Optical quantum computation using cluster states

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We propose an approach to optical quantum computation in which a deterministic entangling quantum gate may be performed using, on average, a few hundred coherently interacting optical elements (beamsplitters, phase shifters, single photon sources, and photodetectors with feedforward). This scheme combines ideas from the optical quantum computing proposal of Knill, Laflamme and Milburn [Nature 409 (6816), 46 (2001)], and the abstract cluster-state model of quantum computation proposed by Raussendorf and Briegel [Phys. Rev. Lett. 86, 5188 (2001)].

Optical approaches to quantum computation are attractive due to the long decoherence times of photons, and the relative ease with which photons may be manipulated. A ground breaking proposal of Knill, Laflamme and Milburn (KLM) demonstrated that all-optical quantum computation is, in principle, possible using just beamsplitters, phase shifters, single photon sources, and photodetectors with feedforward. Experimental demonstrations of several of the basic elements of KLM have since been performed [1–4].

Despite these impressive successes, the obstacles to fully scalable quantum computation with KLM remain formidable. The biggest challenge is to perform a two-qubit entangling gate in the near-deterministic fashion required for scalable quantum computation. KLM propose doing this using a combination of three ideas. (1) Using linear optics, single-photon sources and photodetectors, non-deterministically perform an entangling gate. This gate fails most of the time, destroying the state of the computer, and so is not immediately suitable for quantum computation. (2) By combining the basic non-deterministic gate with quantum teleportation, a class of non-deterministic gates which are not so destructive is found. We denote these gates \( CZ_{n^2/(n+1)^2} \), where \( n \) is a positive integer. \( CZ_{n^2/(n+1)^2} \) has probability of success \( n^2/(n+1)^2 \); the larger \( n \) is, the greater the chance of success, but the more complex the corresponding optical circuit. (3) By using quantum error-correction, the probability of the gate succeeding can be improved until the gate is near-deterministic, allowing scalable quantum computation.

The combination of these three ideas allows quantum computation, in principle. Existing experimental implementations have demonstrated (1), and promise to do (2) (for small values of \( n \)) in the near future. However, to perform \( CZ_{n^2/(n+1)^2} \) for large values of \( n \), or to do step (3), is far more difficult. KLM analyse a scheme in which the \( CZ_{9/16} \) gate is combined with error-correction. To do a single entangling gate with probability of success 95% requires about 300 successful \( CZ_{9/16} \) gate operations, i.e., tens of thousands of optical elements. Higher probabilities of success require more optical elements.

The present paper describes an approach to optical quantum computation that makes use of ideas (1) and (2) (for \( n = 1 \) and \( n = 2 \)), but avoids step (3). The scheme combines KLM’s non-deterministic gates with the cluster-state model of quantum computation proposed by Raussendorf and Briegel [5]. Using a \( CZ_{4/9} \) gate (which uses roughly 2–3 times fewer optical elements than the \( CZ_{9/16} \) gate) a single logical quantum gate in this proposal requires, on average, fewer than 8 successful \( CZ_{4/9} \) gates. In this scheme there is an additional overhead due to the single-qubit gates; even when that is taken into account, fewer than 24 \( CZ_{4/9} \) gates are required to simulate an entangling gate. This is not only substantially simpler than KLM, but the resulting logical gates work deterministically (assuming ideal optical elements), as opposed to the 5% error experienced by KLM’s entangling gates.

Yoran and Reznik [6] have proposed a scheme for optical quantum computation based on KLM, but using substantially simpler resources. This scheme has several elements in common with the current proposal, including offline preparation of a quantum state, which is used to do computation deterministically. (These similarities bear further investigation; although [6] does not use the cluster-state model of computation, their method has many similarities.) For comparison, [6] estimate 20–30 \( CZ_{9/16} \) gates per logical gate, or perhaps 2–3 times as many optical elements as the cluster-state proposal, due to the greater complexity of the \( CZ_{9/16} \) gate.

Cluster-state quantum computation: The cluster-state model of quantum computation [8] is a beautiful alternate model of quantum computation, mathematically equivalent to the standard quantum circuit model, but quite different in physical aspect. We describe briefly the procedure used to simulate a quantum circuit in the cluster-state model; proofs may be found in [7]. Note that this is an abstract model for quantum computation, not a proposal for physical implementation, so we describe it without reference to the details of a specific physical system.

