Transverse confinement in stochastic cooling of trapped atoms

D. Ivanov† and S. Wallentowitz∗
Emmy–Noether Nachwuchsgruppe “Kollektive Quantenmessung und Rückkopplung an Atomen und Molekülen”, Fachbereich Physik, Universität Rostock, Universitätsplatz 3, D-18051 Rostock, Germany

Abstract. Stochastic cooling of trapped atoms is considered for a laser-beam configuration with beam waists equal or smaller than the extent of the atomic cloud. It is shown, that various effects appear due to this transverse confinement, among them heating of transverse kinetic energy. Analytical results of the cooling in dependence on size and location of the laser beam are presented for the case of a non-degenerate vapour.

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1. Introduction

Cooling techniques for atoms play a crucial role in modern physics. They have allowed for the localisation of atomic gases by weak trapping forces and step by step have enabled the reach into the domain of ultracold gases. At such low temperatures the peculiar quantum-statistical properties of the atoms become immanent in various effects of cold atomic collisions. Bose-Einstein condensation [1–5] and the recent production of Fermi-Dirac degenerate gases [6–9] are limiting cases of the now existing experimental feasibilities. Moreover, the implementation of atom-lasers [10–12] and microstructured traps on so called atom chips [13–16] show the vast potential of applications.

The typical strategy to generate a Bose-Einstein condensate from a moderately cold sample of trapped atoms is to apply different cooling techniques in sequence: it usually starts with laser cooling [17–19] and ends with evaporative cooling [20–22]. The latter technique seems to be unbeaten as of yet for the final cooling step. Laser cooling, that relies on cycling transitions where photons are spontaneously emitted, does not provide the ultimate cooling power, due to the reabsorption and scattering of the emitted photons. However, the drawback of evaporative cooling is well known to be its intrinsic loss of atoms. Since hot atoms are released from the trapping potential to reach a colder sample, a substantial atom loss has to be taken into account that ultimately limits the size of the condensed sample. Furthermore, as does sympathetic cooling [23], it requires sufficiently strong atomic collisions for thermal re-equilibration.

Given a prepared sample of condensed atoms, a multitude of technical and possibly fundamental noise effects lead to a finite lifetime of the condensate state. Among these noise effects there are collisions with background vapour, electromagnetic noise sources via the

† Also at: Russian Center of Laser Physics, St.-Petersburg State University, 5 Ulianovskaya, Pedrodvoretz, St. Petersburg, Russia
∗ email: sascha.wallentowitz@physik.uni-rostock.de
trapping potential, scattering of light, etc. The study of these detrimental disturbances and the development of methods to reduce their impact on the condensate will be a challenging task for the future. One way to compensate for such heating effects may be simply the continuous application of cooling during the entire experiment. This may partially compensate the heating and thus extends the lifetime of the condensate. The only technique working so far at these temperatures is evaporative cooling. Its continuous application, however, would be rather unfortunate for the condensate, since though the lifetime of the condensate may be extended, its size in terms of atom number will continuously decrease.

Some years ago, Raizen et al. have proposed the use of stochastic cooling for trapped atoms [24]. It is a successful method in high-energy physics [25, 26] where the transverse motion of a particle beam has to be collimated and cooled. Clearly, the energies involved there are not in the regime important for an application to trapped atoms. However, classical numerical simulations have shown the feasibility of stochastic cooling also for trapped atoms [24, 27]. Furthermore, it has been recently shown, that also at ultralow temperatures this technique reveals cooling [28]. Thus it may perhaps be utilized to stabilise an atomic Bose-Einstein condensate.

It should be pointed out that on the single-atom level feedback control of atomic position has been theoretically studied [29, 30] and experimentally realised in optical lattices [31] and high-quality cavity fields [32].

In this paper we extend our analysis of stochastic cooling of trapped atoms to include also effects due to the transverse confinement of atoms. The latter has its origin in the finite beam waist of the employed control-laser beam. At temperatures above the condensation point we give analytic results of the cooling and discuss its optimisation with respect to size and location of the control-laser beam waist.

In Sec. 2 we explain the method of stochastic cooling of atoms and derive the expression for the single-atom density matrix after the single cooling step. Given this result at hand, in Sec. 3 we calculate the total energy change of atoms due to a single step of stochastic cooling in terms of quantum-statistical averages. In Sec. 4 the regime of a non-degenerate gas is considered for which analytical expressions for the energy contributions are obtained. Moreover, the dependence of cooling on geometrical parameters is discussed. Finally, in Sec. 5 conclusions are given.

