to how noncom-
mutative field theories arise in physics and their applications to string theory, particle
physics and condensed matter systems.

Keywords: Noncommutative field theory, string theory, quantum gravity, quantum Hall
systems.

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1. Introduction

The idea behind spacetime noncommutativity is to replace the coordinates $x^i$ of spacetime by Hermitian operators (also denoted $x^i$) which obey the commutation relations

$$[x^i, x^j] = i \theta^{ij}, \quad (1)$$

where $\theta^{ij}$ is an antisymmetric tensor that may be constant, a function of the coordinates $x^i$ themselves, or a function of both coordinates and momenta. In the first instance the operators $x^i$ essentially define a Heisenberg algebra, while in the last case they generate an algebra of pseudo-differential operators. This idea dates back to the 1930’s and is attributed to Heisenberg, who proposed it as a means to control the ultraviolet divergences which plagued quantum field theory. It was purported to ameliorate the problem of infinite self-energies in a Lorentz-invariant way (for appropriate choices of $\theta^{ij}$). The first phenomenological realization of this idea took place in particle physics but rather in condensed matter physics by Peierls, who applied it to non-relativistic electronic systems in external magnetic fields (the celebrated Peierls substitution).

A toy model of this realization comes about from taking $\theta^{ij}$ in (1) to be real-valued constants. In this case the $\theta^{ij}$ play a role completely analogous to Planck’s constant $\hbar$ in the quantum phase space relation $[x^i, p_j] = i \hbar \delta^i_j$. In particular, there is a spacetime uncertainty relation

$$\Delta x^i \Delta x^j \geq \frac{1}{2} |\theta^{ij}|,$$  

(2)

which implies that $|\theta^{ij}|$ measures the smallest patch of observable area in the ($ij$)-plane. This gives a limit to the resolution in which one may probe spacetime itself, and hence gives insight into short-distance spacetime structure. The spacetime becomes “fuzzy” at very short distances, as there is no longer any definite notion of a ‘point’. Such ideas are very common in models of quantum gravity, which predicts that classical general relativity breaks down at the Planck scale and requires a modification of the classical notions of geometry. The recent surge of excitement in the subject has come about from the discovery that such scenarios are realized explicitly in string theory with D-branes.$^{3,4,5,6,7}$

The purpose of this article is to provide a rudimentary exposition of the interrelationships between the ideas of noncommutative geometry that we have described above. The material is geared at the reader with a reasonable background in theoretical physics, but no detailed prior knowledge of noncommutative geometry or
string theory. We will begin by presenting a very simple quantum mechanical model, the Landau problem, which represents the simplest framework in which one can see noncommutative field theory emerging as an effective description of the dynamics (Section 2). We will then briefly describe how this scenario emerges in string theory (Section 3), and how it leads to the study of noncommutative quantum field theory (Section 4). We also describe various potential applications of this formalism to processes in particle physics and in astrophysics. Finally, in Section 5 we turn our attention back to the framework of Section 2 and describe a novel application of noncommutative field theory in condensed matter physics to the fractional quantum Hall effect. More extensive reviews of noncommutative field theory may be found in Refs. 12, 13, 14, where more complete lists of references are also given.

2. Strong Magnetic Fields

In this Section we will describe how the fundamental notions of noncommutative field theory arise in what is perhaps the simplest possible physical setting, namely the quantum mechanics of the motion of charged particles in two dimensions under the influence of a constant, perpendicularly applied magnetic field. This introduces the main technical points that the string theory inspirations do, as we will describe in the next Section, but within a much simpler framework. It also makes contact with the historical development of the subject described in the previous Section and will be one of the motivations for the application of noncommutative field theory that we describe in Section 5. A similar introduction to noncommutativity is presented in Ref. 15.

2.1. The Landau problem

The Landau problem deals with a system of $N_e$ non-relativistic, interacting electrons moving in two-dimensions. We denote their position coordinates and velocities respectively by

$$ r_I = (x_I, y_I) = (x_I^1, x_I^2), \quad v_I = \dot{r}_I, $$

with $I = 1, \ldots, N_e$. The two-dimensional system is subjected to a constant, external perpendicularly applied magnetic field $B = B \hat{z}$ (Fig. 1). We will work in the gauge where the corresponding vector potential is of the form

$$ A(r_I) = (0, B x_I) $$

with $B = \nabla \times A$. The Lagrangian governing this motion is then given by

$$ L = \sum_{I=1}^{N_e} \left( \frac{m_e}{2} v_I^2 + \frac{e}{c} v_I \cdot A(r_I) - V(r_I) \right) - \sum_{I<J} U(r_I - r_J), $$

where $V$ is the electron self-energy due its interaction, say, with an impurity which is externally introduced into the system, and $U$ is a pair-interaction potential between the electrons with the hard-core condition $U(0) = 0$. 
Fig. 1. Set-up for the Landau problem. A system of electrons moves in two-dimensions under the influence of an externally applied, constant perpendicular magnetic field.

Canonical quantization of this system proceeds in the usual way giving the Hamiltonian operator

$$H = \sum_{I=1}^{N_e} \left( \frac{\pi_I^2}{2m_e} + V(r_I) \right) + \sum_{I<J} U(r_I - r_J),$$

(6)

where

$$\pi_I = m_e \nu_I = p_I - \frac{e}{c} A(r_I)$$

(7)

is the (non-canonical) gauge-invariant kinematical momentum, while $p_I$ is the canonical momentum obeying the usual commutation relations

$$[x_I, p_J^x] = i\hbar \delta_{IJ}, \quad [y_I, p_J^y] = i\hbar \delta_{IJ},$$

$$[x_I, y_J] = [p_I^x, p_J^y] = 0,$$

(8)

and so on. From (4) and (8) it follows that the components of the kinematical momentum (7) have the non-vanishing quantum commutators

$$[\pi_I^x, \pi_J^y] = i\hbar \frac{eB}{c} \delta_{IJ}.$$

(9)

Thus the physical (i.e. gauge invariant) momenta of the electrons in the background magnetic field live in a noncommutative space.

The quantum momenta $\pi_I$ can be written in terms of harmonic oscillator creation and annihilation operators. In the absence of interactions, $V = U = 0$, the energy eigenvalues of the normal-ordered Hamiltonian (6) are thus those of Landau levels

$$E = \sum_{I=1}^{N_e} \hbar \omega_c \left( n_I + \frac{1}{2} \right), \quad n_I = 0, 1, 2, \ldots,$$

(10)

where

$$\omega_c = \frac{eB}{m_e c}$$

(11)

is the cyclotron frequency of the classical electron orbits in the magnetic field. The mass gap between Landau levels is the constant $\Delta$ given by

$$\Delta = \frac{\hbar}{2} \omega_c.$$

(12)
In the next Subsection we will examine the Landau problem in the limit whereby this mass gap becomes very large and all excited Landau levels decouple from the ground state which has quantum numbers $n_I = 0$ for all $I = 1, \ldots, N_e$.

