SPECIAL RELATIVITY WITH TWO INVARIANT SCALES: MOTIVATION, FERMIONS, BOSONS, LOCALITY, AND CRITIQUE

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We present a Master equation for description of fermions and bosons for special relativities with two invariant scales, SR2, (c and λP). We introduce canonically-conjugate variables (χ0, χ0) to (ǫ, π) of Judes-Visser. Together, they bring in a formal element of linearity and locality in an otherwise non-linear and non-local theory. Special relativities with two invariant scales provide all corrections, say, to the standard model of the high energy physics, in terms of one fundamental constant, λP. It is emphasized that spacetime of special relativities with two invariant scales carries an intrinsic quantum-gravitational character. In an addenda, we also comment on the physical importance of a phase factor that the whole literature on the subject has missed and present a brief critique of SR2. In addition, we remark that the most natural and physically viable SR2 shall require momentum-space and spacetime to be non-commutative with the non-commutativity determined by the spin content and C, P, and T properties of the examined representation space. Therefore, in a physically successful SR2, the notion of spacetime is expected to be deeply intertwined with specific properties of the test particle.

1. Motivation for SR2

There is a growing theoretical evidence that gravitational and quantum frameworks carry some elements of incompatibilities. The question is how deep are the indicated changes, and what precise form they may take. One hint comes from the observation that incorporating gravitational effects in quantum measurement of spacetime events leads to a Planck-scale saturation. In the framework of Kempf, Mangano, Mann, and the present author, the gravitationally-induced modification to the de Broglie (dB) wave-particle duality takes the form

$$\lambda_{dB} = \frac{h}{p} \quad \text{grav.} \quad \lambda = \frac{\lambda_P}{\tan^{-1}(\lambda_P/\lambda_{dB})},$$

1The work presented here is an extended version of the informal lectures given by the author in May 2001 in Rome.

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where $\lambda_P$ is the Planck circumference ($= 2\pi \lambda_P$), with $\lambda_P = \sqrt{\hbar G/c^3}$, as the Planck length. The $\lambda$ reduces to $\lambda_{dB}$ for the low energy regime, and saturates to $4\lambda_P$ in the Planck realm. In this way the Planck scale is not merely a dimensional parameter but has been brought in relation to a universal saturation of gravitationally-modified de Broglie wavelengths.

This is a very welcome situation for theories of quantum gravity where for a long time a paradoxical situation had existed. Each inertial observer could measure in his frame the fundamental universal constants, $\hbar, c, G$, and obtain from them a universal fundamental constant, $\lambda_P$. And yet this very $\lambda_P$ — being a length scale — is subject to special-relativistic length contraction which paradoxically makes it loose its universal character.

The indicated saturation then not only resolves this paradoxical situation but also suggests that special relativity must suffer a modification. This modification must be endowed with the property that it carries two invariant scales; one the usual $c$, and the second $\lambda_P$.

The necessity for a SR2 as argued in Refs. [3, 4] is similar to ours, while motivation of Ref. [5] is contained in certain anomalies in astrophysical data; see Refs. [6, 7, 8, 9, 10].

2. Existing SR2 proposals

Simplest of SR2 theories result from keeping the algebra of boost- and rotation-generators intact while modifying the boost parameter in a non-linear manner. Specifically, in the SR2 of Amelino-Camelia the boost parameter, $\varphi$, changes from the special relativistic form

$$\cosh \varphi = \frac{E}{m}, \quad \sinh \varphi = \frac{p}{m}, \quad \hat{\varphi} = \frac{P}{p},$$

(2)

to a new structure:

$$\cosh \xi = \frac{1}{\mu} \left( e^{\lambda_P E} - \frac{cosh (\lambda_P m)}{\lambda_P \cosh (\lambda_P m/2)} \right),$$

(3)

$$\sinh \xi = \frac{1}{\mu} \left( \frac{pe^{\lambda_P E}}{\cosh (\lambda_P m/2)} \right), \quad \hat{\xi} = \frac{P}{p},$$