2. Stochastic cooling

The method of stochastic cooling of trapped atoms consists of the repeated application of two operations: the measurement of the momentum of atoms and the subsequent application of a kick to compensate for the measured momentum. Several aspects are important and should be emphasised for an understanding of the working of this technique. First of all it is not done on a single atom but on a large set of atoms. Those atoms that are subject to measurement and kick are specified by their spatial location in a given volume of space. In the experiment that volume is defined by the spatial extent of the laser beams that implement the required operations.

Since in the experiment, at the time of measurement of momentum, it is usually unknown how many atoms contributed to the measured signal, the momentum per atom averaged over the atomic ensemble is not the measured observable. To obtain this momentum per atom one would in fact need knowledge on the precise number of atoms, that contributed to the measured signal. What can, instead, be assessed by the measurement is the total momentum of the atoms. This is the sum over the atomic momenta, since each atom equally contributes to the signal.
Given the measured total momentum of the set of atoms, for compensating it, a (optical) field is turned on to provide the necessary kick by its interaction with the atoms. Since each atom separately interacts with the field, one can only apply a common kick to each atom. The determination of the required kick per atom necessarily involves a characterisation of the number of atoms in the set, given that only the total momentum is known. Since the atom number will not be measured, a priori information is required for estimating the actual atom number. Clearly, this way atom-number fluctuations, whether classical or quantum in nature, cannot be coped with, which shows an intrinsic source of imperfection of the method.

Furthermore, since a measurement on a single system and not a series of measurements on identically prepared systems is performed, not ensemble averages are measured. Depending on the measurement resolution strong correlations between the atoms are induced by the measurement projection, since a huge number of microstates of the atoms may be associated to the same observed measurement outcome, which form a complicated superposition state.

2.1. Single-atom density matrix

Here we consider a full three-dimensional model and consider spatial confinement of the volume where atoms are manipulated, see Fig. 1. For simplicity we assume only one laser-beam profile, despite the fact that several laser beams are involved in the implementation of the required operations [33–35]. The control-laser beam is directed along the $z$-axis and its transverse profile in $x$ and $y$ directions is described by the beam-waist function $w_\perp(x, y)$. Thus, the $z$ component of the total momentum of atoms inside the beam $\hat{P}_w$ is measured and then compensated to zero by means of a negative feedback loop. Using the atomic field operator $\hat{\phi}(r)$ for bosonic atoms, i.e. with commutator

$$[\hat{\phi}(r), \hat{\phi}^\dagger(r')] = \delta(r-r'),$$

this observable can be written as ($\hbar = 1$)$^{\#}$

$$\hat{P}_w = -i \int dV w_\perp(r) \hat{\phi}^\dagger(r) \partial_z \hat{\phi}(r).$$

The many-body quantum state of the atomic cloud after a single operation of stochastic cooling can be given as an integral over all possible measurement outcomes $P$ for the measured total momentum:

$$\hat{\rho}_+ = \int dP \hat{U}(P) \hat{M}(P) \hat{\rho}_- \hat{M}^\dagger(P) \hat{U}^\dagger(P).$$

In this expression the initial many-body density operator is denoted as $\hat{\rho}_-$ and the final one, after the single feedback operation, is denoted as $\hat{\rho}_+$. It is assumed here, that the measurement and shift of momentum can be performed on a time scale $\Delta t$ much faster than the characteristic dynamics of the free system, i.e. $\Delta t \ll \omega^{-1}$ with $\omega$ being the trap frequency. Then measurement and shift can be taken as instantaneous processes without time delay between them.

The measurement of momentum $\hat{P}_w$ with specific outcome $P$ is described by the resolution amplitude [36, 37]

$$\hat{M}(P) = \sqrt{1 \over \sqrt{2\pi\sigma}} \exp \left\{ -\frac{(P - \hat{P}_w)^2}{4\sigma^2} \right\},$$

$\#$ Throughout the paper we use $\hbar = 1$.  

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Figure 1. Geometry of the feedback setup. The laser beam is aligned along the z axis, it determines the size of the feedback region.

where $\sigma$ denotes the measurement resolution. Applied on a momentum eigenstate $|P_0\rangle$ it gives the probability amplitude to observe the value $P$.\footnote{The set of operators $\hat{M}(P)\hat{M}(P)$ forms a positive operator-valued measure [38].} A quasi canonically conjugate operator for the centre of mass of the atoms in the laser beam, being experimentally accessible, can be defined analogously as [39]

$$\hat{Q}_w = \frac{1}{N_e} \int dV w_\perp(r) \hat{\phi}^\dagger(r) z \hat{\phi}(r). \quad (5)$$

Since $N_e$ is an estimated atom number, the commutator relation of operator $\hat{P}_w$ and $\hat{Q}_w$ reveals a deviation from the usual canonical form:

$$[\hat{Q}_w, \hat{P}_w] = i \hat{N}_w/N_e. \quad (6)$$

Here the true atom-number is defined as the operator

$$\hat{N}_w = \int dV w_\perp^2(r) \hat{\phi}^\dagger(r) \hat{\phi}(r). \quad (7)$$