### 2.2. The lowest Landau level

In the previous Subsection we encountered a very simple situation in which the momentum space of a physical system is noncommutative. To see how a noncommutative coordinate space arises, let us consider the strong field limit $B \to \infty$, i.e. the energy regime $B \gg m_e$, or equivalently the (formal) limit of small electron mass $m_e \to 0$. In this limit the Lagrangian (5) reduces to

$$L \longrightarrow L_0 = \sum_{I=1}^{N_e} \left( \frac{eB}{c} x_I \dot{y}_I - V(x_I, y_I) \right) - \sum_{I<J} U(r_I - r_J).$$

(13)

For each $I = 1, \ldots, N_e$, this Lagrangian is of the form $p \dot{q} - h(p, q)$, and so the coordinates $\left( \frac{eB}{c} x_I, y_I \right)$ form a canonical pair giving

$$[x_I, y_I] = \frac{i}{\hbar} c eB.$$

(14)

These relations also follow formally from (7) and (9) in the limit $B \to \infty$ with the symmetric gauge choice $A(r_I) = \frac{1}{2} (-B y_I, B x_I)$.

Let us now examine the precise meaning of the limit taken above. Since the cyclotronic frequency (11) diverges in the limit $B \to \infty$ (or $m_e \to 0$), the spacing (12) between Landau levels becomes infinite and the lowest $n_I = 0$ level decouples from all of the rest. Thus the strong field limit projects the quantum mechanical system onto the lowest Landau level. This limit is in fact a phase space reductive one. Since the reduced Lagrangian (13) is of first order in time derivatives, it effectively turns the coordinate space into a phase space. In other words, the original four-dimensional phase space (per electron) degenerates into the two-dimensional configuration space. We conclude that noncommuting coordinates arise in electronic systems constrained to lie in the lowest Landau level.

We can write the commutation relations in the form introduced in the previous Section as

$$\left[ x^I_I, x^J_J \right] = i \delta_{IJ} \theta^{ij},$$

(15)

where the noncommutativity parameters $\theta^{ij}$ are given by

$$\theta^{ij} = \frac{\hbar c}{eB} \epsilon^{ij}$$

(16)

with $\epsilon^{ij}$ the antisymmetric tensor. The present context is in fact the one in which the Peierls substitution was originally carried out in 1933. If one introduces an impurity, described by a potential energy function $V$, into the electronic system as in (13), then one can compute the first order energy shift in perturbation theory,
due to the impurity, of the lowest Landau level by taking the components of the position coordinates $r_I = (x_I, y_I)$ in $V(r_I)$ to be noncommuting variables.

Let us remark that one could have also arrived at this conclusion within the Hamiltonian formalism. In the limit described above, the Hamiltonian (6) reduces to

$$H \longrightarrow H_0 = \sum_{I=1}^{N_e} V(r_I) + \sum_{I<J} U(r_I - r_J).$$

(17)

This reduced Hamiltonian describes a topological theory, in that it vanishes in the absence of the potentials, whereby there are no propagating degrees of freedom. On the other hand, the kinematical momenta (7) in this limit become

$$\pi_I = m_e v_I \longrightarrow 0,$$

(18)

and the condition $\pi_I \equiv 0$ should be treated as constraints on the theory. Since according to (9) they do not commute, they are second class constraints in the usual Dirac classification of constrained mechanical systems. This requires us to replace canonical Poisson brackets with Dirac brackets, whose quantization under the correspondence principle of quantum mechanics gives the coordinate noncommutativity (14).

### 2.3. Field theory

We now investigate the consequences of noncommutativity on second quantization of the system, i.e. in its effective non-relativistic field theory description. For this, we introduce the classical electron density

$$\rho(r) = \sum_{I=1}^{N_e} \delta^2(r - r_I)$$

(19)

which defines the number operator for the many-body system with $N_e = \int d^2r \rho(r)$. Using it, we rewrite the Hamiltonian (17) in the lowest Landau level as

$$H_0 = \int d^2r \rho(r) V(r) + \frac{1}{2} \int \int d^2r \ d^2r' \rho(r) U(r - r') \rho(r') .$$

(20)

The quantum density operator is defined in terms of electron creation and annihilation operators $\psi^\dagger(r)$ and $\psi(r)$ as

$$\rho(r) = \psi^\dagger(r) \psi(r).$$

(21)

However, it is difficult to define (19) as a quantum operator. We bypass this problem by working instead in momentum space with the Fourier transform

$$\hat{\rho}(k) = \int d^2r \rho(r) \ e^{ik\cdot r} .$$

(22)
Since $r_I$ is a noncommuting operator, we must specify an ordering for (22). We shall use symmetric or Weyl ordering defined by specifying the Fourier transform as

$$\tilde{\rho}(k) = \sum_{I=1}^{N} e^{ik \cdot r_I},$$

which differs from normal ordering, say, by a momentum dependent phase factor,

$$\tilde{\rho}(k) = e^{i k \cdot q} e^{i k \cdot q} = e^{i (k+q)^2}.$$ (24)

We can compute the commutation relations of the density operators (23) by using (15) and the Baker-Campbell-Hausdorff formula to write

$$e^{i k \cdot r_I} e^{i q \cdot r_I} = e^{-i k \times q} e^{i (k+q) \cdot r_I},$$ (25)

where we have defined the two-dimensional cross-product

$$k \times q = k_i \theta^{ij} q_j$$ (26)

and the noncommutativity parameters $\theta^{ij}$ are given by (16). We thereby find that the operators (23) close the trigonometric algebra

$$[\tilde{\rho}(k), \tilde{\rho}(q)] = 2i \sin \left(\frac{1}{2} k \times q\right) \tilde{\rho}(k + q).$$ (27)

This algebra coincides with the algebra of magnetic translation operators for the fractional quantum Hall effect in the lowest Landau level.18,19

For an arbitrary c-number function $f(r)$ on the plane, we define its classical average using the electron density as

$$\langle f \rangle = \int d^2 r \rho(r) f(r) = \int \frac{d^2 k}{(2\pi)^2} \tilde{\rho}(k) \tilde{f}(-k).$$ (28)

In the quantum theory, we can compute the commutator of two such averages by multiplying the trigonometric algebra (27) on both sides by the convolution product $\tilde{f}(-k) \tilde{g}(-q)$ of Fourier transforms, and then integrate over the Fourier momenta to get

$$[\langle f \rangle, \langle g \rangle] = \langle [f, g] \rangle,$$ (29)

where we have introduced the star-commutator

$$[f, g]_\star(r) = (f \star g)(r) - (g \star f)(r).$$ (30)

The function $f \star g$ is the noncommutative, associative Grönewold-Moyal star-product of the functions $f$ and $g$ from the theory of deformation quantization,
and it may be expressed in position space in terms of a non-local bi-differential operator as

\[(f \star g)(r) = f(r) \exp \left( \frac{i}{2} \sum_{ij} \theta^{ij} \frac{\partial_i}{\partial_j} \right) g(r) \]

\[= f(r) g(r) + \sum_{n=1}^{\infty} \frac{i^n}{2^n n!} \theta^{i_1j_1} \cdots \theta^{i_nj_n} \partial_{i_1} \cdots \partial_{i_n} f(r) \partial_{j_1} \cdots \partial_{j_n} g(r) \]

(31)

with \( \partial_i = \partial/\partial x^i \). The relation (29) thereby describes a very simple physical occurrence of the star-product for fields in a strong magnetic background.