(4)

*Instead of the term “doubly special relativity (DSR)” coined in the work of Amelino-Camelia, we prefer to use the phrase “special relativity with two invariant scales (SR2).” Without in any way questioning physics content of Amelino-Camelia’s proposal, we take this non-semantic issue for the following reason. The special of “special relativity” refers to the circumstance that one restricts to a special class of inertial observers which move with a relative uniform velocity. The general of “general relativity” lifts this restrictions. The “special” of special relativity has nothing to do with one versus two invariants scales. It rather refers to the special class of inertial observers; a circumstance that remains unchanged in special relativity with two invariant scales. The theory of general relativity with two invariant scales would thus not be called “doubly general relativity.”*
while for the SR2 of Magueijo and Smolin the change takes the form:

\[
\cosh \xi = \frac{1}{\mu} \left( \frac{E}{1 - \lambda P E} \right),
\]

\[
\sinh \xi = \frac{1}{\mu} \left( \frac{p}{1 - \lambda P E} \right), \quad \hat{\xi} = \frac{p}{\mu},
\]

(5)

Here, \( \mu \) is a Casimir invariant of SR2 (see Eq. (27) below) and is given by

\[
\mu = \begin{cases} 
\frac{2}{\lambda_P} \sinh \left( \frac{\lambda m}{2} \right) & \text{for Ref. [5]}'s SR2} \\
\frac{1}{1 - \lambda_P m} & \text{for Ref. [3]}'s SR2}
\end{cases}
\]

(6)

The notation is that of Ref. [12]; with the minor exceptions: \( \lambda, \mu_0, m_0 \) there are \( \lambda_P, \mu, m \) here. In what follows we shall generically represent boost parameter associated with special relativities with one, or two, invariant scales by \( \xi \). The former relativity shall be abbreviated as SR1 (to distinguish it from SR2),\(^b\) Note that giving the explicit expressions for both the sinh \( \xi \) and cosh \( \xi \) in Eqs. (3,4) is necessary in order to fix the form of the energy-momentum dispersion relation through the identity:

\[
\cosh^2 \xi - \sinh^2 \xi = 1.
\]

Of course, one may have chosen to work in terms of one of the hyperbolic trigonometric functions and the dispersion relation, instead.

At this early stage it is not clear if there is a unique SR2, or, if the final choice will be eventually settled by observational data, or by some yet-unknown physical principle. Given this ambiguity, this Article addresses itself to presenting a Master equations for fermionic and bosonic representations for generic SR2.

3. Master equation for spin-1/2: Dirac case

Since the underlying spacetime symmetry generators remain unchanged much of the formal apparatus of the finite dimensional representation spaces associated with the Lorentz group remains intact. In particular, there still exist \((1/2, 0)\) and \((0, 1/2)\) spinors. But now they transform from the rest frame, to an inertial frame in which the particle has momentum, \( p \) as:

\[
\phi_{(1/2, 0)} (p) = \exp \left( \frac{\sigma}{2} \cdot \hat{\xi} \right) \phi_{(1/2, 0)} (0)
\]

(8)

\[
\phi_{(0, 1/2)} (p) = \exp \left( - \frac{\sigma}{2} \cdot \hat{\xi} \right) \phi_{(1/2, 0)} (0).
\]

(9)

Since in this Article we do not undertake a study of the behavior of these spinors under the parity operation, or examine the massless limit in detail, we do not identify the \((0, 1/2)\) spinors as left-handed and the \((1/2, 0)\) spinors as right-handed. Since the null momentum vector \( 0 \) is still isotropic, one may assume that (see p. 44 of Ref. [12] and Refs. [14, 15]):

\[
\phi_{(0, 1/2)} (0) = \zeta \phi_{(1/2, 0)} (0),
\]

(10)