The modified commutation relation (6) has an impact on the action of the shift operator $\hat{U}(P)$, that is supposed to produce a shift of $\hat{P}_w$ by $-P$, and that is defined as

$$\hat{U}(P) = \exp(-iP\hat{Q}_w). \quad (8)$$

Transforming the observable $\hat{P}_w$ by use of to the unitary transformation (8) we obtain

$$\hat{U}^\dagger(P) \hat{P}_w \hat{U}(P) = \hat{P}_w - P\hat{N}_w/N_e. \quad (9)$$

Thus an optimal shift by $-P$ is produced only on average when additionally estimating

$$N_e = \langle \hat{N}_w \rangle. \quad (10)$$

Nevertheless, atom-number fluctuations will always deteriorate the perfection of the shift operation. In view of the lack of knowledge on the true atom number $\hat{N}_w$, the estimate (10) represents an optimum. Thus in the following we use this justifiable estimate.
For our further derivations it is convenient to calculate the single-atom density matrix from Eq. (3). It is defined as
\[ \sigma_{+}(r_1, r_2) = \langle \hat{\phi}^\dagger (r_2) \hat{\phi} (r_1) \rangle, \] (11)
and can be calculated as a trace over the many-body density operator given in Eq. (3). In this way the single-atom density matrix after (+) an operation of stochastic cooling reads
\[ \sigma_{+}(r_1, r_2) = \int dP \langle \hat{M}^\dagger (P) \hat{U}^\dagger (P) \hat{\phi} (r_1) \hat{\phi} (r_2) \hat{U} (P) \hat{M} (P) \rangle_{-}, \] (12)
where \langle . . . \rangle_{-} denotes tracing over the many-body density operator \( \hat{\rho}_{-} \), that represents the quantum state before the feedback operation.

The action of \( \hat{U} (P) \) on a field operator results as a c-number exponential factor
\[ \hat{U}^\dagger (P) \hat{\phi} (r) \hat{U} (P) = \hat{\phi} (r) \exp \left[ -i z w \perp (r) P / \langle \hat{N}_w \rangle \right]. \] (13)
Moreover, using the Fourier representation of the resolution amplitude
\[ \hat{M} (P) = \int dq M(q) e^{iq(P-P_w)}, \] (14)
with
\[ M(q) = \sqrt{\frac{2\sigma^2}{\pi}} \exp(-\sigma^2 q^2) \] (15)
the single-atom density matrix can be rewritten as
\[ \sigma_{+}(r_1, r_2) = \int dP \int dq \int dq' \langle \hat{\phi}^\dagger (r_2) \hat{\phi}^\dagger (r_1) \hat{\phi} (r_2) \hat{\phi} (r_1) \rangle_{-} M^*(q) M(q') \times \exp \left\{ i P \left[ \frac{2 z w \perp (r_2) - z_1 w \perp (r_1)}{\langle \hat{N}_w \rangle} + q' - q \right] \right\}. \] (16)
This result can be further simplified using the transformation
\[ \exp \left[ i q P \right] \hat{\phi} (r) \exp \left[ -i q P_w \right] = \hat{\phi} (x, y, z - q w \perp (r)), \] (17)
which results in
\[ \sigma_{+}(r_1, r_2) = \int dP \int dq \int dq' M^*(q) M(q') \exp \left\{ i P \left[ \frac{2 z w \perp (r_2) - z_1 w \perp (r_1)}{\langle \hat{N}_w \rangle} + q' - q \right] \right\} \times \langle \hat{\phi}^\dagger (x_2, y_2, z_2 - q w \perp (r_2)) \hat{\phi} (x_1, y_1, z_1 - q w \perp (r_1)) e^{i(q' - q) P_w} \rangle. \] (18)
Performing then the \( P \) and \( q' \) integrations we finally obtain
\[ \sigma_{+}(r_1, r_2) = 2\pi \int dq M^*(q) M \left( q \left[ z_1 w \perp (r_1) - z_2 w \perp (r_2) \right] / \langle \hat{N}_w \rangle \right) \times \langle \hat{\phi}^\dagger (x_2, y_2, z_2 - q w \perp (r_2)) \hat{\phi} (x_1, y_1, z_1 - q w \perp (r_1)) e^{i P_w \left[ z_2 w \perp (r_2) - z_1 w \perp (r_1) \right]} \rangle_{-}, \] (19)
which shows that the single-atom density matrix after the feedback depends in general on higher atom-atom correlations before the feedback. In Eq. (19) this is encoded by the occurrence of the exponential operator.
3. Feedback-induced change of energy

Important features of the application of a single feedback step can be given in analytical form. For example, the difference of energy after and before the feedback step $\Delta E$ can be calculated from Eq. (19). The information contained therein allows us to recognise noise sources and determine optimal parameters for maximum cooling. The parameters that can be optimised are the measurement resolution $\sigma$ and the geometrical characteristics given by the size and location of the control-laser beam with respect to the trapping potential.

