Generation of entangled squeezed states in atomic Bose-Einstein condensates

Le-Man Kuang*1,2, Ai-Hua Zeng3 and Zhen-Hua Kuang3
1 Department of Physics, Hunan Normal University, Changsha 410081, People’s Republic of China
2 The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, Trieste 34014, Italy
3 Department of Physics, Shaoyang University, Shaoyang 422000, People’s Republic of China

A method for producing entangled squeezed states (ESSs) for atomic Bose-Einstein condensates (BECs) is proposed by using a BEC with three internal states and two classical laser beams. We show that it is possible to generate two-state and multi-state ESSs under certain circumstances. PACS number(s): 03.75.Fi, 03.65.Ud, 03.65.Ta, 42.50.Dv

I. INTRODUCTION

Quantum entanglement has been the focus of much work in the foundations of quantum mechanics, being particularly with quantum nonseparability, the violation of Bell’s inequalities, and the so-called Einstein-Pololsky-Rosen (EPR) paradox. Beyond this fundamental aspect, creating and manipulating of entangled states are essential for quantum information applications. Among these applications are quantum computation [1], quantum teleportation [2], quantum dense coding [3], and quantum cryptography [4]. Hence, quantum entanglement has been viewed as an essential resource for quantum information processing.

In recent years, much progress has been made on creating quantum entanglement between macroscopic atomic samples [5, 6, 7, 8, 9, 10, 11]. There are several proposals to generate quantum entanglement between macroscopic atomic ensembles [6] and to explore its applications to quantum communication [8, 10, 11] and quantum computation [11]. In particular, quantum entanglement between two separate macroscopic atomic samples [6] has been demonstrated experimentally. On the aspect of atomic Bose-Einstein condensates (BECs) it has been shown that substantial many-particle entanglement can be generated directly in a two-component weakly interacting BEC using the inherent inter-atomic interactions [6, 11] and a spinor BEC using spin-exchange collision interactions [8, 10, 11].

Based on an effective interaction between two atoms from coherent Raman processes, Helmerson and You [12] proposed a coherent coupling scheme to create massive entanglement of BEC atoms. An entanglement swapping scheme between trapped BECs [17] has also been proposed. Indeed, nowadays manipulation and control of quantum entanglement between BEC atoms has become one of important goals for experimental studies with BECs. As well known, one of the key problems in the experimental explorations of quantum entanglement is to coherently control interaction between the relevant particles. The strength of the inter-atomic interactions in atomic BECs can vary over a wide range of values through changing external fields. This kind of control and manipulation of inter-atomic interactions has been experimentally realized through magnetical-field-induced Feshbach resonances in atomic BECs [18]. Therefore, atomic BECs provide us with an ideal experimental system for studying quantum entanglement.

On the other hand, recently much attention has been paid to continuous variable quantum information processing in which continuous-variable-type entangled pure states play a key role. For instance, two-state entangled coherent states are used to realize efficient quantum computation [10] and quantum teleportation [20]. Two-mode squeezed vacuum states have been applied to quantum dense coding [21]. In particular, following the theoretical proposal of Ref. [22], continuous variable teleportation has been experimentally demonstrated for coherent states of a light field [23] by using entangled two-mode squeezed vacuum states produced by parametric down-conversion in a sub-threshold optical parametric oscillator. It is also has been shown that a two-state entangled squeezed vacuum state can be optically created and used to realize quantum teleportation of an arbitrary coherent superposition state of two equal-amplitude and opposite-phase squeezed vacuum states [24, 25]. Therefore, it is an interesting topic to create entangled squeezed states in atomic BECs.

In this paper, we present a scheme to produce entangled squeezed states for atomic BECs. The proposed system consists of an atomic BEC with three internal states and two classical laser beams with appropriate frequencies. They form a three-level lambda configuration. We show that it is possible to generate entangled squeezed states for atomic BECs. This paper is organized as follows. In Sec. II, we present the physical system under our consideration, establish our model, and give an approximate analytic solution of the model. In Sec. III, we show how to produce entangled squeezed vacuum states for atomic BECs. We shall conclude our paper with discussions and remarks in the last section.