\(^b\)In this notation the Galilean relativity is denoted by SR0.
where \( \zeta \) is an undetermined phase factor. The analysis presented in Ref. [16] also convinces us that the validity of the identity [16] is independent of the “right-left” identification of the standard argument. In general, the phase \( \zeta \) encodes C, P, and T properties. The interplay of Eqs. (8-9) and (10) yields the Master equation for the \((1/2, 0) \oplus (0, 1/2)\) spinors,

\[
\psi(p) = \begin{pmatrix} \phi_{(1/2, 0)}(p) \\ \phi_{(0, 1/2)}(p) \end{pmatrix},
\]

(11)
to be

\[
\begin{pmatrix} -\zeta & \exp(\sigma \cdot \xi) \\ \exp(-\sigma \cdot \xi) & -\zeta^{-1} \end{pmatrix} \psi(p) = 0.
\]

(12)

This is one of the central results of this Article. As a check, taking \( \xi \) to be \( \varphi \), and after some simple algebraic manipulations, the Master equation (12) reduces to:

\[
\begin{pmatrix} -m\zeta & EI_2 + \sigma \cdot p \\ EI_2 - \sigma \cdot p & -m\zeta^{-1} \end{pmatrix} \psi(p) = 0,
\]

(13)

where \( I_n \) stands for \( n \times n \) identity matrix (and \( 0_n \) shall represent the corresponding null matrix). With the given identification of the boost parameter we are in the realm of SR1. There, the operation of parity is well understood. Demanding parity covariance for Eq. (13), we obtain \( \zeta = \pm 1 \). Identifying

\[
\begin{pmatrix} 0_2 \ I_2 \\
I_2 \ 0_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 -\sigma \\
\sigma \ 0_2 \end{pmatrix},
\]

(14)

with the Weyl-representation \( \gamma^0 \), and \( \gamma^1 \), respectively, Eq. (13) reduces to the Dirac equation of SR1

\[
(\gamma^\mu p_\mu \mp m) \psi(p) = 0.
\]

(15)

The linearity of the Dirac equation in, \( p_\mu = (E, -p) \), is now clearly seen to be associated with two observations:

\( O_1 \). That, \( \sigma^2 = I_2 \); and

\( O_2 \). That in SR1, the hyperbolic functions – see Eq. (2) – associated with the boost parameter are linear in \( p_\mu \).

In SR2, observation \( O_1 \) still holds. But, as Eqs. (3-4) show, \( O_2 \) is strongly violated. For this reason the Master equation (12) cannot be cast in a manifestly covariant form with a finite number of contracted Lorentz indices of SR2 as long as we mark spacetime events by \( x^\mu \) of SR1.

The last inference is also a welcome result as it indicates a possible intrinsic non-locality in SR2s. Since in all SR2s the shortest spatial length scales that can be probed are bound from below by \( \lambda_P \), the naively-expected \( \delta^3(\mathbf{x} - \mathbf{x'}) \) in the anticommutators of the form \( \{ \Psi_i(x, t), \Psi_j^\dagger(x', t) \} \) should be replaced by a highly, but not infinitely, peaked Gaussian-like functions with half-width of the order of \( \lambda_P \).
Note that, the spinors are obtained without reference to a wave equation:

\[ \psi(p) = \begin{pmatrix} \exp\left(\frac{\sigma_2}{2} \cdot \xi\right) & 0_2 \\ 0_2 & \exp\left(-\frac{\sigma_2}{2} \cdot \xi\right) \end{pmatrix} \begin{pmatrix} \phi_{(1/2, 0)}(0) \\ \phi_{(0, 1/2)}(0) \end{pmatrix}. \]  

(16)

The \( \phi_{(1/2, 0)}(0) \) as well as \( \phi_{(0, 1/2)}(0) \) are taken as eigenstates of the helicity operator,

\[ \frac{\sigma_2}{2} \cdot \hat{\xi}. \]  

(17)

The choice \( \zeta = +1 \), in Eq. (10), yields the “particle” spinors, while, \( \zeta = -1 \), gives the “antiparticle” spinors. The extension of the presented formalism for Majorana spinors is more subtle.\[ 17, 18, 19, 20 \] We hope to present it an extended version of this Article.