In the following we use the thermal equilibrium state of the atomic ensemble to calculate the average energy change. This allows us to obtain a natural description of cooling in terms of energy $\Delta E(T)$ that is subtracted or, possibly, added by a feedback operation at a certain temperature point. However, this approach does not necessarily reflect the most general experimental situation, since specific correlations generated step by step in the feedback process are not taken into account. These correlations may possibly lead to enhancement of cooling via various effects, as shown in Refs [24, 27]. Thus the results of this paper, where we assume a thermal equilibrium state, represent the leading cooling/heating mechanisms for the quasi-equilibrium case.

Having the expression for the single-atom density matrix (19) the change of total energy due to the application of a single feedback step can be formulated. The Hamiltonian of the system of non-interacting atoms in the isotropic, harmonic trap potential is

$$\hat{H} = \int dV \hat{\phi}^\dagger (r) \left[ -\nabla^2 + \frac{m\omega^2}{2} r^2 \right] \hat{\phi}(r),$$

(20)

where $m$ and $\omega$ are the atomic mass and the vibrational trap frequency, respectively. We divide this Hamiltonian into parts describing the energy of the motion in $z$ direction, i.e. in longitudinal direction with respect to the measured momentum $\hat{P}_w$, and into parts related to the energy of the transverse motion in the $xy$ plane.

The average total energy of the system in a given many-body quantum state can be written as the sum of these different contributions as

$$E = \langle \hat{H} \rangle = T_\parallel + T_\perp + V_\parallel + V_\perp.$$

(21)

Using the definition of the single-atom density matrix (11), the corresponding kinetic parts of Eq. (21) are given as

$$T_\parallel = -\frac{1}{2m} \int dV \int dV' \delta(r-r') \nabla^2 \sigma(r,r'),$$

(22)

$$T_\perp = -\frac{1}{2m} \int dV \delta(r-r') \nabla^2 \sigma(r,r'),$$

(23)

where $\nabla^2 = \partial^2_x + \partial^2_y$, and the potential-energy contributions read

$$V_\parallel = \frac{m\omega^2}{2} \int dV z^2 \sigma(r,r),$$

(24)

$$V_\perp = \frac{m\omega^2}{2} \int dV (x^2 + y^2) \sigma(r,r).$$

(25)

A change of the average energy due to the application of a single step of stochastic cooling is dominantly generated by the energy exchange with the externally applied optical fields that implement the momentum shift of atoms. The unidirectional flow of energy from the system to the optical fields is determined by the irreversibility introduced in the quantum measurement process. Apart from that, however, there are several sources of heating, among them also the back-action noise of the measurement itself. Especially at ultralow temperatures
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a decay or even reversal of the net energy flow may be expected, when the detrimental heating terms compensate the sought cooling effect of the momentum shift. In the following we will extract all these energetic terms by considering the feedback-induced change of energy based on Eqs (21)–(25). More specifically, we consider the change of the average energy, i.e. the difference between the energy after a single step of stochastic cooling and that before,

$$\Delta E = E_+ - E_- = \Delta T_{\parallel} + \Delta T_{\perp} + \Delta V_{\parallel} + \Delta V_{\perp}. \tag{26}$$

### 3.1. Energy change in the longitudinal motion

The dominant change of energy will occur in the potential and kinetic energies associated with the longitudinal motion in the $z$ direction. The longitudinal potential energy $V_{\parallel}$ is given by Eq. (24) and together with Eq. (19) it can be shown that the change of longitudinal potential energy is

$$\Delta V_{\parallel} = \frac{m\omega^2}{8\sigma^2} \langle \tilde{N}_w \rangle. \tag{27}$$

This positive energy contribution arises from the back-action noise of the total-momentum measurement in the $z$ direction. The centre-of-mass $z$ coordinate of the affected atoms, associated with the total mass $m\langle \tilde{N}_w \rangle$, is then subject to an increased uncertainty of the size $(2\sigma)^{-1}$, which Eq. (27) shows to introduce a heating term in the potential energy.