To simulate a quantum circuit like that in Fig. 1, we first prepare the cluster state, an entangled network of qubits defined as in Fig. 4. Each qubit in the quan-
Quantum circuit is replaced by a horizontal line of qubits in the cluster state. Different horizontal qubits represent the original qubit at different times, with the progress of time being from left to right. Each single-qubit gate in the quantum circuit is replaced by two horizontally adjacent qubits in the cluster state. (Alternately, one or three horizontally adjacent qubits could be used; that would correspond to slightly different classes of single-qubit unitaries being simulated.) CPHASE gates in the original circuit are simulated using a vertical “bridge” connecting the appropriate qubits.

\[
\begin{align*}
|+\rangle & \quad U_{\alpha_1,\alpha_2} \quad U_{\alpha_3,\alpha_4} \\
|+\rangle & \quad U_{\beta_1,\beta_2} \quad U_{\beta_3,\beta_4}
\end{align*}
\]

FIG. 1: A two-qubit quantum circuit. Without loss of generality we assume the computation starts in the \(|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}\) state, since single-qubit gates can be prepended to the circuit if we wish to start in some other state. The boxes are single-qubit gates, \(U_{\alpha,\alpha'} \equiv X_{\alpha'}Z_{\alpha}\), denoting a rotation by \(\alpha\) about the \(z\) axis of the Bloch sphere, followed by a rotation by \(\alpha'\) about the \(x\) axis. The two-qubit gate is a controlled-phase (CPHASE) gate, whose action in the computational basis is \(|ab\rangle \to (-1)^{ab}|ab\rangle\). CPHASE and the single-qubit operations \(U_{\alpha,\alpha'}\) are together universal for quantum computation.

With the cluster state prepared, simulation of the circuit is accomplished using single-qubit measurements, and feedforward of measurement results to control the basis in which later measurements are performed. The sequence of measurements is illustrated in Fig. 3. Each circle represents a single qubit. The cluster state is constructed by preparing each qubit in the state \(|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}\), and then applying CPHASE between any two qubits joined by a line. Since the CPHASE gates commute with one another, it does not matter in what order they are applied.

For convenience we have presented the cluster-state model in a slightly different form than \(\mathbb{R}\). In \(\mathbb{R}\) the vertical bridges contain two additional intermediate qubits in order to simulate a CPHASE gate. The reason \(\mathbb{R}\) has this more complicated bridge is because they assume that the quantum circuit one wishes to simulate is not known until after preparation of the cluster state. \(\mathbb{R}\) make this assumption in order to show that a single cluster state can simulate an arbitrary quantum computation of a given breadth and depth. In implementation one knows the circuit beforehand (e.g. Shor’s circuit for factoring \(\mathbb{R}\)), and the intermediate qubits in the bridge can be dispensed with. The simpler bridge, while not essential, does simplify the optical implementation.

To combine the cluster-state model with KLM we need one final observation about the properties of cluster states. Using the definition of the cluster state and the CPHASE gate we obtain the following expression for the cluster state, up to normalization,

\[
\sum_{z_1, z_2, \ldots} (-1)^{\text{sign}(z_1, z_2)} |z_1, z_2, \ldots\rangle,
\]

where the first sum is over all configurations \(z_1, z_2, \ldots\) \((z_j = 0, 1)\) of the qubits making up the cluster state, and the sum in the exponent is over all pairs \((j, k)\) of neighboring qubits in the cluster. Suppose we measure one of the cluster qubits in the computational basis, with outcome \(m\). It follows from Eq. (1) that the posterior state is just a cluster state with that node deleted, up to a local \(Z^m\) operation applied to each qubit neighbouring the deleted qubit. These are known local unitaries, whose effect may be compensated in subsequent operations, so we may effectively regard such a computational basis measurement as simply removing the qubit from the cluster.

KLM optical quantum computation: KLM encodes a single qubit in two optical modes, \(A\) and \(B\), with logical qubit states \(|0\rangle_L \equiv |0\rangle_{AB}\) and \(|1\rangle_L \equiv |1\rangle_{AB}\). State preparation is done using single-photon sources,
While measurements in the computational basis may be achieved using high-efficiency photodetectors. Such sources and detectors make heavy demands not entirely met by existing optical technology, although encouraging progress on both fronts has been reported recently. Arbitrary single-qubit operations are achieved using phase shifters and beamsplitters.

The main difficulty in KLM is achieving near-deterministic entangling interactions between qubits. KLM propose two basic constructions, one building upon the other, for implementing a non-deterministic CPHASE gate, that is, a gate which with some probability succeeds, and with some probability fails, and whether the gate succeeds or fails is known. The two constructions differ in their success probability, and in whether failure results in the destruction of the qubits, or in some incorrect operation being applied. We now summarize the basic properties of the two constructions.

The destructive non-deterministic CPHASE gate: We describe a construction of Knill 8 that slightly simplifies the original KLM construction. Knill’s construction takes two logical qubits as input, and with probability 2/27 applies a CPHASE gate, or else fails, destroying the state of the qubits. The gate uses two phase shifters, four beamsplitters, two single-photon ancillas, and two photodetectors measuring the ancillas; these must be capable of distinguishing 1 photon from 0 or 2 photons.