4. Master equation for higher spins

The above-outlined procedure applies to all, bosonic as well as fermionic, \((j, 0) \oplus (0, j)\) representation spaces. It is not confined to \( j = 1/2 \). A straightforward generalization of the \( j = 1/2 \) analysis immediately yields the Master equation for an arbitrary-spin,

\[ \begin{pmatrix} -\zeta & \exp(2J \cdot \xi) \\ \exp(-2J \cdot \xi) & -\zeta^{-1} \end{pmatrix} \psi(p) = 0, \]  

(18)

where

\[ \psi(p) = \begin{pmatrix} \phi_{(j, 0)}(p) \\ \phi_{(0, j)}(p) \end{pmatrix}. \]  

(19)

Equation (18) contains the central result of the previous section as a special case. For studying the \( \text{SR1} \) limit it is convenient to bifurcate the \((j, 0) \oplus (0, j)\) space into two sectors by splitting the \( 2(2j + 1) \) phases, \( \zeta \), into two sets: \( (2j + 1) \) phases \( \zeta_+ \), and the other \( (2j + 1) \) phases \( \zeta_- \). Then, in particle’s rest frame the \( \psi(p) \) may be written as:

\[ \psi_h(0) = \begin{cases} u_h(0) & \text{when } \zeta = \zeta_+ \\ v_h(0) & \text{when } \zeta = \zeta_- \end{cases} \]  

(20)

The explicit forms of \( u_h(0) \) and \( u_h(0) \) which we shall use (see Eq. (10)) are:

\[ u_h(0) = \begin{pmatrix} \phi_h(0) \\ \zeta_+ \phi_h(0) \end{pmatrix}, \quad v_h(0) = \begin{pmatrix} \phi_h(0) \\ \zeta_- \phi_h(0) \end{pmatrix}, \]  

(21)

where \( \phi_h(0) \) are defined as: \( J \cdot \hat{p} \phi_h(0) = h \phi_h(0) \), and \( h = -j, -j + 1, \ldots, +j \). In the parity covariant \( \text{SR1} \) limit, we find \( \zeta_+ = +1 \) while \( \zeta_- = -1 \).

*Even though they may also be obtained as a solution of the relevant wave equation.
As a check, for \( j = 1 \), identification of \( \xi \) with \( \varphi \), and after implementing parity covariance, yields

\[
(\gamma_{\mu\nu} p_{\mu} p_{\nu} \mp m^2) \psi(p) = 0 .
\]  \hspace{1cm} (22)

The \( \gamma_{\mu\nu} \) are unitarily equivalent to those of Ref. \[21\], and thus we reproduce bosonic matter fields with \( \{C, P\} = 0 \). A carefully taken massless limit then shows that the resulting equation is consistent with the free Maxwell equations of electrodynamics.

Since the \( j = 1/2 \) and \( j = 1 \) representation spaces of \( \text{SR}_2 \) reduce to the Dirac and Maxwell descriptions, it is apparent, that the \( \text{SR}_2 \) contains physics beyond the linear-group realizations of \( \text{SR}_1 \). To the lowest order in \( \lambda_P \), Eq. (12) yields

\[
(\gamma^\mu p_\mu + \tilde{m} + \delta_1 \lambda_P) \psi(p) = 0 ,
\]  \hspace{1cm} (23)

where

\[
\tilde{m} = \begin{pmatrix} -\zeta & 0_2 \\ 0_2 & -\zeta^{-1} \end{pmatrix} m
\]  \hspace{1cm} (24)

and

\[
\delta_1 = \begin{cases} 
\gamma^0 \left( \frac{E^2 - m^2}{2} \right) + \gamma^i p_i E & \text{for Ref. \[5\]'s SR}_2 \\
\gamma^\mu p_\mu (E - m) & \text{for Ref. \[3\]'s SR}_2 
\end{cases}
\]  \hspace{1cm} (25)

Similarly, the presented Master equation can be used to obtain \( \text{SR}_2 \)'s counterparts for Maxwell’s electrodynamic. Unlike the Coleman-Glashow framework \[22\], the principle of special relativity with two invariant scales provides all corrections, say, to the standard model of the high energy physics, in terms of one – and not forty six – fundamental constant, \( \lambda_P \).