II. MODEL AND SOLUTION

Consider a cloud of BEC atoms which have three internal states labelled by |1⟩, |2⟩, and |3⟩ with energies...
$E_1$, $E_2$, and $E_3$, respectively. The two lower states $|1\rangle$ and $|3\rangle$ are Raman coupled to the upper state $|2\rangle$ via, respectively, two classical laser fields of frequencies $\omega_1$ and $\omega_2$ in the Lambda configuration. The interaction scheme is shown in Fig. 1. The atoms in these internal states are subject to isotropic harmonic trapping potentials $V_i(\mathbf{r})$ for $i = 1, 2, 3$, respectively. Furthermore, the atoms in BEC interact with each other via elastic two-body collisions with the $\delta$-function potentials $V_{ij}(\mathbf{r} - \mathbf{r'}) = U_{ij}\delta(\mathbf{r} - \mathbf{r'})$, where $U_{ij} = 4\pi\hbar^2a_{ij}/m$ with $m$ and $a_{ij}$, respectively, being the atomic mass and the s-wave scattering length between atoms in states $i$ and $j$. A good experimental candidate of this system is the sodium atom condensate for which there exist appropriate atomic internal levels and external laser fields to form the Lambda configuration which has been used to demonstrate ultraslow light propagation and amplification of light and atoms in atomic BECs.

The second quantized Hamiltonian to describe the system at zero temperature is given by

$$H = \hat{H}_a + \hat{H}_{a-l} + \hat{H}_c,$$

where $\hat{H}_a$ gives the free evolution of the atomic fields, $\hat{H}_{a-l}$ describes the dipole interactions between the atomic fields and laser fields, and $\hat{H}_c$ represents inter-atom two-body interactions.

The free atomic Hamiltonian is given by

$$\hat{H}_a = \sum_{i=0}^{3} \int dx \hat{\psi}^\dagger_i(x) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_i(x) + E_i \right] \hat{\psi}_i(x),$$

where $E_i$ are internal energies for the three internal states, $\hat{\psi}_i(x)$ and $\hat{\psi}^\dagger_i(x)$ are the boson annihilation and creation operators for the $|i\rangle$-state atoms at position $x$, respectively, they satisfy the standard boson commutation relation $[\hat{\psi}_i(x), \hat{\psi}^\dagger_{i'}(x')] = \delta_{ii'}\delta(x - x')$ and $[\hat{\psi}_i(x), \hat{\psi}_{i'}(x')] = 0$.\][(5)]

The atom-laser interactions in the dipole approximation can be described by the following Hamiltonian

$$\hat{H}_{a-l} = \frac{1}{2} \int dx \left[ \Omega_1 \hat{\psi}^\dagger_i(x) \hat{\psi}^\dagger_{1}(x)e^{i(k_1 \cdot x - \omega_1 t)} + \Omega_2 \hat{\psi}^\dagger_i(x) \hat{\psi}^\dagger_{2}(x)e^{i(k_2 \cdot x - \omega_2 t)} + H.c. \right],$$

where $\Omega_1 = -\mu_{12}E_1/\hbar$ and $\Omega_2 = -\mu_{12}E_2/\hbar$ are the Rabi frequencies of the two laser beams with $\mu_{ij}$ denoting a transition dipole-matrix element between states $|i\rangle$ and $|j\rangle$, $k_1$ and $k_2$ are wave vectors of correspondent laser fields.

The collision Hamiltonian is taken to be the following form

$$\hat{H}_c = \frac{2\pi\hbar^2}{m} \int dx \left[ \sum_{i=1}^{3} a^sc_i \hat{\psi}^\dagger_i(x) \hat{\psi}^\dagger_{1}(x) \hat{\psi}_i(x) \hat{\psi}_{1}(x) \right] + \sum_{i\neq j} \left[ 2a^sc_i \hat{\psi}^\dagger_i(x) \hat{\psi}^\dagger_{j}(x) \hat{\psi}_i(x) \hat{\psi}_{j}(x) \right] + \sum_{i=1}^{3} \lambda_i \hat{b}^\dagger_i \hat{b}^\dagger_i + \sum_{i \neq j} \lambda_{ij} \hat{b}^\dagger_i \hat{b}^\dagger_j \hat{b} i \hat{b} j,$$

where $a^sc_i$ is $s$-wave scattering length of condensate in the internal state $|i\rangle$ and $a^sc_{ij}$ that between condensates in the internal states $|i\rangle$ and $|j\rangle$.