Non-destructive non-deterministic CPHASE gates: The gate just described can be improved by combining it with the idea of gate teleportation 3, 11, 12. The result is a gate $CZ_{n^2/(n+1)^2}$ which with probability $n^2/(n+1)^2$ applies a CPHASE to two input qubits, where $n$ is a positive integer. When the gate fails, the effect is to perform a measurement of those qubits in the computational basis. Increasing values of $n$ correspond to increasingly complicated teleportation circuits. The only two values of $n$ we shall need are $n = 1$ and $n = 2$, both of which use relatively simple teleportation circuits, with just a few optical elements — for $n = 1$, 8 beamsplitters, 4 photodetectors, and 4 single-photon preparations; for $n = 2$ least 70 beamsplitters, 30 photodetectors, and 12 single-photon preparations.

The basic $CZ_{n^2/(n+1)^2}$ gate involves two teleportation steps performed in parallel on the two qubits, succeeding with independent probabilities $n/(n+1)$. It is possible to perform these teleportations sequentially, with the result 3 that if the first teleportation fails, we can abort the gate, without harming the second qubit. More generally, if we wish to perform CPHASE gates between a single qubit $S$, and several other qubits $A, B, \ldots$, it is possible to first perform all the teleportation steps involving just qubit $S$, and abort if any fail, preserving qubits $A, B, \ldots$. If they all succeed, the remaining teleportation steps involving the other qubits are performed, each with probability of success $n/(n+1)$. Doing the gates in this sequential way will have considerable advantages in the cluster-state model of quantum computation.

KLM achieves scalable quantum computation by combining quantum error-correction and the elements we have described to develop a CPHASE gate that succeeds with much higher probability. This construction is avoided in the cluster-state implementation of optical quantum computation, and so we omit a description.

**Optical quantum computation with cluster states:** The idea is to build up the cluster state by non-deterministically adding extra qubits to the cluster using $CZ_{4/9}$ or $CZ_{1/4}$ gates. If this can be done, the other operations in the cluster-state model can be done following KLM’s prescription. To simplify preparation it helps to suppose that each qubit in the cluster is involved in at most a single vertical bridge. The only reason more vertical connections might be required is if the quantum circuit being simulated involves the same qubit in multiple parallel CPHASE gates. We may assume this does not occur, without affecting the ability of a cluster-state computation to efficiently simulate a quantum circuit.

We will build the cluster up by interleaving two types of operation: attempting to add a site connected to the current cluster through a single bond, and attempting to add a site connected to the current cluster through a double bond. It is not difficult to see that any cluster can be built up by alternating operations of this type. We analyse the two cases separately.

The procedure to add a site connected by a single bond to the cluster is illustrated in Fig. 3. With probability 2/3 this succeeds, and a site is added to the cluster, while with probability 1/3 it fails, and a measurement in the computational basis removes a qubit from the cluster, namely, the qubit with which a CPHASE was attempted. Thus, the expected number of sites added to the cluster is $2/3 \times 1 + 1/3 \times (-1) = 1/3$.

![FIG. 4: Attempting to add a site connected by a single bond to the current cluster, using a $CZ_{4/9}$ gate. By performing the gate with sequential teleportations we ensure that the probability of success is 2/3.](image)

The procedure used to add a site connected to the current cluster by a double bond is illustrated in Fig. 4. We sequentially attempt $CZ_{4/9}$ gates between qubits $S$ and $A$, and $S$ and $B$. As described earlier, this can be done so that each gate works with probability 2/3. If the gate between $S$ and $A$ fails, then qubit $A$ is removed from the cluster, and we abort the procedure. This occurs with probability 1/3. If it succeeds, then we attempt $CZ_{4/9}$ between qubit $S$ and $B$. If this fails, then qubit $B$ is removed from the cluster, and we abort the procedure. This occurs with probability 2/9. If both gates succeed then we add qubit $S$ to the cluster. This occurs with probability 4/9. The expected number of sites added to the cluster is thus $-1/9$.

Observe that any cluster may be built up by alternating two steps: (a) attempting to add one or more sites
that are connected to the current cluster by a single bond, and (b) attempting to add just one site that is connected by a double bond. We conclude that for every two attempts to add a site, the average number of sites added by a double bond. We conclude that for every two attempts to add a site, the average number of sites added by a double bond.

We have described a scheme for optically preparing cluster states and using them for computation. Many alternate approaches to preparation may be conceived. One interesting approach is illustrated in Fig. 6. Microclusters” are non-deterministically prepared and then “glued” together using $CZ_{1/4}$ gates, in order to create the cluster. An advantage of this approach is that the basic elements are $CZ_{1/4}$ gates, instead of the more complex $CZ_{4/9}$ gates. In the short term this is likely to be simpler to implement, and to offer proof-of-principle experimental demonstrations. Over the long run, however, the polynomial overhead incurred by this scheme means that the scheme based on $CZ_{4/9}$ gates is more promising.

Conclusion: By combining the abstract cluster-state model of computation with KLM we obtain a scheme for optical quantum computation significantly less demanding than in existing schemes based on single-photon preparation, linear optics, and photodetectors. How it compares with schemes using different basic elements, like the coherent-state scheme of Ralph et al. [12], depends on future technological developments. Work is underway to simplify the scheme further, and to address the question of fault-tolerance.

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