5. Spin-1/2 and Spin-1 description in Judes-Visser Variables

We now take the tentative position, that the ordinary energy-momentum \( p^\mu \) is not the natural physical variable in \( \text{SR}_2 \).s. The Judes-Visser variables \[12\]: \( \eta^\mu = (\epsilon(E, p), \pi(E, p)) = (\eta^0, \eta^i) \) appear more suited to describe physics sensitive to Planck scale. The \( \epsilon(E, p) \) and \( \pi(E, p) \) relate to the rapidity parameter \( \xi \) of \( \text{SR}_2 \) in same functional form as do \( E \) and \( p \) to \( \varphi \) of \( \text{SR}_1 \):

\[
cosh (\xi) = \frac{\epsilon(E, p)}{\mu} , \quad \sinh (\xi) = \frac{\pi(E, p)}{\mu} ,
\]  \hspace{1cm} (26)

where

\[
\mu^2 = [\epsilon(E, p)]^2 - [\pi(E, p)]^2 .
\]  \hspace{1cm} (27)

They provide the most economical and physically transparent formalism for representation space theory in \( \text{SR}_2 \). For \( j = 1/2 \) and \( j = 1 \), Eq. (18) yields the exact \( \text{SR}_2 \) equations for \( \psi(\pi) \):

\[
(\gamma^\mu \eta_\mu + \tilde{\mu}) \psi(\pi) = 0 ,
\]  \hspace{1cm} (28)

\[
(\gamma_{\mu\nu} \eta_\mu \eta_\nu + \tilde{\mu}^2) \psi(\pi) = 0 ,
\]  \hspace{1cm} (29)
where
\[
\hat{\mu} = \begin{pmatrix}
-\xi^{-1} & 0_2 \\
0_2 & -\xi
\end{pmatrix} \mu.
\] (30)

6. Concluding Remarks

Our task in this Article was to provide a description of fermions and bosons at the level of representation space theory in \(\text{SR}^2\). However, we confined entirely to the representations of the type \((j, 0) \oplus (0, j)\) – these types are important for matter fields, and to study gauge-field strength tensors. To study \(\text{SR}^2\)'s effect on the gauge fields and weak-field gravity the present Article’s formalism needs to be extended to \((j, j)\) representation spaces. In view of Weinberg’s earlier works [23] it is known that there is a deep connection between local quantum field theory, \(\text{SR}^1\) \((j, j)\) spaces [16], and the equality of the inertial and gravitational masses. Therefore, the suggested study must answer \(\text{SR}^2\)'s effect on the equivalence principle.

In quantum field theoretic framework, the special relativity’s spacetime \(x^\mu\) is canonically conjugate to \(p_\mu\), and appears in the field operators as:
\[
\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{h=-j}^{+j} \left[ a_h(p) u_h(p) e^{-ip_\mu x^\mu} + b_h(p) v_h(p) e^{ip_\mu x^\mu} \right],
\] (31)
where the particle-antiparticle spinors, \(u_h(p)\) and \(v_h(p)\) (generically represented by \(\psi_h(p)\)), are solutions of the Master equations (but with \(\xi \rightarrow \varphi\)) introduced above, and can be readily obtained from:
\[
\psi_h(p) = \begin{pmatrix}
\exp(+J \cdot \varphi) & 0_{2j+1} \\
0_{2j+1} & \exp(-J \cdot \varphi)
\end{pmatrix} \psi_h(0).
\] (32)