There are two important comments we should make about this derivation. First of all, only the commutators of averages coincide with star-commutators as in (29), and in general one has

\[\langle f \rangle \langle g \rangle \neq \langle f \star g \rangle. \] (32)

Secondly, the expansion of the star-commutator (30) for \( \theta^{ij} \to 0 \) (equivalently \( B \to \infty \)) yields, from (16) and (31), to lowest order the result

\[[f,g]_\star = \frac{\hbar}{2\pi} \{ f,g \} + O \left( \frac{1}{B^2} \right) \],

(33)

where \( \{ f,g \} = \epsilon^{ij} \partial_i f \partial_j g \) is the usual Poisson bracket of the functions \( f \) and \( g \). This “classical limit” is the general foundation for the deformation quantization programme, in which the quantum phase space is constructed by deforming the usual commutative product of functions on classical phase space into a noncommutative star-product. The star-commutator (30) thereby encodes the usual correspondence principle of quantum mechanics.

3. String Theory and D-Branes

In this Section we will describe a very precise realization of spacetime noncommutativity which arises in string theory, and which has sparked the enormous amount of activity in the subject over the past few years. It is a direct generalization of the example described in the previous Section. Its main virtue is that it naturally induces what is known as a (relativistic) noncommutative field theory, the subject of the next Section. We will first describe heuristically why noncommutative geometry is expected to play a role in string theory, and then move our way towards a quantitative derivation of its appearance. Unless explicitly written, in the remainder of this paper we will assume natural units in which \( \hbar = c = e = 1 \).

3.1. Noncommutative geometry in string theory

String theory is often regarded as the best candidate for a quantum theory of gravitation, or more generally as a unified theory of all the fundamental interactions. Within the framework of quantum gravity, a noncommutative spacetime geometry is expected on quite general grounds in any theory incorporating gravity into a
quantum field theory. At a semi-classical level, suppose we try to localize a particle to within a Planck length $\lambda_P \sim 10^{-33}$ cm in any given plane of a spacetime. This would require that an energy equal to the Planck mass $\sim 10^{19}$ GeV/c$^2$ must be available to the particle. But such a process has enough energy to create a black hole and swallow the particle. We may avoid this paradox by requiring the spacetime uncertainty principle:

$$\sum_{i<j} \Delta x^i \Delta x^j \geq \lambda_P^2.$$  (34)

This distorts the surrounding spacetime at very short distance scales in the manner explained in Section 1. We conclude from this simple analysis that spacetime non-commutativity is required when trying to quantize the Einstein theory of general relativity.

A similar scenario emerges directly from string theory. From the analysis of ultra-high energy string scattering amplitudes, one is led to postulate the string-modified Heisenberg uncertainty relation:

$$\Delta x \geq \frac{h}{2} \left( \frac{1}{\Delta p} + \ell_s^2 \Delta p \right),$$  (35)

where $\ell_s$ is the intrinsic string length (Fig. 2). This relationship reflects the inherent non-locality of string theory, since it implies that the extent of an object grows with its momentum. At large distance scales $\gg \ell_s$ (formally the limit $\ell_s \rightarrow 0$), wherein the strings effectively look like point particles, it reduces to the standard phase space relation in quantum mechanics with the spread decreasing with momentum. Generically, by minimizing it with respect to $\Delta p$ one finds that there is an absolute lower bound $\Delta x \geq (\Delta x)_{\text{min}}$ on the measurability of distances in the spacetime given by the length of the strings,

$$(\Delta x)_{\text{min}} = \ell_s.$$  (36)

This simply means that strings cannot probe distances smaller than their intrinsic size. String theory thereby requires a modification of classical general relativity.

More generally, basic conformal symmetry arguments in string theory lead to the anticipation of space/time uncertainty relations:

$$\Delta x \Delta t \geq \lambda_P^2.$$  (37)

It is possible to realize such length scales using as probes not the strings themselves, but rather certain non-perturbative open string degrees of freedom known as D-branes. In fact, these objects allow one to probe even shorter, sub-Planckian distance scales in string theory, and they enable microscopic derivations of fairly
Fig. 3. A pair of D-branes with open string excitations which may start and end on the same brane, or stretch between the two of them.

They are therefore the natural degrees of freedom which capture phenomena related to quantum gravitational fluctuations of the spacetime. The beautiful aspect of this point of view is that these phenomena can be treated systematically and at a completely quantitative level in string theory.

3.2. D-branes

Motivated by the discussion of the previous Subsection, let us now systematically look at D-branes. A D-brane may be defined as a hypersurface in spacetime onto which open strings attach (with Dirichlet boundary conditions). A schematic picture may be found in Fig. 3. These degrees of freedom are actually required for the overall consistency of the string theory, which we require to be unitary. The quantum theory of the open string excitations induces a spectrum of fields which reside on the branes. In the massless sector these include a gauge field $A_i$, adjoint scalar fields $X^m$ describing the transverse fluctuations of the D-branes in spacetime, and fermion fields $\psi_\alpha$. Integrating out the massive string modes on $N$ coincident D-branes leaves a low-energy effective field theory which can be obtained as the dimensional reduction, to the brane worldvolume, of ten-dimensional $U(N)$ Yang-Mills gauge theory (more precisely, its supersymmetric extension).

Let us study some features of this low-energy field theory description. The reduction of the $F^2$ term of the Yang-Mills action in ten spacetime dimensions (the critical superstring target space dimension) leads to the Yang-Mills potential

$$V_{YM}(X) = -\frac{1}{4g^2} \sum_{m \neq n} \text{Tr} [X^m, X^n]^2,$$

where $g$ is the Yang-Mills coupling constant and $X^m$ are $N \times N$ Hermitian matrices. If $N = 1$ then the $X^m$ correspond to the fields which embed the D-brane into the Euclidean target spacetime. For $N > 1$ they lose this geometric interpretation, and in this way the appearance of noncommuting spacetime coordinates arises via a dynamical mechanism. The potential (38) is a sum of non-negative terms, with $V_{YM}(X) \geq 0$ (note that $\text{Tr} [X^m, X^n]^2 = -\text{Tr} [X^m, X^n][X^m, X^n] \dagger$). Its global
minimum $V_{YM}(X) = 0$ is attained when the Hermitian matrices obey

$$[X^m, X^n] = 0$$

for each $m, n$. This means that the Hermitian matrices $X^m$ are simultaneously diagonalizable in the vacuum state. Their simultaneous real eigenvalues describe collective coordinates for the $N$ D-branes. Thus the classical ground state corresponds to an ordinary classical geometry. However, quantum fluctuations about the vacuum (39) describe a spacetime with a noncommutative geometry. The fluctuations correspond to turning on off-diagonal matrix elements of the $X^m$’s and are associated with short open string excitations between pairs of D-branes,\(^{35}\) as depicted in Fig. 3.