For a weakly interacting BEC at zero temperature one may neglect all modes except for the condensate mode and approximately factorize the atomic field operators as a product of a single mode operator $\hat{b}_i$ and a normalized wavefunction for the atoms in the BEC $\phi_i(x)$, i.e., $\hat{\psi}_i(x) \approx \hat{b}_i \phi_i(x)$ where $\phi_i(x)$ is given by the ground state of the following Schrödinger equation

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V_i(x) + E_i \right] \phi_i(x) = \hbar \nu_i \phi_i(x),$$

where $\hbar \nu_i$ is the energy of the mode $i$.

The valid conditions of the single-mode approximations were demonstrated in Refs. [28, 29], which indicate that this approximation provides a reasonably accurate picture for weak many-body interactions, i.e., for small number of condensed atoms. For large condensates, the mode functions of condensates are altered due to the collision interactions, and the two-mode approximation breaks down. A simple estimate shows that this happens when the number of atoms $N$ satisfies $Na^sc \gg r_0$, where $a^sc$ is a typical scattering length and $r_0$ is a measure of the trap size. If we consider a large trap with the size $r_0 = 100 \mu m$ and the typical scattering length $a^sc = 5 \text{ nm}$, the single mode approximation is applicable for $N \leq 20000$. Substituting the single-mode expansions of the atomic field operators into Eqs. [28, 29], we arrive at the following three-mode approximate Hamiltonian

$$\hat{H} = \hbar \sum_{i=1}^{3} \mu_i \hat{b}^\dagger_i \hat{b}_i - \hbar \left[ g_{1i} \hat{b}^\dagger_1 \hat{b}_1 e^{-i\omega_1 t} + g_{2i} \hat{b}^\dagger_2 \hat{b}_2 e^{-i\omega_2 t} + H.c. \right] + \sum_{i \neq j} \lambda_{ij} \hat{b}^\dagger_i \hat{b}^\dagger_j \hat{b}_i \hat{b}_j,$$
where $g_i$ are the linear interacting constants defined by
\[ g_i = \frac{\Omega_i}{2} \int dx \phi_i^*(x) \phi_1(x) e^{ik_x x}. \] (7)

And $\lambda_i$ and $\lambda_{ij}$ are nonlinear interacting constants given by
\[ \lambda_i = \frac{2\pi \hbar^2 a_{sc}}{m} \int dx |\phi_i(x)|^4, \] (8)
\[ \lambda_{ij} = \frac{4\pi \hbar^2 a_{sc}}{m} \int dx |\phi_i(x)|^2 |\phi_j(x)|^2, \quad (i \neq j). \] (9)

Going over to an interaction picture with respect to
\[ H_0 = \hbar \nu_1 \sum_{i=1}^3 \hat{b}_i^\dagger \hat{b}_i + h(\omega_1 - \omega_2) \hat{b}_1^\dagger \hat{b}_3 + \hbar \omega_1 \hat{b}_2 \hat{b}_2, \] (10)
we can transfer the time-dependent Hamiltonian \[9\] to the following time-independent Hamiltonian
\[ \hat{H}_I = \hbar (\Delta_1 - \Delta_2) \hat{b}_1^\dagger \hat{b}_3 + \hbar \Delta_1 \hat{b}_1^\dagger \hat{b}_2 \]
\[ - h (g_1 \hat{b}_1^\dagger \hat{b}_2 + g_2 \hat{b}_2^\dagger \hat{b}_1 + H.c.) \]
\[ + \sum_{i=1}^3 \lambda_i \hat{b}_i^\dagger \hat{b}_i + \sum_{i \neq j} \lambda_{ij} \hat{b}_i^\dagger \hat{b}_i \hat{b}_j, \] (11)
where $\Delta_1 = \nu_2 - \nu_1 - \omega_1$ and $\Delta_2 = \nu_2 - \nu_3 - \omega_2$ are the detunings of the two laser beams, respectively.