Now, as our discussion on non-locality indicates \(x^\mu\) of \(\text{SR}^1\) is perhaps not the natural physical spacetime variable at the Planck scale. The spacetime at Planck scale, we suggest, is represented by new event vectors \(\chi^\mu\) (to be treated as “canonically conjugate” to Judes-Visser variable \(\eta^\mu\)); and suggests the following definition for the field operators built upon the \(\text{SR}^2\)'s spinors:
\[
\Psi(\chi) = \int \frac{d^3\eta}{(2\pi)^3} \frac{\mu}{\eta_0} \sum_{h=-j}^{+j} \left[ a_h(\eta) u_h(\eta) e^{-i\eta_\mu \chi^\mu} + b_h(\eta) v_h(\eta) e^{i\eta_\mu \chi^\mu} \right],
\] (33)
with
\[
\psi_h(\eta) = \begin{pmatrix}
\exp(+J \cdot \xi) & 0_{2j+1} \\
0_{2j+1} & \exp(-J \cdot \xi)
\end{pmatrix} \psi_h(0).
\] (34)

Immediately, we verify that for spin-1/2 fermions in \(\text{SR}^2\)
\[
\left\{ \Psi_i(\chi, \chi') \right\} = \delta^3(\chi - \chi') \delta_{ij}.
\] (35)
What appears as non-locality in the space of events marked by \(x^\mu\) now, in the space of events marked by \(\chi^\mu\), exhibits itself as locality. This is a rather unexpected
observation and it calls for a deeper understanding of the $\eta_\mu$ and $\chi_\mu$ description of SR2. The Planck length is intrinsically built in the latter spacetime variables, and it may carry significant relevance for extending SR2 to the gravitational realm.

The evolution of special relativity in the sequence

$$\text{SR}_0 \xrightarrow{c} \text{SR}_1 \xrightarrow{c, \lambda, P} \text{SR}_2$$

(translates to giving spacetime, first, a relativistic and, then, a quantum-gravitational character. The work initiated here, and in Ref. [24], gives concrete shape to modifications that one may expect in the standard model of high-energy physics and theory of gravitation.

7. Addenda: A brief Critique

This Article was penned sometime ago and requires some remarks in the form of an addenda and a critique. These are enumerated below:

1. If all turns out as claimed in this Article then SR2 shall constitute a fundamentally new program for a theory of quantum gravity. At present, there exist serious questions on physical distinguishability of SR2’s from SR1. I have already written on the subject elsewhere [24] and a number of other authors have raised similar questions, see, e.g., Refs. [25, 26, 27, 28, 29]. To brush aside these issues with an argument, such as, “Mathematical triviality by no means implies physical equivalence, and one may argue that it is in fact an asset,” only delays resolution of the issues involved. As has been noted by Czerhoniak [31] the question of distinguishability of SR2 and SR1 is deeply connected with observations of Lukierski and Nowicki [32, 33]—i.e., whether or not the underlying momentum-space/spacetime is commutative, or non-commutative. I have argued in Ref. [34] that the latter is not a matter of choice but a logical implication of the interface of the gravitational and quantum frameworks. The question then is what precise form this non-commutativity takes and if this too, in some way, can also be deciphered from a critical study of representation spaces associated with the spacetime symmetries. A tentative answer, to be presented elsewhere, is in the affirmative. Surprisingly, the non-commutativity seems to depend on the spin content and C, P, and T properties of the examined representation space. Therefore, in a physically successful SR2, the notion of spacetime is expected to be deeply intertwined with specific properties of the test particle.

2. In Ref. [24], the authors have brushed aside the importance of a relative phase difference—specifically in the notation of Eq. (10), they ignore $\zeta = -1$. This is of more than an academic interest. Without it, all antiparticle are absent from the theory.

The symbols above the arrows indicate the invariants for the subsequent SRn.

Unless one ignores a set of two mistakes which partly cancel out the effect, but it then comes
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References


back to plague when one considers Maxwell’s field, or gravitation.
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