In this way the worldvolume field theories on the D-branes get altered by quantum gravitational effects.\(^{33}\) To understand this modification, it is instructive to examine other classical vacua associated with the potential (38). Generally, the equations of motion resulting from variation of $V_{YM}(X)$ are given by

$$[X_m, [X^m, X^n]] = 0.$$  \((40)\)

A natural class of solutions is then provided by $X^m_0$ satisfying the commutation relations

$$[X^m_0, X^n_0] = i \theta^{mn}$$  \((41)\)

with $\theta^{mn}$ real-valued c-numbers, as in (1). Taking the trace of both sides of (41) and using cyclicity shows that, for $\theta^{mn} \neq 0$, the relations (41) can only be satisfied by $N \times N$ matrices in the limit $N \to \infty$, i.e. by operators acting on a separable Hilbert space which are not trace-class, $\text{Tr} (X^m_0) = \infty$. This is the usual situation for a Heisenberg algebra. The expansion of the large $N$ matrices in (38) about these more general classical vacua as

$$X^m = X^m_0 + A^m(X_0)$$  \((42)\)

then determines a field theory for the $A^m$’s on a noncommutative space.\(^{3,36}\) This field theory in fact corresponds to the noncommutative Yang-Mills gauge theory which we will describe in the next Section.

### 3.3. String theory in magnetic fields

We can make the appearance of noncommutative geometry in string theory yet even more precise by considering the analog in string theory\(^{37}\) of the Landau problem for strong magnetic fields that we studied in detail in the previous Section. Let us consider the worldsheet field theory for open strings attached to D-branes, which is defined by a $\sigma$-model on the string worldsheet $\Sigma$ whose fields $y^i$ are maps from $\Sigma$ into the Euclidean target spacetime describing the propagation of the strings. The geometry of the target space is characterized by closed string supergravity fields,
most notably the spacetime metric $g_{ij}$ and the Neveu-Schwarz two-form $B_{ij}$ which we assume is non-degenerate. The action is

$$S_{\Sigma} = \frac{1}{4\pi\ell_s^2} \int_{\Sigma} d^2\xi \left( g_{ij} \partial^a y^i \partial_a y^j - 2\pi i \ell_s^2 B_{ij} \epsilon^{ab} \partial_a y^i \partial_b y^j \right),$$

(43)

where $\xi^a, a = 1, 2$ are local coordinates on the surface $\Sigma$ and $\partial_a = \partial/\partial \xi^a$. The two-form $B_{ij}$ may be regarded as a magnetic field on the D-branes.

In the case that the $B_{ij}$ are constant, the second term in (43) is a total derivative which may be integrated by parts to give the boundary action

$$S_{\partial\Sigma} = -\frac{i}{2} \oint_{\partial\Sigma} dt \, B_{ij} \dot{y}^i(t) \dot{y}^j(t),$$

(44)

where $t$ is the coordinate of the boundary $\partial\Sigma$ of the string worldsheet residing on the D-brane worldvolume, and $\dot{y}^i = \partial y^i/\partial t$. This is formally the same action that arose in the Landau problem in the strong field limit, and hence we can expect that quantization of the open string endpoint coordinates $y^i(t)$ will induce a noncommutative geometry on the D-brane. One needs to be somewhat careful though as this is not a theory of particles. There is still the first term present in the $\sigma$-model action (43) which describes the dynamics of the interiors of the open strings, or equivalently the closed string sector of the theory. It reminds us that the point particles of (44) are really the endpoints of open strings. In particular, the two ends of an open string couple directly to a background $B$-field and the string becomes polarized as a dipole.

The remarkable observation is that there is a consistent low-energy limit, called the Seiberg-Witten limit,\(^7\) which decouples all massive string modes at the same time as scaling away the bulk part of the string worldsheet dynamics from its boundary. It is defined by scaling the spacetime metric as

$$g_{ij} \sim \ell_s^4 \sim \varepsilon \longrightarrow 0$$

(45)

while keeping fixed the Neveu-Schwarz two-form field $B_{ij}$. Then the worldsheet field theory is effectively described by the boundary action (44) alone and canonical quantization produces the commutation relations

$$[y^i, y^j] = i \theta^{ij}, \quad \theta = B^{-1}$$

(46)

on $\partial\Sigma$. Thus the D-brane worldvolume becomes a noncommutative space. Because of the point particle limit $\ell_s \rightarrow 0$ taken in (45), the effective dynamics is governed in this limit as usual by a field theory for the massless open string modes. Following the analysis of the previous Section, we thus find that the low-energy effective field theories on D-branes get modified now to those defined with noncommuting coordinates, or equivalently by star-products of the fields. In this way, string theory in the presence of D-branes naturally leads to field theories on noncommutative spaces. These models are the subject of the next Section. It should be stressed that, as in the previous Section, noncommutative field theories emerge here as effective
descriptions of the string dynamics. An equivalent description is possible using ordinary theories on commutative spacetime. However, the noncommutative setting is much more natural and both conceptually and computationally useful, and it is from this formalism that the true Planck scale physics of string theory may be captured by quantum field theory.

4. Noncommutative Quantum Field Theory

One of the main interests in the emergence of field theories on noncommutative spaces in the manner described above is that they retain some of the non-locality of string theory, yet seem to be well-defined as field theories. They thus present the remarkable situation that many novel stringy features could be studied within the simpler language of quantum field theory. To what extent this is true is still to a large extent an unresolved issue. For instance, at present, there still lacks a general, systematic renormalization programme for handling such non-local field theories. These issues have addressed to all orders of perturbation theory in Refs. 38, 39, 40, 41. Nevertheless, they can be studied, and as field theories the non-locality gives them rather exotic features which challenge the conventional wisdom of ordinary quantum field theory. This is perhaps the greatest motivation for the extensive study that they have seen, in that they are interesting on their own as potentially well-defined examples of non-local field theories. In this Section we shall briefly survey some of these interesting new features, indicating how they capture some of the non-locality of quantum gravity and highlighting some of the main potential implications they could have on the structure of spacetime. We will assume throughout that the noncommutative field theories live on Euclidean spacetime. In Minkowski signature, making time a noncommuting coordinate in the present context leads to severe acausal effects and conceptual difficulties such as the precise interpretation of Hamiltonian evolution. It also leads to violations of unitarity and Lorentz invariance, as we discuss in Section 4.4. A possible cure for this violation is suggested in Ref. 42 by integrating over all background fields corresponding to noncommutativity parameters $\theta^{ij}$. This sum over backgrounds is of course the natural recipe dictated by string theory and quantum gravity, which are both unitary and covariant.