For the kinetic energy of the motion in the $z$ direction a more involved calculation of the second-order $z$-derivative of the density matrix (19) is required. After some lengthy but straightforward calculation the following expression for the feedback-induced change of longitudinal kinetic energy is then obtained:

$$\Delta T_{\parallel} = \frac{\sigma^2}{2m\langle \tilde{N}_w \rangle} - \frac{\langle \tilde{P}_w^2 \rangle}{2m\langle \tilde{N}_w \rangle} + \frac{\langle \Delta \tilde{N}_w \tilde{P}_w^2 \rangle}{2m\langle \tilde{N}_w \rangle^2}. \tag{28}$$

Here $\Delta \tilde{N}_w = \tilde{N}_w - \langle \tilde{N}_w \rangle$ is the fluctuation of the actual atom number in the control beam around its average. The first term in Eq. (28) is the kinetic energy left in the system by the imprecise total-momentum measurement with resolution $\sigma$. The second term is the sought cooling effect, where the centre-of-mass kinetic energy of the affected atoms is removed from the system. The last term, though, arises from quantum fluctuations of the number of atoms in the control-laser beam. This heating term appears since in the momentum-shift operation the actual atom number $\tilde{N}_w$ is not known but only estimated by $\langle \tilde{N}_w \rangle$. Thus this term represents a quantum-statistical imperfection of the feedback loop of stochastic cooling.

### 3.2. Energy change in the transverse motion

At first sight one may guess that the transverse motion is not affected by the feedback loop, since only momentum in the $z$ direction is measured and shifted. However, since the atoms that contribute to the measured signal are confined within the laser beam waist $w_{\perp}(r)$, the measurement of $P_w$ also contains an indirect measurement of the transverse position of atoms with a resolution roughly given by the diameter of the beam. Thus a back-action noise in the transverse momenta can be expected that may lead to further contributions to the kinetic energy. It is now left to show how large these energy contributions are compared with those emerging from the longitudinal motion.

From Eqs (19) and (25) it can be easily seen that the potential energy in transverse $x$ and $y$ directions is unchanged, i.e.

$$\Delta V_{\perp} = 0. \tag{29}$$
This result is obvious since only the momentum in $z$ direction with a transverse spatial confinement is measured without affecting the noise in the transverse coordinates. Let us therefore consider the kinetic energy of the transverse coordinates as defined in Eq. (23). Calculating the required second-order derivatives of Eq. (19) and performing the integrations, after some lengthy but straightforward calculus, we obtain for the change in transverse kinetic energy

$$
\Delta T_{\perp} = \frac{1}{2m} \int dV [\nabla w_\perp(r)]^2 \left\{ \frac{m}{2\sigma^2} \langle \dot{T}_\parallel(r) \rangle + \frac{1}{\langle N_w \rangle^2} \left[ \sigma^2 z^2 + \frac{3}{4} w_\perp^2(r) \right] \langle \dot{\hat{\phi}}(r) \dot{\hat{\phi}}(r) \rangle \right. \\
+ \frac{1}{2\sigma^2 \langle N_w \rangle^2} \left[ \sigma^2 z^2 + \frac{3}{4} w_\perp^2(r) \right] \left\{ \langle \dot{\hat{\phi}}(r) \dot{\hat{\phi}}(r), \hat{P}_w^2 \rangle \right\} \\
+ \frac{1}{4\sigma^2 \langle N_w \rangle} \langle \hat{\rho}_z(\mathbf{r}), \hat{P}_w \rangle \right\} - \frac{1}{2m \langle N_w \rangle} \int dV z \nabla w_\perp(r) \cdot \langle \{ \hat{p}(\mathbf{r}), \hat{P}_w \} \rangle,
$$

where $\{ \hat{A}, \hat{B} \} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is the anti-commutator and the momentum density reads

$$
\hat{p}(\mathbf{r}) = -\frac{i}{2} \left\{ \dot{\hat{\phi}}(\mathbf{r}) \nabla \hat{\phi}(\mathbf{r}) - \left[ \nabla \hat{\phi}(\mathbf{r}) \right] \hat{\phi}(\mathbf{r}) \right\}.
$$

For a thermal equilibrium state, space dependent averages will have a symmetry with respect to $z \rightarrow -z$, and thus the second integral in Eq. (30) can be shown to vanish as an odd moment of $z$.

### 3.3. Gaussian beam-waist function

At this point the specific form of the beam-waist function shall be introduced. We consider here a Gaussian beam with the following definition for $w_\perp(r)$:

$$
w_\perp(\mathbf{r}) = \exp \left[ -\frac{(x-x_0)^2 + (y-y_0)^2}{r_0^2} \right].
$$

(32)

In this way the area $A_0$ of the integrated beam intensity,

$$
A_0 = \int dA w_\perp^2(\mathbf{r}) = \pi r_0^2,
$$

(33)

allows us to interpret $r_0$ as an effective radius of the laser beam. The squared gradient of $w_\perp(\mathbf{r})$ results from Eq. (32) as

$$
[\nabla w_\perp(\mathbf{r})]^2 = w_\perp^2(\mathbf{r}) \frac{(x-x_0)^2 + (y-y_0)^2}{r_0^4},
$$