We consider the situation of the exact two-photon resonance (i.e., $\Delta_1 = \Delta_2 = \Delta$), and suppose that the large two-photon detuning $\Delta \gg \nu_3 - \nu_1$. In this case from the Hamiltonian \[10\] the atomic field operators $\hat{b}_2$ and $\hat{b}_2^\dagger$ can be adiabatically eliminated. Then we arrive at the following effective two-mode Hamiltonian containing only atomic field operators in internal states $|1\rangle$ and $|3\rangle$
\[ \hat{H}_{eff} = \omega_1 \hat{b}_1^\dagger \hat{b}_1 + \omega_3 \hat{b}_3^\dagger \hat{b}_3 + (g_1^2 \hat{b}_1^\dagger \hat{b}_1 + g_2^2 \hat{b}_2^\dagger \hat{b}_2) \]
\[ + \lambda_0 \hat{b}_1^\dagger \hat{b}_1^\dagger \hat{b}_1^\dagger \hat{b}_1 + \lambda_1 \hat{b}_1^\dagger \hat{b}_1 \hat{b}_3^\dagger \hat{b}_3 + \lambda_3 \hat{b}_3^\dagger \hat{b}_3 \hat{b}_3^\dagger \hat{b}_3, \] (12)
where we have set $\hbar = 1$ and introduced
\[ \omega_1 = -\frac{|g_1|^2}{\Delta}, \quad \omega_3 = -\frac{|g_2|^2}{\Delta}, \quad g = -\frac{g_1 g_2}{\Delta}. \] (13)

From Eqs. \[12\] and \[13\] we see that the laser-atom interactions are converted as an atomic effective tunneling interaction between state $|1\rangle$ and state $|3\rangle$ with the tunneling coupling strength being determined by strengths of the laser-atom interactions and the detuning. In the derivation of Eq. \[12\], all terms involving $\hat{b}_1^\dagger \hat{b}_2$ have been ignored since the atomic population in the internal state $|2\rangle$ approaches zero under conditions of our consideration.

For the sake of simplicity, we consider a symmetric interaction situation in which inter-atomic interactions in condensates in the internal states $|1\rangle$ and $|3\rangle$ have the same interacting strengths and two applied lasers have the same Rabi frequencies. So that we have $g_1 = g_2$ and $\lambda_1 = \lambda_3 = g$. Then from Eq. \[13\] we can obtain $\omega_1 = \omega_3 = -|g_1|^2/\Delta = g$. Hence the effective Hamiltonian \[12\] reduces to the following simple form
\[ \hat{H}_{eff} = g(\hat{b}_1^\dagger \hat{b}_1 + \hat{b}_3^\dagger \hat{b}_3) + q(\hat{b}_1^\dagger \hat{b}_1^\dagger \hat{b}_1^\dagger \hat{b}_1 + \hat{b}_3^\dagger \hat{b}_3^\dagger \hat{b}_3^\dagger \hat{b}_3) \]
\[ + g(\hat{b}_1 \hat{b}_3 + \hat{b}_3 \hat{b}_1) + 2 \chi \hat{b}_1^\dagger \hat{b}_1 \hat{b}_3, \] (14)
where we have set $\chi = \lambda_{13}/2$. When $q$ and $\chi$ are much less than $|g|$, which is the case of weak inter-atomic nonlinear interactions, the effective Hamiltonian can be solved approximately under the rotating-wave approximation. In order to do this, one introduces the following unitary transformation
\[ \hat{b}_1 = \frac{1}{\sqrt{2}} (\hat{B}_1 - i \hat{B}_3), \quad \hat{b}_3 = \frac{1}{\sqrt{2}} (\hat{B}_1 + i \hat{B}_3), \] (15)
where $\hat{B}_1$ and $\hat{B}_3$ satisfy the usual boson commutation relations: $[\hat{B}_1, \hat{B}_3] = 0 = [\hat{B}_1^\dagger, \hat{B}_3^\dagger]$, and $[\hat{B}_1, \hat{B}_3^\dagger] = \delta_{ij}$ with $\hat{B}_1^\dagger$ being the hermitian conjugate of $\hat{B}_1$. Under the rotating-wave approximation \[31\], we get the following approximate Hamiltonian
\[ \hat{H}_{eff} \approx \omega \hat{N} + g(\hat{B}_1^\dagger \hat{B}_1 - \hat{B}_3^\dagger \hat{B}_3) \]
\[ + \frac{1}{4} q \hat{N}^2 = (\hat{B}_1^\dagger \hat{B}_1 - \hat{B}_3^\dagger \hat{B}_3)^2 \]
\[ + \frac{1}{4} \sqrt{2} \hat{N}^2 - \chi \hat{B}_1^\dagger \hat{B}_1 \hat{B}_3^\dagger \hat{B}_3, \] (16)
where the total number operator $\hat{N}$ is a conserved constant which is given by $\hat{N} = \hat{b}_1^\dagger \hat{b}_1 + \hat{b}_3^\dagger \hat{b}_3 = \hat{B}_1^\dagger \hat{B}_1 + \hat{B}_3^\dagger \hat{B}_3$, and we have introduced a new parameter
\[ \omega = g - \frac{1}{2} (\chi + q). \] (17)