4.1. Fundamental aspects

For our purposes, we will define a noncommutative field theory as an ordinary field theory whose commutative pointwise products of fields are replaced with the noncommutative, associative star-product introduced in Section 2, i.e. we replace

$$f(x) g(x) \longrightarrow (f \star g)(x) = f(x) \exp \left( \frac{i}{2} \theta^{ij} \frac{\partial_i}{\partial_j} \right) g(x) ,$$

with $(\theta^{ij})$ an invertible antisymmetric matrix. With respect to this product, an elementary computation shows that the spacetime coordinates $x = (x^i)$ obey the required commutation relations (1),

$$[x^i, x^j]_* = x^i \star x^j - x^j \star x^i = i \theta^{ij} .$$

(47)
Under Fourier transformation of fields, the star-product (47) corresponds to the modification of the usual Fourier convolution product as

$$\tilde{f}(k) \tilde{g}(q) \rightarrow \tilde{f}(k) \tilde{g}(q) e^{i k \times q} , \quad k \times q = k_i \theta^{ij} q_j ,$$

(49)

where the tildes denote Fourier transforms. The alteration (49) in momentum space exemplifies the inherent non-locality of the star-product of fields. If $\theta$ is the average magnitude of a matrix element of $(\theta^{ij})$, then $1/\sqrt{\theta}$ is the energy threshold beyond which a particle moves and interacts in a distorted spacetime. The product deformation thus becomes effective at energies $E$ with $E \sqrt{\theta} \ll 1$, wherein not only are the interactions between the fields significantly modified, but so are the quanta which mediate these interactions.

As an explicit example, let us consider the noncommutative $\phi^4$ theory which is defined by the Euclidean action

$$S_\phi = \int d^4 x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi * \phi * \phi * \phi \right] ,$$

(50)

where $\phi(x)$ is a real scalar field on $\mathbb{R}^4$. Note that only interactions are modified by noncommutativity. The free field theory is unaffected owing to the fact that the spacetime average of the product of two fields is unchanged by the deformation,

$$\int d^4 x (f * g)(x) = \int d^4 x f(x) g(x) ,$$

(51)

which is easily derived from the representation (47) via an integration by parts (assuming appropriate decay behaviour of the fields at infinity in $\mathbb{R}^4$). In scalar field theories such as this one, by using (49) one can easily compute interaction vertices in momentum space as

$$\lambda \left( \frac{\lambda}{4} \sum_{I<J} k_I \times k_J \right) \left( \frac{\lambda}{4} \sum_{I<J} k_I \times k_J \right)$$

(52)

along with the usual momentum conservation constraint

$$k_1 + k_2 + \ldots + k_n = 0 .$$

(53)

We see that the noncommutative vertex (52) is momentum dependent, in contrast to the usual case whereby the tree-level vertex function would simply be equal to the coupling constant $\lambda$. In particular, this modified interaction vertex is only cyclically invariant under permutations of the momenta $k_I$, and so one needs to keep careful track of the order of the momenta flowing into the vertex, again in contrast to what would occur in ordinary scalar field theory.

It is well-known from multi-colour gauge theories and matrix models how to keep track of the cyclic ordering.\(^{43,44}\) One doubles each line of a Feynman graph
into ribbons. The ribbon graphs now have topology associated to them, and one can characterize them into two sets, called planar and non-planar.\textsuperscript{45-47} The planar diagrams are those which can be drawn on the surface of a sphere or the plane without crossing any of the ribbons. They are dynamically characterized by the fact that the totality of the momenta of internal line contractions vanishes. In this case the noncommutative phase factor \((52)\) does not significantly alter the analytic expression for the corresponding amplitude. It is equal to the ordinary, \(\theta = 0\) commutative diagram, times a possible \(\theta\)-dependent phase factor coming from the external momenta of the graph. In particular, there is no change in the convergence properties as compared to the commutative case. Much more interesting are the non-planar diagrams, those which cannot be drawn on the surface of a sphere or the plane. Dynamically, non-trivial internal momentum contractions remain, and these graphs typically modify the ultraviolet behaviour in a significant way.\textsuperscript{47} Naively, the rapid phase oscillations of \((52)\) suppress large momentum modes and would appear to make the amplitude ultraviolet finite. As we will discuss in the next Subsection, this is a subtle point, because the phase factors \((52)\) become ineffective at vanishing momenta, or equivalently the commutative ultraviolet divergence must reappear at \(\theta = 0\) and the amplitude exhibits non-analytic behaviour as a function of the noncommutativity parameter. This is a surprising feature of the quantum field theory, as the classical field theory smoothly reduces to its commutative counterpart at \(\theta = 0\).

\subsection*{4.2. UV/IR mixing}

Let us now explore in a bit more detail the non-locality induced by noncommutativity, and in particular the convergence properties of Feynman diagrams in the quantum field theory. At a semi-classical level, the non-locality of the star-product \((47)\) itself already produces exotic effects. Suppose that \(f\) and \(g\) are wavepackets which are supported in a region of small size \(\Delta \ll \sqrt{\theta}\). One can then show, essentially from the momentum representation of the star-product, that \(f \star g\) is non-zero in a large region of size \(\theta/\Delta\). An extreme example of this non-locality is provided by the star-product of two infinitely-localized delta-functions, which is constant throughout space,

\begin{equation}
\delta(x) \star \delta(x) = \frac{1}{|\det(\pi \theta)|}.
\end{equation}

In other words, the behaviour of the field theory at very short distances, where the effects of spacetime noncommutativity are significant, influences its long wavelength properties.

The effect just described has rather profound consequences in the quantum field theory, and it leads to the famous \(\text{UV/IR mixing}\) property of noncommutative field theories.\textsuperscript{47} If a Feynman diagram requires an ultraviolet cutoff \(\Lambda\) to regularize it,
then this automatically induces an effective infrared cutoff

$$\Lambda_0 = \frac{1}{\theta \Lambda}$$

(55)
on the graph. We will describe below some of the remarkable consequences of this mixing of energy scales, but let us first point out a simple physical picture of this effect. As we did in (25), from the Baker-Campbell-Hausdorff formula and the commutation relation (48), one can straightforwardly compute

$$e^{i k \cdot x} \star e^{i q \cdot x} \star e^{-i k \cdot x} = e^{-\frac{1}{2} k \times q} \star e^{i (k+q) \cdot x} \star e^{-i k \cdot x} = e^{i q (x^j - \theta^{ij} k_j)}.$$  

(56)

By Fourier transformation, it then follows that star-conjugation of fields by plane waves induces a non-local spacetime translation as

$$e^{i k \cdot x} \star f(x^i) \star e^{-i k \cdot x} = f(x^i - \theta^{ij} k_j).$$  

(57)

We interpret (57) to mean that the quanta in noncommutative field theory include “dipoles”, i.e. extended, rigid objects whose length or electric dipole moment $\Delta x^i$ grows with its center of mass momentum $p_j$,

$$\Delta x^i = \theta^{ij} p_j.$$  

(58)

These quanta are responsible for many of the stringy effects that noncommutative field theories exhibit (c.f. (35)), and they are just like the electron-hole bound states which arise in a strong magnetic field. The dipoles interact by joining at their ends, and this gives a simple picture of the non-local nature of the interactions in noncommutative quantum field theory.