(34)

and using these results, Eq. (30) reads

$$
\Delta T_{\perp} = \frac{1}{2m} \int dV w_\perp^2(\mathbf{r}) \left\{ \frac{m}{2\sigma^2} \langle \dot{T}_\parallel(\mathbf{r}) \rangle + \frac{1}{\langle N_w \rangle^2} \left[ \sigma^2 z^2 + \frac{3}{4} w_\perp^2(\mathbf{r}) \right] \langle \dot{\hat{\phi}}(\mathbf{r}) \dot{\hat{\phi}}(\mathbf{r}) \rangle \right. \\
+ \frac{1}{2\sigma^2 \langle N_w \rangle^2} \left[ \sigma^2 z^2 + \frac{3}{4} w_\perp^2(\mathbf{r}) \right] \left\{ \langle \dot{\hat{\phi}}(\mathbf{r}) \dot{\hat{\phi}}(\mathbf{r}), \hat{P}_w^2 \rangle \right\} \\
+ \frac{1}{4\sigma^2 \langle N_w \rangle} \langle \hat{\rho}_z(\mathbf{r}), \hat{P}_w \rangle \left\{ \frac{(x-x_0)^2 + (y-y_0)^2}{r_0^4} \right\}.
$$

(35)

This kinetic-energy change will determine the heating effect due to the transverse confinement of atoms in the laser beam.
4. Non-degenerate atomic vapour

For ultracold temperatures near the condensation temperature $T_0$ the longitudinal energy change has been discussed already in the approximation of a rectangular shape of the beam waist in Ref. [28]. In the following we evaluate the complete energy change (26), with the contributions given by Eqs. (27), (28), (29) and (35). We consider the regime of a non-degenerate gas, where the thermal de Broglie wavelength is much smaller than the interatomic distance. In that case the feature of indistinguishability of atoms can be neglected, keeping however the full wave mechanics of the single atom. That is, the single atom’s position and momentum still obey the canonical commutator relation, from which several important effects emerge.

Calculating expectation values for a thermal state in the canonical ensemble at temperature $T$ and total atom number $N$, treating the atoms as distinguishable particles, the number of atoms in the laser beam, for example, results as

$$\langle \hat{N}_w \rangle = N \frac{s^2}{2 + s^2} \exp\left(-\frac{d^2}{2 + s^2}\right).$$  

Here we used the scaled distance $d$ of the control beam from the trap origin and the scaled beam radius $s$, defined by

$$d = \sqrt{\left(\frac{x_0^2 + y_0^2}{L_{th}}\right)}, \quad s = \frac{r_0}{L_{th}}.$$  

The rms extension of the atomic cloud is given by

$$L_{th} = \Delta x_0 \left[\tanh\left(\frac{\omega}{2k_B T}\right)\right]^{-1/2},$$  

with $\Delta x_0 = \sqrt{1/(2m\omega)}$ being the ground-state position uncertainty in the trap potential and $k_B$ being the Boltzmann constant. In the following we also use the size of the atomic cloud in units of the ground-state uncertainty:

$$l_{th} = \frac{L_{th}}{\Delta x_0}.$$  

4.1. Longitudinal energy change

The complete change of energy in the longitudinal motion in units of vibrational energy quanta reads

$$\Delta E_{\parallel}/\omega = \frac{1}{4} \left[ \frac{(\sigma/\Delta p_0)^2}{\langle \hat{N}_w \rangle} + \frac{\langle \hat{N}_w \rangle}{(\sigma/\Delta p_0)^2} \right] - \frac{l_{th}^2}{4}$$

$$+ \frac{l_{th}^2}{4N} \left\{ \frac{(2 + s^2)^2}{s^2(4 + s^2)} \exp\left[\frac{4d^2}{(2 + s^2)(4 + s^2)}\right] - 1 \right\},$$  

where $\Delta p_0 \Delta x_0 = \frac{\hbar}{2}$ so that $\Delta p_0 = \sqrt{m\omega}/2$. The first term represents the measurement-induced noise leading to an increase of kinetic and potential energies. Taking into account only the energy change as given here this heating effect can be minimised by adapting the measurement resolution to the number of atoms in the control-laser beam as

$$\sigma = \Delta p_0 \sqrt{\langle \hat{N}_w \rangle}.$$  

The minimum heating due to this noise results then as one half energy quantum.

The sought cooling effect is represented by the second term in Eq. (40), which is the centre-of-mass kinetic energy of the atoms addressed by the feedback. This value does neither
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Figure 2. Boundaries between cooling (−) and heating (+) for the temperature $T = 10 T_0$ and varying total numbers of atoms. $\sigma$ is chosen as the optimal value given in Eq. (41).

depend on size nor location of the feedback region. For large temperatures $l_\perp \to \sqrt{2k_B T/\omega}$, so that the subtracted kinetic energy reduces to $k_B T/(2\omega)$, manifesting the removed energy as being given by the equipartition theorem.