The bases of the Fock spaces in the $(\hat{b}_1, \hat{b}_3)$ and $(\hat{B}_1, \hat{B}_3)$ representations are defined, respectively, by
\[ |n, m\rangle = \frac{1}{\sqrt{n! m!}} \hat{b}_1^m \hat{b}_3^n |0, 0\rangle, \] (18)
\[ |n, m\rangle = \frac{1}{\sqrt{n! m!}} \hat{B}_1^m \hat{B}_3^n |0, 0\rangle, \] (19)
where $n$ and $m$ take non-negative integers. Obviously, $\hat{H}_{eff}$ is diagonal in the Fock space of $(\hat{B}_1, \hat{B}_3)$, and we have
\[ \hat{H}_{eff} |n, m\rangle = E(n, m) |n, m\rangle, \] (20)
where eigenvalues of the Hamiltonian are given by the following expression
\[ E(n, m) = \omega (n + m) + g(n - m) + \frac{1}{4} q (\chi + n) (n + m)^2 \]
\[ + (q - \chi) nm. \] (21)
III. ENTANGLED SQUEEZED STATES

In this section we shall show that entangled squeezed vacuum states for atomic BECs can be produced when atomic BECs are initially in a product squeezed vacuum state through properly manipulating laser-atom interactions and inter-atomic interactions in the BECs.

Consider a product squeezed vacuum state of two squeezed vacuum states defined in Fock spaces of \( |b_1, b_3\rangle \) and \( |B_1, B_3\rangle \), respectively,

\[
|ξ_1, ξ_3\rangle = \hat{S}_{b_1}(ξ_1)\hat{S}_{b_3}(ξ_3)|0,0\rangle,
\]

\[
|η_1, η_3\rangle = \hat{S}_{B_1}(η_1)\hat{S}_{B_3}(η_3)|0,0\rangle,
\]

where the single mode squeezing operators in the \( |b_1, b_3\rangle \) and \( |B_1, B_3\rangle \) representations with arbitrary complex squeezing parameters \( ξ \) and \( η \) \((i = 1, 3)\) are defined by

\[
\hat{S}_{b_i}(ξ_i) = \exp \left[ -\frac{1}{2} \left( ξ_i j b_i^\dagger - ξ_i^* b_i \right) \right],
\]

\[
\hat{S}_{B_i}(η_i) = \exp \left[ -\frac{1}{2} \left( η_i B_i^\dagger - η_i^* B_i \right) \right].
\]

For the convenience in later use we here introduce a two-mode squeezed state in the \( |B_1, B_3\rangle \) representation

\[
|ξ\rangle_{B_1B_3} = \hat{S}_{B_1}(ξ_1)\hat{S}_{B_3}(ξ_3)|0,0\rangle,
\]

where the two-mode squeezing operator is defined by

\[
\hat{S}_{B_1B_3}(ξ) = \exp \left[ -ξ\hat{B}_1^\dagger\hat{B}_3^\dagger + ξ^* \hat{B}_1\hat{B}_3 \right],
\]

where \( ξ \) is an arbitrary complex number.