Let now examine the UV/IR mixing property at a more quantitative level. Consider again, for definiteness, the noncommutative $\phi^4$ field theory with action (50). Using the steps described in the previous Subsection, one can work out the one-particle irreducible effective action at one-loop order in momentum space as

$$S_{1PI} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2} \phi(k) \partial(-k) \left[ k^2 + \tilde{m}^2 + \lambda \frac{\Lambda^2_{eff}}{96\pi^2} - \lambda \frac{\Lambda^2}{96\pi^2} \ln \left( \frac{\Lambda^2_{eff}}{\Lambda^2} \right) \right],$$  

(59)

where

$$\tilde{m}^2 = m^2 + \lambda \frac{\Lambda^2}{48\pi^2} - \lambda \frac{\Lambda^2}{48\pi^2} \ln \left( \frac{\Lambda^2}{m^2} \right)$$  

(60)

is the usual $\phi^4$ mass renormalization at one-loop order, and

$$\Lambda^2_{eff} = \frac{1}{\Lambda^2 + k_i (\theta^{ij} k_j)}$$  

(61)

is the momentum-dependent effective ultraviolet cutoff. From these expressions one clearly sees that the limits $\theta \to 0$ or $k \to 0$ (the infrared limit) and $\Lambda \to \infty$ (the ultraviolet limit) do not commute. Taking $\Lambda \to \infty$ leaves infrared singularities as $k \to 0$, as then the noncommutative phase factors (52) become ineffective at damping the ultraviolet behaviour in this momentum range. This feature makes standard Wilsonian renormalization treacherous, as it would normally require a
clear separation of high and low momentum scales. The low-energy effective field theory here does not decouple from the high-energy dynamics. The higher the cutoff \( \Lambda \) is, the more sensitive are the amplitudes to the lowest energies available.

4.3. **Gauge interactions**

Let us now study the example of noncommutative gauge theory which can be expected to tie in to the properties of our observable world, and which is also inspired by the string theory applications that we described in the last Section.\(^3,5,7\) The noncommutative Yang-Mills action for a \( U(N) \) gauge field \( A_i(x) \) on \( \mathbb{R}^4 \) is given by

\[
S_{NCYM} = -\frac{1}{4g^2} \int d^4x \ \text{Tr} \ F_{ij}(x)^2 ,
\]

where

\[
F_{ij} = \partial_i A_j - \partial_j A_i - i [A_i, A_j]_*,
\]

is the noncommutative field strength tensor. The curvature (63) is a non-local deformation of the usual \( U(N) \) field strength

\[
F_{ij} = \partial_i A_j - \partial_j A_i - i [A_i, A_j] + O(\theta, (\partial A)^2).
\]

The action (62) is invariant under the noncommutative version of the usual gauge transformations, leading to the star-gauge invariance

\[
A_i \mapsto U \star A_i \star U^{-1} + i U \star \partial_i U^{-1},
\]

where \( U(x) \) is an \( N \times N \) star-unitary matrix field on \( \mathbb{R}^4 \),

\[
U \star U^\dagger = U^\dagger \star U = 1.
\]

The presence of the star-product in the gauge transformation rule (65) mixes spacetime and colour degrees of freedom in an intricate way. In fact, noncommutative gauge transformations in a certain sense generate the infinite unitary group \( U(\infty) \).\(^{50}\) This implies that star-gauge invariance will contain geometrical symmetries of the spacetime, in particular the symplectomorphisms of \( \mathbb{R}^4 \) with respect to the Poisson bi-vector \( \theta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \). To understand this point, let us consider the basic plane wave fields in the case \( N = 1 \),

\[
U_a(x) = e^{i(\theta^{-1})_{ij} a^j x^i}.
\]

From (56) one easily checks that they are star-unitary,

\[
U_a \star U_a^\dagger = U_a^\dagger \star U_a = 1,
\]

while from (57) it follows that they implement translations of fields by the vector \( a = (a^i) \in \mathbb{R}^4 \),

\[
U_a(x) \star g(x) \star U_a^\dagger(x) = g(x + a).
\]
From (65) and (67) one then finds that the corresponding star-gauge transformations are given by

\[ A_i(x) \mapsto A_i(x + a) - (\theta^{-1})_{ij} a^j. \quad (70) \]

Up to a global translation of \( A_i \), which leaves invariant the field strength tensor (63), we see that spacetime translations are equivalent to gauge transformations in noncommutative Yang-Mills theory.\(^{51}\) The only other known theory with such a geometrical gauge symmetry is gravity, and we may thereby conclude that noncommutative gauge theory provides a toy model of general relativity.

This translational symmetry can be naturally gauged within this framework, and the noncommutative gauge theory can be thereby shown to reduce to a teleparallel formalism of general relativity.\(^ {52}\) This feature fits in well with the hope that noncommutative gauge theories capture important stringy properties.\(^ {53,54}\) A particularly important consequence of this spacetime-colour mixing is that there are no local gauge-invariant operators in noncommutative Yang-Mills theory.\(^ {46,47,51}\) From the appropriate analogs of Wilson line operators, it is in fact possible to derive closed string, gravitational degrees of freedom from these open string noncommutative gauge theories.\(^ {55,56}\)

### 4.4. Violations of special relativity

In this Subsection we will give an overview of some of the broad phenomenological applications that spacetime noncommutativity may have. Let us first observe that noncommutative field theories violate Lorentz invariance. In the string picture, this violation is due to the expectation value of the supergravity field \( B_{ij} \).

Focusing on the four-dimensional situation, the noncommutativity tensor \( \theta_{ij} \) provides a directionality \( \theta_i = \epsilon_{ijk} \theta^{jk} \) in space, within any fixed inertial frame. Thus noncommutative field theory is not invariant under rotations or boosts of localized field configurations within a fixed observer inertial frame of reference.

The orientation \( \theta \) can be used to provide stringent constraints on the observable magnitude of the noncommutativity parameters \( \theta^{ij} \). Let us briefly summarize a few of the analyses that have been made:

(i) One can compare the noncommutative extension of the standard model with certain Lorentz-violating extensions. Noncommutative field theories are CPT symmetric,\(^ {57}\) hence so should be these commutative extensions. Comparison with the known literature can be used to derive the bound\(^ {58}\)

\[ \theta < (10 \text{ TeV})^{-2} \quad (71) \]

by using atomic clock comparison studies and a model for the \(^9\)Be nucleus wavefunction.