Finally, the third term in Eq. (40) represents quantum noise due to the transverse spatial confinement of atoms subject to the feedback. It is produced by atom-number fluctuations that crucially depend via the average atom number on the size and location of the control-laser beam. This noise is always positive, leading thus to an unavoidable heating contribution, and does vanish only for $s \to \infty$. The limit $s \to \infty$ realizes the situation where all atoms are inside the control-laser beam, thus containing exactly $N$ atoms with vanishing atom-number fluctuations. Moreover, the strength of this heating term diminishes with increasing total number of atoms, as can be observed in Fig. 2.

There the contours $\Delta E_\parallel = 0$ have been plotted in the parameter space $(s, d)$ for varying total atom numbers at fixed temperature $T = 10 T_0$, where

$$T_0 = \frac{\omega}{k_B} \left( \frac{N}{\zeta(3)} \right)^{1/3},$$

is the condensation temperature in the thermodynamic limit [40] with $\zeta(n)$ being the Riemann $\zeta$ function. These contours represent the boundaries between cooling (−) on the right-hand side of the contour and heating (+) on its left-hand side. They show that finer spatial resolutions $s$ and larger distances from the trap centre $d$ are allowed for cooling when the total atom number increases. Whereas the measurement-induced noise is constant and negligibly small, the major contribution leading to a restriction of the parameters $s$ and $d$ comes from the atom-number fluctuations in the chosen beam waist.
4.2. Transverse energy change

The energy change in the transverse motion can be obtained in the same way as that for the longitudinal one, cf. Eq. (40), and reads

\[
\Delta E_{\perp}/\omega = \frac{N}{4\langle \hat{N}_w \rangle} \left[ \frac{\langle \hat{N}_w \rangle}{\sigma/\Delta p_0} + \frac{(\sigma/\Delta p_0)^2}{\langle \hat{N}_w \rangle} \right] \frac{4 + s^2(2 + d^2)}{(2 + s^2)^3} \exp \left[ -\frac{d^2}{2 + s^2} \right] \\
+ \frac{N}{4\langle \hat{N}_w \rangle} \left[ \frac{l_{th}^2 + l_{th}^{-2}}{\langle \hat{N}_w \rangle} + \frac{2}{(\sigma/\Delta p_0)^2} \right] \frac{8 + s^2(2 + d^2)}{(4 + s^2)^3} \exp \left[ -\frac{2d^2}{4 + s^2} \right] \\
+ \frac{N(N-1)}{\langle \hat{N}_w \rangle^2} \frac{l_{th}^2 s^2}{4} \left[ 4 + s^2(2 + d^2) \right] \exp \left[ -\frac{2d^2}{2 + s^2} \right] \\
+ \frac{1}{4(\sigma/\Delta p_0)^2\langle \hat{N}_w \rangle^2} \left\{ N \frac{12 + s^2(2 + d^2)}{(6 + s^2)^3} \exp \left[ -\frac{3d^2}{6 + s^2} \right] \\
+ N(N-1) \frac{s^2}{2 + s^2} \frac{8 + s^2(2 + d^2)}{(4 + s^2)^3} \exp \left[ -\frac{d^2(8 + 3s^2)}{(4 + s^2)(2 + s^2)} \right] \left\} \right. \right. 
\]

As mentioned before these heating contributions vanish in the limit \( s \rightarrow \infty \). Moreover, they depend on the total number of atoms \( N \), which is also mediated by the dependence on \( \langle \hat{N}_w \rangle \), see Eq. (36), and possibly on \( \sigma \).

Let us first consider the emerging changes in the boundary between cooling and heating, when now also the transverse energy change is taken into account. That is, we look for the contour in \( (s, d) \) parameter space that satisfies \( \Delta E = 0 \), where \( \Delta E = \Delta E_\parallel + \Delta E_{\perp} \). In Fig. 3 this contour (solid curve) is shown and compared to the boundary based only on longitudinal terms (dashed curve) for a total atom number of \( N = 10^6 \). It is clearly observed that the additional heating terms due to the transverse confinement of atoms in the control beam leads to a shift of the boundary to larger values of \( s \) and smaller values of \( d \). The former is due to the fact that the measurement of the total momentum in \( z \) direction of atoms in the beam,
indirectly also represents a measurement of transverse coordinates within the beam-waist size. This leads again to measurement back-action noise, which is naturally reduced by increasing $s$.