It is straightforward to see that a direct-product state of two squeezed vacuum states in the \( |b_1, b_3\rangle \) and \( |B_1, B_3\rangle \) representations is transferred to an entangled state in the correspondent representation, respectively. In general, the entangled state in correspondent representation cannot be explicitly expressed as a product squeezed vacuum state for general squeezing parameters \( ξ \) and \( η \). However, a product squeezed state of two squeezed vacuum states with the same squeezing parameters in the \( |b_1, b_3\rangle \) representation may be transferred to a product squeezed vacuum state of two squeezed vacuum states with the same squeezing amplitudes but opposite phases in the \( |B_1, B_3\rangle \) representation, while a product squeezed vacuum state of two squeezed vacuum states with the same squeezing amplitudes but opposite phases in the \( |b_1, b_3\rangle \) representation is transferred to a two-mode squeezed state in the \( |B_1, B_3\rangle \) representation. And a product squeezed vacuum state of two squeezed vacuum states with the same squeezing parameters in the \( |B_1, B_3\rangle \) representation is transferred to a two-mode squeezed state with the same squeezing parameter in the \( |b_1, b_3\rangle \). These transformation relations are explicitly expressed as

\[
|ξ, ξ\rangle = |ξ, -ξ\rangle, \quad |ξ, -ξ\rangle = i|ξ\rangle_{B_1B_3}, \quad |ξ, ξ\rangle = |ξ\rangle_{b_1b_3}.
\]

In what follows we shall investigate generation of entangled squeezed vacuum states for the case in which BECs are initially in the two product squeezed vacuum states in the \( |b_1, b_3\rangle \) representation \( |ξ_i, -ξ\rangle \).

In this case, two BECs in the \( |b_1, b_3\rangle \) modes are initially in a product squeezed vacuum state of two squeezed vacuum states with the same squeezing amplitudes and the \( π \) phase difference. From Eq. (15) we know that after transferring to the \( |B_1, B_3\rangle \) representation, the system under our consideration is initially in a two-mode squeezed vacuum state. This initial state can be explicitly written as

\[
|Φ(0)\rangle = \frac{1}{\cosh r} \sum_{n=0}^{∞} [-ie^{iθ} \tanh r]^n |n, n\rangle,
\]

where \( \xi = r \exp(iθ) \), with \( r \) and \( θ \) real and positive. Then making use of Eqs. (29), (30), and (31) we know that at time \( t \) the system will be a state

\[
|Φ(t)\rangle = \frac{1}{\cosh r} \sum_{n=0}^{∞} \exp \{ it \left[ (q + \chi - 2g)n - (3q + \chi)n^2 \right] \}
\]

\[
\times [-ie^{-iθ} \tanh r]^n |n, n\rangle.
\]

When relevant parameters satisfy the conditions \( q = 2λ \) and \( 4g = -19q \), the wavefunction of the system (31) becomes

\[
|Φ(τ)\rangle = \frac{1}{\cosh r} \sum_{n=0}^{∞} \exp \left[ -\frac{i}{2} τn(n - 3) \right]
\]

\[
\times [-ie^{-iθ} \tanh r]^n |n, n\rangle,
\]

where we have set \( τ = 7qt \).

We note that the wavefunction of the system (31) differs from a conventional two-mode squeezed state (29) by an extra phase factor appearing in its decomposition into a superposition of Fock states. It can always be represented as a continuous sum of two-mode squeezed states. And under appropriate periodic conditions, it can reduce to discrete superpositions of two-mode squeezed states. It is this point that we use in present paper to create entangled squeezed states what we expect. Actually, the state (31) can be expressed as a continuous superposition of two-mode squeezed states

\[
|Φ(τ)\rangle = \int_{0}^{2π} \frac{dφ}{2π} g(φ)|ie^{iφ} ξ\rangle_{B_1B_3},
\]

where the phase \( g(φ) \) function is given by

\[
g(φ) = \sum_{n=0}^{∞} \exp \left[ -\frac{1}{2} τn(n - 3) - inφ \right]
\]