(ii) We can also compare noncommutative quantum electrodynamics with some of the more standard QED processes, by taking into account the motion of the laboratory frame relative to \( \theta \). Among the many scenarios considered have
been high-energy $e^+e^-$ and hadron scattering, CP-violation, the anomalous magnetic moment $(g-2)\mu$, and so on. A review of these phenomenological applications is found in Ref. 59, where a complete list of references is also given.

(iii) Finally, we can compare noncommutative QED with low-energy atomic transitions. For example, in the noncommutative version of the Lamb shift in hydrogen,\(^60\) the leading modification of the Coulomb potential is given by

$$V_C(r) = -\frac{e^2}{r} - \frac{e^2 (r \times k) \cdot \theta}{r^3} + O(\theta^2).$$

(72)

A recent overview of the various bounds obtained on spacetime noncommutativity is presented in Ref. 61.

Let us now turn to the phenomenological implications of UV/IR mixing. From (59)–(61) we see that noncommutativity modifies the standard dispersion relation of special relativity to

$$E^2 = k^2 + m^2 + \Delta M^2 \left( \frac{1}{k \theta} \right),$$

(73)

where $\Delta M^2(\Lambda)$ is the ultraviolet divergent mass renormalization. The deformation $\Delta M^2$ in (73) induces a violation of classical special relativity. We can compare this formula to experimental measurements in the energy range

$$\Lambda_0 < E < \Lambda = \frac{1}{\theta \Lambda_0},$$

(74)

where $\Lambda_0$ is an experimentally determined phenomenological infrared scale. This implies that one can only compare the effects of UV/IR mixing with very high energy experimental data.

For example, the photon dispersion relation in noncommutative electrodynamics is given by\(^62\)

$$\omega = ck - ck \theta_\perp \cdot B_\perp,$$

(75)

where $\theta_\perp, B_\perp$ are the components of $\theta$ and a constant background magnetic induction $B$ transverse to the direction of light propagation $k$, i.e. $k \cdot \theta_\perp = k \cdot B_\perp = 0$. To reproduce the bound (71), one would need to arrange a Michelson-Morley type interferometry experiment with visible light, i.e. $B$ of the order of a Tesla, and with leg lengths $l_1, l_2$ obeying $l_1 + l_2 \geq 10^{18}$ cm, which is of the order of a parsec. This is probably impractical for galactic magnetic fields.\(^15,62\)

Finally, various cosmological comparisons can be made based on the deformed dispersion relation (73).\(^63\) For example, in certain models of astrophysical gamma-ray bursts, spacetime foam induces dispersion. This comes about from ultra-high energy cosmic ray thresholds (the GZK cutoff) on cosmic proton energies due to the photopion production reaction $p + \gamma \rightarrow p + \pi$ with cosmic microwave background radiation. Relations such as (73) can be used to explain the various puzzling and paradoxical observations of cosmic rays.
5. The Fractional Quantum Hall Effect

Having dispelled with our tour of the significance of noncommutative field theory in high-energy physics, we will now go back to our motivating example of Section 2 and apply what we have learnt about these novel field theories. A very precise application of noncommutative field theory is in fact to the fractional quantum Hall effect. A particular such model provides a mean field theory description that is far superior to its commutative version and which displays the correct quantitative features expected from condensed matter physics.

5.1. The Laughlin wavefunction

Let us begin with a brief review of some basic and well-known facts about the mean field theory of the Landau problem. In the fractional quantum Hall effect, the interactions lead to a state similar to the filled lowest Landau level, but allowing for fractionally charged quasi-particle excitations. Let \( m \) be a positive integer. The ratio of the electron density to the density of the lowest Landau level is the filling fraction \( \nu \), and at \( \nu = \frac{1}{m} \), a good microscopic description of such a state is provided by the \( N_e \)-electron Laughlin wavefunction

\[
\Psi_{1/m}(z_1, \ldots, z_{N_e}) = \prod_{I<J} (z_I - z_J)^m \ e^{-\frac{1}{2\pi} \sum_{I=1}^{N_e} |z_I|^2}, \tag{76}
\]

where \( \theta = \hbar c/eB \) and \( z_I = x_I + iy_I \) are complex coordinates on the plane for each \( I = 1, \ldots, N_e \). In canonical quantization, the pairs \((z_I, \bar{z}_I)\) define essentially \( N_e \) decoupled harmonic oscillators. The state (76) has charge density equal to \( \frac{1}{m} \) times the density of a filled Landau level.

A quasi-particle at the point \( z_0 \) is created from the state (76) by acting with the operator

\[
\mathcal{Q}(z_0) = \prod_{I=1}^{N_e} (\bar{z}_I - z_0), \tag{77}
\]

where we represent the oscillator algebra by \( \bar{z}_I = 2\theta \frac{\partial}{\partial z_I} \). The quasi-particle states are characterized by the two properties they have:

(i) Fractional charge \( \frac{1}{m} \).
(ii) Fractional exchange statistics, i.e. a \( 2\pi \) rotation of the relative coordinate of a two-quasi-particle state multiplies the state by the phase factor \( e^{2\pi i / m} \).

The low-energy excitations of the ground state may be described by a Landau-Ginzburg theory of a superfluid density \( \phi \) minimally coupled to a fictitious abelian vector potential \( A_i \) in \( 2 + 1 \) dimensions. The original quasi-particles are magnetic vortex solutions of this model, while their fractional statistics is reproduced by including in the action an abelian Chern-Simons term for the gauge potential,

\[
S_{CS} = \frac{im}{2\pi} \int d^3x \ e^{ijk} A_i \partial_j A_k. \tag{78}
\]
The Gauss law for this gauge-matter coupled theory implies that a vortex of unit magnetic charge also carries electric charge $\frac{1}{n}$. The Aharonov-Bohm phase factors about the magnetic vortex then lead to the appropriate fractional statistics. This model effectively describes the Landau problem as a quantum Hall fluid in terms of a hydrodynamical gauge theory.\textsuperscript{67,68}

5.2. Noncommutative Chern-Simons theory

We will now demonstrate that the noncommutative version of the Chern-Simons action (78) leads directly to a very efficient description of the quasi-particle excitations,\textsuperscript{9} in which the elevation of the hydrodynamic gauge theory to a non-commutative gauge theory captures the graininess of the quantum Hall fluid. The primary motivation \textit{a priori} for making the spatial directions $r = (x^1, x^2)$ noncommuting variables resides in our analysis of Section 2. The time coordinate $x^0 = t$ is left as an ordinary commutative variable. The action is defined by

$$S_{NCS} = \frac{im}{2\pi} \int d^3x \, \epsilon^{ijk} \left( A_i \partial_j A_k + \frac{2}{3} A_i \star A_j \star A_k \right). \quad \text{(79)}$$