The fact that the boundary is shifted to smaller values of $d$ is due to relative atom-number fluctuations that decrease for increasing atom numbers found near to the trap centre. It should be noted that either curve actually contains two different curves at temperatures $T = 10 T_0$ and $T = 10000 T_0$, which however cannot be distinguished in the plot. The only explicit dependence in Eqs. (40) and (43) is due to the occurrence of $l_{\text{th}}$. For $N \gg 1$, however, this dependence is very weak. Nevertheless, all features discussed here and in the following implicitly depend on temperature via the chosen scaling of $s$ and $d$ by $L_{\text{th}}$.

Assuming that for increasing total atom number the optimised measurement resolution $\sigma$ increases, as for the value given in Eq. (41) for longitudinal terms, for large atom numbers $N \gg 1$ an asymptotic expansion can be given for Eq. (43):

$$\Delta E_\perp / \omega = \frac{1}{4} \left\{ \frac{\langle \hat{N}_w \rangle}{\langle \sigma / \Delta p_0 \rangle^2} + \frac{\langle \sigma / \Delta p_0 \rangle}{\langle \hat{N}_w \rangle} + \frac{l_{\text{th}}^2}{s^2} \right\} \frac{4 + s^2 (2 + d^2)}{s^2 (2 + s^2)^2},$$

(44)

where Eq. (36) has been used. In the same limit the longitudinal energy change can be approximated to finally obtain the total change of energy as

$$\Delta E / \omega = \frac{1}{4} \left\{ \frac{\langle \sigma / \Delta p_0 \rangle^2}{\langle \hat{N}_w \rangle} + \frac{\langle \hat{N}_w \rangle}{\langle \sigma / \Delta p_0 \rangle^2} \right\} \left[ 1 + \frac{4 + s^2 (2 + d^2)}{s^2 (2 + s^2)^2} \right] - \frac{l_{\text{th}}^2}{4} \left[ 1 - \frac{4 + s^2 (2 + d^2)}{s^2 (2 + s^2)^2} \right].$$

(45)

This asymptotic expansion represents a good approximation starting already from atom numbers $N > 100$, as can be seen from Fig. 4. There it is observed that for fixed temperature the boundaries converge quickly to the asymptotic one for $N \to \infty$.

Equation (45) shows that again the optimal value for the measurement resolution is given by (41), for which the final result reads

$$\Delta E / \omega = \frac{1}{2} \left[ 1 + \frac{4 + s^2 (2 + d^2)}{s^2 (2 + s^2)^2} \right] - \frac{l_{\text{th}}^2}{4} \left[ 1 - \frac{4 + s^2 (2 + d^2)}{s^2 (2 + s^2)^2} \right].$$

(46)
Figure 5. Dependence of $s_{\min}$ on $l_{th}^{2}$ for large total number of atoms $N \rightarrow \infty$.

For this expression the solution of the condition $\Delta E = 0$ can be analytically given as

$$d(s) = (2 + s^2) \sqrt{l_{th}^2 - \frac{2}{l_{th}^2 + 2}} - \frac{2}{s^2(2 + s^2)}.$$

For $d = 0$ the corresponding value for $s$ is the minimal beam-waist radius. This minimal radius is obtained as

$$s_{\min} = \left( \sqrt{1 + \frac{l_{th}^2}{l_{th}^2 - 2}} - 1 \right)^{1/2}.$$

In Fig. 5 this function is shown in dependence on $l_{th}^2$, which for high temperatures is proportional to $T$. At $l_{th}^2 = 2$ the removed centre-of-mass kinetic energy exactly compensates the measurement-induced heating, which requires $s_{\min} \rightarrow \infty$. For values $l_{th}^2 < 2$ cooling does not occur, since the unavoidable measurement-induced noise can no longer be compensated. However, the corresponding temperatures for $l_{th}^2 \leq 2$ are below the condensation temperature $T_0$, where our approach for a non-degenerate gas is no longer valid. For this regime see Ref. [28].

For larger values of $l_{th}^2$, or $T$ correspondingly, the limiting value $s_{\min} \rightarrow (\sqrt{3} - 1)^{1/2} \approx 0.86$ is reached. In this regime only a fraction of the atomic cloud needs to be subject to the feedback loop since $s_{\min} < 1$. The unscaled minimum beam-waist radius is thus $86\%$ of the rms extension of the atomic cloud.

5. Conclusions

In summary we have studied the effects of transverse confinement in stochastic cooling of trapped atoms. It could be clearly shown that these effects are substantial for the cooling process and that minimum values for both the size and location of the control-laser beam exist.

In the regime of non-degenerated gases analytical expressions could be derived, that contain the full quantum-fluctuation effects. Among these effects are atom-number fluctuations that appear due to the finite volume of the control beam. They appear in form of an imperfection of the feedback loop and in form of back-action noise due to the indirect measurement of transverse coordinates.
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References