Since \( n(n - 3) \) is always even, the exponential function \( \exp[-iτn(n - 3)/2] \) in Eq. (33) is periodic function with the period \( T = 2π \). When \( τ = (M/N)2π \) with \( N \) and \( M \) being mutually prime integers, the phase function \( g(φ) \) is
a periodic function with respect to \( n \) with the period 2\( N \). Hence, the wavefunction may be expressed as a discrete superposition state of two-mode squeezed states

\[
\Phi \left( \tau = \frac{M}{N} 2\pi \right) = \sum_{n=0}^{2N-1} c_r \left[ ie^{i\varphi_r} \xi \right]_{B_1 B_3},
\]

where the running phase is defined by

\[
\varphi_r = \frac{\pi}{N}, \quad (r, s = 0, 1, 2, \cdots, 2N - 1).
\]

The coefficients in Eq. (34) are given by

\[
c_r = \frac{1}{(2N)^2} \sum_{n=0}^{2N-1} \exp \left\{ -\frac{i\pi}{N} \right\} \left[ nr - Mn(n - 3) \right].
\]

We now give two nontrivial examples of entangled squeezed vacuum states. The first one is the case of \( N = 2 \) and \( M = 1 \), i.e., \( \tau = \pi \) in Eq. (34). In this case, from Eq. (36) we find that there exist only two nonzero \( c \)-coefficients \( c_1 = c_2 = \frac{1}{2}\sqrt{2} \exp(\pi i/4) \), which leads to the following superposition state of two two-mode squeezed states with the same squeezing amplitudes but opposite phase in the \((\hat{B}_1, \hat{B}_3)\) representation

\[
\Phi (\tau = \pi) = \frac{1}{\sqrt{2}} \left[ -\xi_{B_1 B_3} - i|\xi_{B_1 B_3}| \right],
\]

where we have discarded the common phase factor \( \exp(-i\pi/4) \) on the right-hand side of above equation.

After transferring to the \((\hat{b}_1, \hat{b}_3)\) representation, we obtain an entangled state of two product squeezed vacuum states

\[
\Phi (\tau = \pi) = \frac{1}{\sqrt{2}} \left[ i\xi_{B_1 B_3} - i|\xi_{B_1 B_3}| \right].
\]

IV. CONCLUDING REMARKS

We have presented a scheme for the generation of entangled squeezed states for atomic BECs with the Raman-coupled configuration. In the proposed scheme quantum entanglement is created through laser-atom interactions and inter-atomic interactions in the BECs. Under the large detuning and exact two-photon resonant condition, the atomic field operators at the upper level is adiabatically eliminated, the system becomes an effective two-mode system. In this process the laser-atom interactions are converted as an atomic effective tunnelling interaction between two lower levels with the tunnelling coupling strength to be determined by strengths of the laser-atom interactions and the laser detunings. We have discussed how to create two-state and multi-state entangled squeezed states and superposition states of two-mode squeezed states. When the initial state of the two-mode system is a product squeezed state with the same squeezing amplitudes and phases, superposition states of two-mode squeezed states can be created, while when the initial state of the system is a product squeezed state with the same squeezing amplitudes but opposite phases, entangled squeezed states can be generated. We have found that generation of different entangled states are strongly manipulated by varying the initial states of the system. Thus, one can create a variety of entangled states by preparing different initial states.

In our scheme the essential requirements to achieve entangled squeezed states include the exact two-photon resonance, the large detuning of laser frequencies with respect to relevant atomic transitions, and manipulation of strengths of laser-atom interactions and inter-atomic weak nonlinear interactions. The former two can be realized through adjusting frequencies of lasers. Laser-atom interaction strengths can be changed through controlling polarizations and intensities of lasers. Finally, inter-atomic nonlinear interactions can be manipulated through changing atomic scattering lengths in BECs. Recent experiments on Feshbach resonances in a Bose condensate have indicated that the scattering length of ultracold atoms can be altered through Feshbach resonance. It is also worth noting that the Yale group has successfully produced squeezed-state atomic BECs. These experimental advances together with current mature detecting techniques for atomic BECs provide us with the possibility to create and to observe experimentally entangled squeezed states in atomic condensates. However, it should be mentioned that quantum entanglement discussed in present paper is of particular nature: the entangled subsystems are not spatially separated. This characteristic may limit its use. How to make use of such kind of quantum entanglement as a resource to carry out quantum information processing is an interesting topics for further study.
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