It is invariant under the usual noncommutative gauge transformations $U$ in (65,66) which are trivial at spatial infinity,\textsuperscript{69} i.e. $U(t, r) \rightarrow \mathbb{1}$ at $|r| \rightarrow \infty$. In the temporal gauge $A_0 = 0$, the action (79) is formally the same as its commutative counterpart (78), i.e.

$$S_{NCS}[A_0 = 0] = \frac{im}{2\pi} \int dt \int d^2r \, \epsilon^{0ij} A_i \partial_t A_j. \quad \text{(80)}$$

However, now the Gauss law, i.e. the equation of motion for $A_0$, involves the non-commutative field strength tensor and is given by

$$F_{ij} = \partial_i A_j - \partial_j A_i - i [A_i, A_j] = 0. \quad \text{(81)}$$

The crucial observation now is that the action and constraint arise from a matrix model in $0 + 1$ dimensions with action\textsuperscript{9,10}

$$S_{MCS} = \frac{2m}{\theta} \int dt \text{ Tr } \left( \frac{1}{2} \epsilon^{ij} X_i D_t X_j + \theta A_0 \right), \quad \text{(82)}$$

where $X^i$, $i = 1, 2$ and $A_0$ are $N \times N$ time-dependent Hermitian matrices, and $D_t = \partial_t - i A_0$. This is a gauged $U(N)$ matrix quantum mechanics which we will call \textit{matrix Chern-Simons theory}. To establish its equivalence with the noncommutative gauge theory defined by (79), we write (82) in the $A_0 = 0$ gauge to get

$$S_{MCS}[A_0 = 0] = \frac{m}{\theta} \int dt \, \epsilon^{ij} X_i \partial_t X_j, \quad \text{(83)}$$

and note that the constraint arising from varying the action (82) with respect to the non-dynamical variable $A_0$ is given by the commutation relation

$$[X^1, X^2] = i \theta. \quad \text{(84)}$$
In particular, as we discussed earlier, the $X^i$ are necessarily infinite-dimensional matrices with $N \to \infty$, i.e. operators acting on a separable Hilbert space.

Let us now expand the action (83) and constraint (84) about a particular time-independent solution $y^i$, as we did in (42), i.e. we write

$$X^i = y^i + \theta \epsilon^{ij} A_j$$

with

$$[y^i, y^j] = i \theta \epsilon^{ij}$$

and $A_i$ functions of the noncommuting coordinates $y^i$. From (86) it follows that the $y^i$'s act on functions of $y$ alone as

$$[y^i, f(y)] = i \theta \epsilon^{ij} \frac{\partial f(y)}{\partial y^j} .$$

We then use the usual Weyl-Wigner correspondence of noncommutative field theory, which generally reflects the fact that noncommutative fields are most naturally thought of as operators acting on a separable Hilbert space. It makes the association

$$\text{Tr} \left( f_1(y) \cdots f_n(y) \right) \mapsto \frac{1}{2\pi \theta} \int d^2 r \ (f_1 \star \cdots \star f_n)(r) ,$$

where the left-hand side of (88) is a trace over infinite-dimensional operators while the right-hand side is a spatial integration over functions on $\mathbb{R}^2$. By substituting (85)–(88) into (83) and (84), we arrive at the noncommutative Chern-Simons action (80) with its constraint (81). Thus the matrix model (82) expanded about the background $X^i = y^i$ as above is exactly equivalent to noncommutative $U(1)$ Chern-Simons gauge theory. This equivalence is completely analogous to the way in which noncommutative Yang-Mills theory (62) is derived from the Yang-Mills potential (38) via expansion of matrices about a noncommutative background.

However, as it presently stands, the matrix quantum mechanics expanded about the noncommutative background describes a system with infinitely many degrees of freedom. We can remedy the situation, and hence describe a quantum Hall droplet of finite size, by introducing a finite-dimensional version of the matrix Chern-Simons theory (82) defined by the matrix-vector $U(N)$ gauged quantum mechanics with action

$$S_N = \frac{m}{\theta} \int dt \ \text{Tr} \left( \epsilon^{ij} X_i D_t X_j - \frac{1}{2 \theta^2} X_i^2 + 2 A_0 \right) + \int dt \ \Phi^\dagger D_t \Phi ,$$

where again $X^i, i = 1, 2$ are $N \times N$ time-dependent Hermitian matrices. The second term in the action (89) is a harmonic oscillator potential for the $X^i$'s, while $\Phi$ transforms as a complex $N$-vector under the gauge group $U(N)$. The crucial point here is that finite $N$ dimensional representations of the classical vacua are now possible, with $N = N_c$ identified as the number of electrons residing in the lowest Landau level. To see this, we note that the $A_0$ constraint now selects a sector of
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particular Φ-charge equal to $m$. We can solve these constraints classically using the $U(N)$ symmetry of the model to make $X^1$ diagonal and $Φ_I$ real. This results in a system with $N$ real degrees of freedom and a residual permutation symmetry generated by the Weyl subgroup of $U(N)$, acting by permuting the entries of $X^1$ and $Φ_I$.

In this way, the constrained finite $N$ matrix Chern-Simons theory (89) reduces to the Calogero model for $N$ identical particles moving in one dimension,\textsuperscript{70,71} which is defined by the quantum mechanical Hamiltonian

$$H_C = \sum_{I=1}^{N} \left( \frac{1}{2} \dot{q}_I^2 + \frac{1}{2} \dot{p}_I^2 \right) + \frac{1}{2} \sum_{I<J} \frac{m(m+1)}{(x_I - x_J)^2}.$$  \hspace{1cm} (90)

A ground state of this Hamiltonian is precisely the Laughlin wavefunction (76), with $x_I = \text{Re}(z_I)$. One can continue this and identify all excited Calogero states with excited Laughlin quasi-particle wavefunctions in a one-to-one manner.\textsuperscript{11} We conclude that the finite $N \times N$ matrix Chern-Simons theory is a theory of $N$ composite fermions in the lowest Landau level. The quasi-particles are well-defined excitations of the noncommutative Chern-Simons gauge field.

Coming at the noncommutative gauge theory from the string theory side, it can be shown that certain configurations of D-branes in superstring theory exhibit the physics of the fractional quantum Hall effect.\textsuperscript{72} The D2-brane effective gauge theory, in a certain generalization of the Seiberg-Witten scaling limit,\textsuperscript{7} implies the effective noncommutative gauge theory described above. In particular, the role of the electrons in the quantum Hall fluid is played by D0-branes, and the large $N$ D0-brane matrix model in this way induces the finite matrix Chern-Simons theory. This provides a string theory derivation of the proposal that the ground state of the fractional quantum Hall fluid is described by a noncommutative Chern-Simons gauge theory.\textsuperscript{9} In this way, string theory can present effective long wavelength descriptions of certain condensed matter phenomena, and noncommutative field theory provides a surprising bridge between these two seemingly disparate developments in physics.

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